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Hypo-efficient domination and hypo-unique domination

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Abstract: For a graph G let $\gamma(G)$ be its domination number. We define a graph G to be (i) a hypo-efficient domination graph (or a hypo-$\mathcal{ED}$ graph) if G has no efficient dominating set (EDS) but every graph formed by removing a single vertex from G has at least one EDS, and (ii) a hypo-unique domination graph (a hypo-$\mathcal{UD}$ graph) if G has at least two minimum dominating sets, but $G - v$ has a unique minimum dominating set for each $v \in V(G)$. We show that each hypo-$\mathcal{UD}$ graph G of order at least 3 is connected and $\gamma(G - v) < \gamma(G)$ for all $v \in V$. We obtain a tight upper bound on the order of a hypo-$\mathcal{P}$ graph in terms of the domination number and maximum degree of the graph, where $\mathcal{P} \in \{\mathcal{UD}, \mathcal{ED}\}$. Families of circulant graphs, which achieve these bounds, are presented. We also prove that the bondage number of any hypo-$\mathcal{UD}$ graph is not more than the minimum degree plus one.

Keywords: domination number, efficient domination, unique domination, hypo-property.

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1. Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes et al. [15]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The *complement* $\overline{G}$ of G is the graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of G. The *join* of graphs G and H, written $G \vee H$, is the graph obtained from the

disjoint union of G and H by adding the edges $\{xy \mid x \in V(G),\ y \in V(H)\}$. In a graph G, for a subset $S \subseteq V(G)$ the *subgraph induced* by S is the graph $\langle S \rangle$ with vertex set S and edge set $\{xy \in E(G)\colon x, y \in S\}$. We write K_n for the *complete graph* of order n and C_n for the *cycle* of length n. Let P_m denote the *path* with m vertices. For any vertex x of a graph G, $N_G(x)$ denotes the set of all neighbors of x in G, $N_G[x] = N_G(x) \cup \{x\}$ and the *degree* of x is $deg_G(x) = |N_G(x)|$. The *minimum* and *maximum* degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A *leaf* of a graph is a vertex of degree 1, while a *support vertex* is a vertex adjacent to a leaf. For a subset $A \subseteq V(G)$, let $N_G[A] = \bigcup_{x \in A} N_G[x]$. The *coalescence* of disjoint graphs H and G is the graph $H \cdot G$ obtained by identifying one vertex of H and one vertex of G.
A set D of vertices in a graph G *dominates* a vertex $u \in V(G)$ if either $u \in D$ or u is adjacent to some $v \in D$. If D dominates all vertices in a subset T of $V(G)$ we say that D *dominates* T. When D dominates $V(G)$, D is called a *dominating set* of the graph G. That is, D is a dominating set if and only if $N[D] = V(G)$. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G, and a dominating set of G with cardinality $\gamma(G)$ is called a γ*-set* of G. A dominating set D is called an *efficient dominating set* (EDS) if D dominates every vertex exactly once [2]. A vertex v of a graph G is γ*-critical* if $\gamma(G - v) < \gamma(G)$. We denote by $V^-(G)$ the set of all γ-critical vertices of G. A graph G is a *vertex domination-critical graph* (or a *vc-graph*) if $V^-(G) = V(G)$ [6]. The concept of domination in graphs has many applications to several fields. Domination naturally arises in facility location problems, in monitoring communication or electrical networks, in land surveying, and in problems involving finding sets of representatives. Many variants of the basic concepts of domination have appeared in the literature. We refer to [12, 14–16] for a survey of the area.
Let $\mathcal{I}$ denote the set of all mutually nonisomorphic graphs. A graph property is any nonempty subset of $\mathcal{I}$. We say that a graph G has the property $\mathcal{P}$ whenever there exists a graph $H \in \mathcal{P}$ wich is isomorphic to G. Any set $S \subseteq V(G)$ such that the induced subgraph $\langle S \rangle$ possesses the property $\mathcal{P}$ is called a $\mathcal{P}$-set.
If a graph G does not possess a given property $\mathcal{P}$, and for each vertex v of G the graph $G - v$ has property $\mathcal{P}$, then G is said to be a *hypo-*$\mathcal{P}$ *graph*. A number of studies have been made where $\mathcal{P}$ stands for the graph being hamiltonian (see [28] and references therein), traceable (see [1] and references therein), planar [26], outerplanar [21], eulerian and randomly-eulerian [18]. Let us also mention hypomatchable graphs (for a survey up to 2003 see [24]). Here we focus on the case when $\mathcal{P} \in \{\mathcal{ED}, \mathcal{UD}\}$, where

- $\mathcal{ED} = \{H \in \mathcal{I} : H$ *has an efficient dominating set*$\}$, and
- $\mathcal{UD} = \{H \in \mathcal{I} : H$ *has exactly one* γ*-set*$\}$.

More formally, we define:

- A graph G is an *efficient domination graph* (or an *$\mathcal{ED}$-graph*) if G has an EDS [19].
- A graph G is a *unique domination graph* (or a *$\mathcal{UD}$-graph*) if G has exactly one γ-set.

For results on graphs with a unique minimum dominating set see [10] and references therein.

- A graph G is a *hypo-efficient domination graph* (or a *hypo-$\mathcal{ED}$ graph*) if G has no EDS but every graph formed by removing a single vertex from G has at least one EDS.
- A graph G is a *hypo-unique domination graph* (or a *hypo-$\mathcal{UD}$ graph*) if G has at least two γ-sets, but $G - v$ has a unique minimum dominating set for each $v \in V(G)$.

One measure of the stability of the domination number of G under edge removal is the bondage number $b(G)$, defined in [9] as the smallest number of edges whose removal from G results in a graph with larger domination number. In general it is hard to determine the bondage number $b(G)$ (see Hu and Xu [17]), and thus useful to find bounds for it. The interested readers can see [27] for a survey on this topic. The concept of vc-graphs plays an important role in the study of the bondage number. The reason for this is at least the fact that if G is a graph and $b(G) > \Delta(G)$, then G is a vc-graph [25]. It is well known that any vc-graph G has at most $(\Delta(G)+1)(\gamma(G)-1)+1$ vertices [6]. Hence $b(G) \leq \Delta(G)$ for any graph G with more than $(\Delta(G)+1)(\gamma(G)-1)+1$ vertices. In order to find graphs G with a high bondage number (i.e., higher than $\Delta(G)$), we, therefore, have to look at vc-graphs. In the process of studying vc-graphs G having $(\Delta(G)+1)(\gamma(G)-1)+1$ vertices, the author has found that for every vertex x of G, $G - x$ has exactly one γ-set and the unique γ-set of $G - x$ is efficient dominating. This fact motivated the author to begin the study of the hypo-efficient domination graphs and hypo-unique domination graphs.
The paper is organized as follows. Section 2 contains some known results which are used in what follows. In Section 3 we prove that each hypo-$\mathcal{UD}$ graph of order at least 3 is a connected vc-graph and we obtain sharp upper bounds in terms of (a) domination number, and (b) domination number and maximum degree for the order of a hypo-$\mathcal{P}$ graph, where $\mathcal{P} \in \{\mathcal{UD}, \mathcal{ED}\}$. Families of circulant graphs which achieve these bounds are presented. We also prove that the bondage number of any hypo-$\mathcal{UD}$ graph is not more than the minimum degree plus one. We conclude in Section 4 with some open problems.

2. Known results

Theorem 1. *[3] Let G be a graph. If G has vertex set $V(G) = \{v_1, v_2, .., v_n\}$, then G has an EDS if and only if some subcollection of $\{N[v_1], N[v_2], .., N[v_n]\}$ partitions $V(G)$. If G has an EDS, then the cardinality of any EDS of G equals the domination number of G.*

Lemma 1. *[5] Let G be a graph and $x, y \in V(G)$. If x is γ-critical, then $\gamma(G-x) = \gamma(G)-1$ and no vertex in $N_G(x)$ is in a γ-set of $G-x$. If $\gamma(G-y) > \gamma(G)$, then y is in all γ-sets of G.*

Theorem 1 and Lemma 1 will be used in the sequel without specific reference.

Theorem 2. *Let G be a graph.*

(i) [6] G is a vc-graph if and only if each block of G is a vc-graph.

(ii) [6] If G is a vc-graph then $|V(G)| \leq (\Delta(G)+1)(\gamma(G)-1)+1$.

(iii) [11] If G is a vc-graph and $|V(G)| = (\Delta(G)+1)(\gamma(G)-1)+1$, then G is regular.

Remark 1. By Theorem 2(i), if G is a connected nontrivial vc-graph, then G is 2-edge connected and $\delta(G) \geq 2$.

The *corona* of graphs H and K_1 is the graph $H \circ K_1$ constructed from a copy of H, where for each vertex $v \in V(H)$, a new vertex v' and a pendant edge vv' are added. Hence $H \circ K_1$ has even order.

Theorem 3. *Let a graph G have no isolated vertices.*

(a) [22] Then $\gamma(G) \leq |V(G)|/2$.

(b) [9, 23] $\gamma(G) = |V(G)|/2$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H.

Let $\mathcal{A} = \{H_1, ..., H_7\}$ be the collection of graphs in Figure 1.

Theorem 4. *[20] If G is a connected graph with $\delta(G) \geq 2$ and $G \notin \mathcal{A}$, then $\gamma(G) \leq \frac{2}{5}|V(G)|$.*

Let n be a positive integer and $S = \{n_1, n_2, ..., n_k\}$ a set of integers such that $0 < n_1 < ... < n_k \leq \lfloor n/2 \rfloor$. The *circulant graph* $C(n, S)$ is a graph with $V(C(n, S)) = \{0, 1, ..., n-1\}$, and such that each vertex i, $0 \leq i \leq n-1$, is adjacent to all the vertices $i \pm n_1, i \pm n_2, ..., i \pm n_k \pmod{n}$. If $n_k \neq n/2$, then

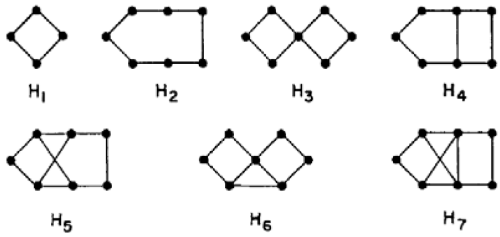

Figure 1. $\delta(H_i) \geq 2$ **and** $\gamma(H_i) > 2|V(H_i)|/5$**, for** $i = 1, ..., 7$ **[20].**

$C(n, S)$ is regular of degree $2k$. When $n_k = n/2$, $C(n, S)$ is regular of degree $2k-1$.

Theorem 5. *Let* $G = C(n; \{1, 2, ..., k\})$*, where* $n \geq 3$ *and* $1 \leq k < \lfloor n/2 \rfloor$*. Then (a) [13]* $\gamma(G) = \left\lceil \frac{n}{2k+1} \right\rceil$*, and (b) [8]* G *is a vc-graph if and only if* $2k+1$ *divides* $n-1$*.*

Lemma 2. *[25] If* G *is a nontrivial graph with a unique minimum dominating set, then* $b(G) = 1$*.*

3. Hypo-unique and hypo-efficient domination

We begin with results on hypo-$\mathcal{UD}$ graphs. Our first theorem shows that each hypo-$\mathcal{UD}$ graph of order at least 3 is a connected vc-graph.

Theorem 6. *If* G *is a hypo-*$\mathcal{UD}$ *graph, then either* $G = K_2$ *or* G *is a connected vc-graph with* $|V(G)| \geq 4$*.*

Proof. Let us assume that G is not connected. Then G has at least 2 connected components, say G_1 and G_2. Let $v_i \in V(G_i)$, $i = 1, 2$. Since each of $G_1 - v_1$ and $G_2 - v_2$ either is order-zero graph or has a unique γ-set, G has exactly one γ-set, which is a contradiction. Thus G is connected.
To proceed we need the following claim.
Claim 1. If $G = H \circ K_1$, where H is a connected graph of order at least 2, then $V^-(G) = V(G) - V(H)$ and G is not a hypo-$\mathcal{UD}$ graph.

Proof of Claim 1. Recall that $\gamma(G) = |V(G)|/2$ for any corona G (Theorem 3). If $x \in V(H)$ and y is the leaf neighbor of x, then (a) $V(H-x)$ is a γ-set of $G-y$, which implies $V(G)-V(H) \subseteq V^-(G)$, and (b) $G-x$ is disjoint union of K_1 and $(H-x) \circ K_1$ which leads to $\gamma(G-x) = \gamma(G)$. Thus $V^-(G) = V(G) - V(H)$.

Since $V(G)-V(H)$ and $\{y\}\cup V(H-x)$ are γ-sets of $G-x$, G is not a hypo-$\mathcal{UD}$ graph. $\square$

Case 1: $V^-(G) \neq \emptyset$. For any $x \in V^-(G)$ let D_x be the unique γ-set of $G-x$. Then $D_x \cup \{y\}$ is a γ-set of G for every $y \in N[x]$. This implies that $\gamma(G-z) \leq \gamma(G)$ for any $z \in V(G) - D_x$, in particular when $z \in N[x]$. Now since G is a hypo-$\mathcal{UD}$ graph, (a) $V(G) - (D_x \cup N(x)) \subseteq V^-(G)$, and (b) if $x \in V^-(G)$ and $deg(x) \geq 2$, then $N[x] \subseteq V^-(G)$.
From (b) we conclude that, if G has a γ-critical vertex of degree at least 2, then $V^-(G) = V(G)$, as required. So, let each γ-critical vertex of G be a leaf. Let $x \in V^-(G)$ and $N(x) = \{y\}$. Since x is a leaf, $y \notin V^-(G)$. Since D_x is the unique γ-set of $G-x$, there is no leaf in D_x. Now by (a), $D_x \cup \{y\}$ and $V^-(G)$ form a partition of $V(G)$. As $D_x \cup \{y\}$ is a γ-set of G, $V^-(G)$ is a dominating set of G. This implies that each element of $D_x \cup \{y\}$ is adjacent to a leaf. Assume that there is a vertex $z \in D_x$ which is adjacent to at least 2 leaves. Then z is in all γ-sets of G which implies that all leaf neighbors of z are outside $V^-(G)$, a contradiction. Thus G is a corona of a connected graph of order at least 2. But this is again a contradiction because of Claim 1.
Case 2: $V^-(G) = \emptyset$. Since G is a hypo-$\mathcal{UD}$ graph, there are at least 2 different γ-sets of G, say D_1 and D_2. If there is $x \in V(G) - (D_1 \cup D_2)$, then since $\gamma(G-x) = \gamma(G)$, both D_1 and D_2 are γ-sets of $G-x$ - a contradiction. Hence $D_1 \cup D_2 = V(G)$ which implies $2\gamma(G) \geq |V(G)|$. By Theorem 3, $2\gamma(G) = |V(G)|$ and either G is a connected corona or $G = C_4$. Now Claim 1 and $V(C_4) = V^-(C_4)$ together lead to $G = K_2$. Clearly K_2 is a hypo-$\mathcal{UD}$ graph. $\square$

Not all vc-graphs are hypo-$\mathcal{UD}$ graphs. For example any coalescence $C_{3k+1} \cdot C_{3l+1}$ is a vc-graph which is not a hypo-$\mathcal{UD}$ graph.

Corollary 1. *If G is a hypo-$\mathcal{UD}$ graph of order $n \geq 4$, then G is 2-edge connected and $\delta(G) \geq 2$. Moreover, all hypo-$\mathcal{UD}$ unicyclic graphs are C_{3k+1}, $k \geq 1$.*

Proof. By Theorem 6, G is a vc-graph. Now by Remark 1, G is 2-edge connected and $\delta(G) \geq 2$. Hence if G is unicyclic, then $G = C_n$. Since all paths P_m, $m \geq 2$, having a unique minimum dominating set are P_{3k}, $k \geq 1$, it follows that $G = C_n$ is a hypo-$\mathcal{UD}$ graph if and only if $n = 3k+1$. $\square$

Corollary 2. *Let G be a hypo-$\mathcal{UD}$ graph of order at least 4.*

(i) For any $x \in V(G)$, the graph $G-x$ has no γ-critical vertices.

(ii) For any pair x, y of vertices of G, $\gamma(G-\{x,y\}) \geq \gamma(G)-1$. The equality holds at least when y does not belong to the unique γ-set of $G-x$.

Proof. If $x \in V(G)$, then $\gamma(G-x) = \gamma(G) - 1$ (by Theorem 6). Assume that there is $u \in V^-(G-x)$. Then for any $v \in N_{G-x}[u]$ and any γ-set D of $G - \{x, u\}$, the set $\{v\} \cup D$ is a γ-set of $G-x$. Since $G-x$ has exactly one γ-set and $\delta(G-x) \geq 1$ (by Corollary 1), we arrive to a contradiction. Thus, (i) holds and for any pair x, y of vertices of G, $\gamma(G - \{x, y\}) \geq \gamma(G) - 1$. Finally, since the removal of a vertex which belongs to no γ-set of a graph has no effect on the domination number, $\gamma(G - \{x, y\}) = \gamma(G) - 1$ whenever y does not belong to the unique γ-set of $G - x$. □

Proposition 1. *Let G be a connected vc-graph of order $n \geq 4$. Then $\gamma(G) \leq \lfloor 2n/5 \rfloor + 1$. The equality holds if and only if $G \in \mathcal{A}$.*

Proof. It is easy to check that if $G \in \mathcal{A}$, then G is a vc-graph and $\gamma(G) = \lfloor 2n/5 \rfloor + 1$. By Remark 1, if G is a vc-graph, then $\delta(G) \geq 2$. Now by Theorem 4 we have $\gamma(G) \leq \lfloor 2n/5 \rfloor$ when $G \notin \mathcal{A}$. □

Proposition 2. *Let G be a hypo-$\mathcal{UD}$ graph of order n. Then $1 \leq \gamma(G) \leq \lfloor 2n/5 \rfloor + 1$. Furthermore, (i) $\gamma(G) = 1$ if and only if $G = K_2$, (ii) $\gamma(G) = 2$ if and only if $n \geq 4$ is even and G is K_n minus a perfect matching, and (iii) $\gamma(G) = \lfloor 2n/5 \rfloor + 1$ if and only if $G \in \{K_2, C_4, C_7\}$.*

Proof. By Theorem 6, either $G = K_2$ or G is a connected vc-graph. Now $\gamma(K_2) = 1$ and Proposition 1 lead to $\gamma(G) \leq \lfloor 2n/5 \rfloor + 1$.
(i) Let G be a hypo-$\mathcal{UD}$ graph with $\gamma(G) = 1$. Then G has $r \geq 2$ vertices of degree $n - 1$. If $v \in V(G)$ and $deg(v) \leq n - 2$, then $G - v$ has r γ-sets, a contradiction. Thus, $G = K_r$. But clearly, among all complete graphs, only K_2 is a hypo-$\mathcal{UD}$-graph.
(ii) Each vc-graph G with $\gamma(G) = 2$ can be obtained from a complete graph of even order by removing a perfect matching [6]. Obviously, every such a graph is a hypo-$\mathcal{UD}$-graph. The result now follows by Theorem 6.
(iii) Let $\gamma(G) = \lfloor 2n/5 \rfloor + 1$. Then either $G = K_2$ or $G \in \mathcal{A}$ (by Proposition 1). It is easy to see that among all these graphs only K_2, C_4 and C_7 are hypo-$\mathcal{UD}$-graphs. □

Proposition 3. *If G is a hypo-$\mathcal{UD}$ n-order graph, then*

$$n \leq (\Delta(G) + 1)(\gamma(G) - 1) + 1.$$

Proof. By Theorem 6, G is a vc-graph or $G = K_2$. The result now follows by Theorem 2. □

The bound in the above corollary is attainable. This is shown in Proposition 6.

Theorem 7. *If G is a hypo-$\mathcal{UD}$ graph, then $b(G) \leq \delta(G) + 1$.*

Proof. If $G = K_2$, then the result is obvious. So, let $G \neq K_2$. By Theorem 6, G is a vc-graph of order at least 4. Denote by G_x the graph obtained from G by removal of all edges incident to $x \in V(G)$, where $deg(x) = \delta(G)$. Since G is a hypo-$\mathcal{UD}$ graph, G_x has a unique minimum dominating set. Since $\delta(G) \geq 2$ (by Corollary 1), G_x has edges. Lemma 2 now implies that there is an edge of G_x, say e, such that $\gamma(G_x - e) > \gamma(G_x)$. But then $\gamma(G) = \gamma(G - x) + 1 = \gamma(G_x) < \gamma(G_x - e)$. Thus $b(G) \leq deg(x) + 1 = \delta(G) + 1$. □

The bound stated in Theorem 7 is tight at least when $G \in \{C_{3k+1} \mid k \geq 1\}$. We now concentrate on hypo-$\mathcal{ED}$ graphs.

Proposition 4. *Let G be a hypo-$\mathcal{ED}$ n-order graph. Then G is connected, $n \geq 4$, and $2 \leq \gamma(G) \leq n/2$. Furthermore, $\gamma(G) = n/2$ if and only if $G = C_4$.*

Proof. Let G_1 and G_2 be connected components of G and $v_i \in V(G_i)$, $i = 1, 2$. Since each of $G - v_1$ and $G - v_2$ has an EDS, G has an EDS - a contradiction. Thus G is connected. It is easy to check that C_4 is the unique hypo-$\mathcal{ED}$ graph of order at most 4. If G has a vertex of degree $n-1$, then G has an EDS. Hence $\gamma(G) \geq 2$. Finally, by Theorem 3 we have that $\gamma(G) \leq n/2$ and if the equality holds, then either G is C_4 or G is a corona of a connected graph. Since the set of all leaves of any corona is an EDS, the result immediately follows. □

Next we present a tight upper bound on the order of a hypo-$\mathcal{ED}$ graph in terms of the domination number and maximum degree of the graph.

Theorem 8. *Let G be a graph without efficient dominating sets. Then $|V(G)| \leq \gamma(G)(\Delta(G) + 1) - 1$.*

(i) Let the equality holds. Then (a) for every γ-set D of G there is exactly one vertex $y_D \in V(G) - D$ such that D is an efficient dominating set of $G - y_D$ and y_D is adjacent to exactly 2 vertices in D, and (b) each vertex belonging to some γ-set of G has maximum degree. In particular, if each vertex of G belongs to some γ-set of G, then G is regular.

(ii) If there are a γ-set D of G and a vertex y of $G - D$ such that D is an efficient dominating set of $G - y$, y is adjacent to exactly 2 vertices of D and all vertices of D have maximum degree, then $\gamma(G)(\Delta(G) + 1) - 1 = |V(G)|$.

Proof. Let $D = \{x_1, x_2, ..., x_k\}$ be an arbitrary γ-set of G. If $x_i x_j \in E(G)$, then

$$|V(G)| \leq \Sigma_{r=1}^{k} |N[x_r]| - |\{x_i, x_j\}| = \Sigma_{r=1}^{k} (deg(x_r) + 1) - 2 \leq \gamma(G)(\Delta(G) + 1) - 2.$$

If $x_i x_j \notin E(G)$ and $y \in V(G) - D$ is a common neighbor of both x_i and x_j, then

$$|V(G)| \leq \Sigma_{r=1}^{k} |N[x_r]| - |\{y\}| = \Sigma_{r=1}^{k} (deg(x_r) + 1) - 1 \leq \gamma(G)(\Delta(G) + 1) - 1.$$

(i) Suppose $|V(G)| = \gamma(G)(\Delta(G) + 1) - 1$. Then $|V(G)| = \Sigma_{r=1}^{k} |N[x_r]| - |\{y\}|$ and $\Sigma_{r=1}^{k}(deg(x_r) + 1) - 1 = \gamma(G)(\Delta(G) + 1) - 1$. Since D is a γ-set, (a) by the first equality we have that D is independent, each vertex in $V(G) - (D \cup \{y\})$ is adjacent to exactly one vertex of D, and y is adjacent to exactly 2 vertices in D, and (b) by the second equality, it follows that $deg(x_r) = \Delta(G)$ for all $r = 1, 2, ..., k$. The rest is obvious.
(ii) Assume now that there is a γ-set $D = \{x_1, x_2, ..., x_k\}$ of G such that $deg(x_1) = ... = deg(x_k) = \Delta(G)$, D is an efficient dominating set of $G - y$ for some vertex $y \in V(G) - D$ and y has exactly 2 elements of D as neighbors. Then $|V(G)| = \Sigma_{r=1}^{k} |N_{G-y}[x_r]| + |\{y\}| = \Sigma_{r=1}^{k} |N_G[x_r]| - |\{y\}| = \gamma(G)(\Delta(G) + 1) - 1$. $\square$

Corollary 3. *Theorem 8 is valid when G is a hypo-$\mathcal{ED}$ graph.*

We give the following examples to illustrate the sharpness of the bound in Theorem 8.

Example 1. All hypo-$\mathcal{ED}$ cycles are C_{3k+1} and C_{3k+2}, $k \geq 1$. Moreover, $|V(C_{3k+2})| = \gamma(C_{3k+2})(\Delta(C_{3k+2}) + 1) - 1$, $k \geq 1$.

Example 2. If $G \in \{C(8k + 5, \{1, ..., k\} \cup \{3k + 2, ..., 4k + 2\}) \mid k \geq 1\}$, then G is a hypo-$\mathcal{ED}$ graph with $|V(G)| = \gamma(G)(\Delta(G) + 1) - 1$.

Proof. First note that G is $(4k + 2)$-regular graph of order $8k + 5$. Hence $\gamma(G) \geq 2$. Since for any $r \in V(G)$ the vertex set $\{r, r + 2k + 1\}$ is dominating for G and $N[r] \cap N[r + 2k + 1] = \{r + 5k + 3\}$, it follows that $\gamma(G) = \gamma(G - \{r + 5k + 3\}) = 2$ and $\{r, r + 2k + 1\}$ is an efficient dominating set for $G - \{r + 5k + 3\}$ (where addition is taken mod $8k + 5$). Thus G is a hypo-$\mathcal{ED}$ graph and clearly $|V(G)| = \gamma(G)(\Delta(G) + 1) - 1$ holds. $\square$

Example 3. Let $G \in \{C(t(2k + 1) - 1, \{1, ..., k\}) \mid k \geq 1, t \geq 2\}$. Then G is a hypo-$\mathcal{ED}$ graph with $|V(G)| = \gamma(G)(\Delta(G) + 1) - 1$.

Proof. A graph G is $2k$-regular of order $n = t(2k + 1) - 1$ and by Theorem 5, $\gamma(G) = t$. Assume first t is odd. Then the set $D_r = \{r \pm l(2k + 1) \pmod{n} \mid l \in \{0, 1, ..., (t - 1)/2\}\}$ is a γ-set of G for any vertex r of G. Furthermore, the distance between any pair of distinct vertices of D_r is at least 3, except for the pair $a_1 = r + (t-1)(2k+1)/2$, $a_2 = r - (t-1)(2k+1)/2$. Since $N[a_1]$ and $N[a_2]$ have exactly the vertex $a_1 + k$ in common, D_r is an EDS of $G - \{a_1 + k\}$ for any vertex r of G.

Assume now t is even. Then the set $U_r = \{r \pm s(2k+1) \pmod{n} \mid s \in \{0, 1, ..., (t-2)/2\}\} \cup \{r + t(2k+1)/2 - 1\}$ is a γ-set of G for any vertex r of G. Note that the distance between any pair of distinct vertices of U_r is at least 3, except for the pair $b_1 = r + (t-2)(2k+1)/2$, $b_2 = r + t(2k+1)/2 - 1$. Since $N[b_1] \cap N[b_2] = \{b_1 + k\}$, U_r is an EDS of $G - \{b_1 + k\}$ for any vertex r of G. □

Now we turn our attention to the hypo-$\mathcal{ED}$ graphs having γ-critical vertices.

Proposition 5. *A connected vc-graph G is a hypo-$\mathcal{E}D$ graph if and only if $G - v$ has an efficient dominating set for all $v \in V(G)$.*

Proof. $\Rightarrow$ Obvious.
$\Leftarrow$ If D is an EDS of G and $v \in V(G) - D$, then D is an EDS $G - v$. Now by Theorem 1, $\gamma(G) = |D| = \gamma(G - v)$, a contradiction. □

Theorem 9. *Let G be a hypo-$\mathcal{ED}$ vc-graph. Then for every vertex $v \in V(G)$, $G - v$ has exactly one efficient dominating set. If in addition G is regular, then G is a hypo-$\mathcal{UD}$ graph.*

Proof. Let $x \in V(G)$, D_x an EDS of $G - x$, $y \in D_x$ and let D_y be an EDS of $G - y$. Note that D_y and $N[y]$ are disjoint and $|D_y| = \gamma(G - y) = \gamma(G) - 1 = \gamma(G - x) = |D_x|$. Hence there exists exactly one vertex of D_y, say z, which is not dominated by D_x. But D_x is a γ-set of $G - x$. Thus $z \equiv x$. As D_y was chosen arbitrarily, x belongs to all EDS of $G - y$. By symmetry y belongs to all EDS of $G - x$. This allow us to deduce that D_x is the unique EDS of $G - x$. Finally, let G be k-regular. Then all vertices of D_x have degree k in $G - x$ and $|V(G - x)| = |D_x|(k+1) = \gamma(G - x)(\Delta(G - x) + 1)$. This implies that all γ-sets of $G - x$ are efficient dominating. But we already know that $G - x$ has exactly one EDS. Thus G is a hypo-$\mathcal{UD}$ graph. □

Theorem 10. *Let a hypo-$\mathcal{E}D$ graph G have a γ-critical vertex. Then*

$$(\delta(G) + 1)(\gamma(G) - 1) + 1 \leq |V(G)| \leq (\Delta(G) + 1)(\gamma(G) - 1) + 1.$$

Proof. If x is a γ-critical vertex of G and $D = \{u_1, ..., u_k\}$ is an EDS of $G - x$, then the sets $\{x\}, N[u_1], ..., N[u_k]$ form a partition of $V(G)$. Since $\delta(G) + 1 \leq |N[u_i]| = deg(u_i) + 1 \leq \Delta(G) + 1$, $i = 1, ..., k$, we have

$$1+(\delta(G)+1)(\gamma(G)-1) \leq |\{x\}|+\Sigma_{i=1}^{k}|N[u_i]| = |V(G)| \leq 1+(\Delta(G)+1)(\gamma(G)-1).$$

□

Corollary 4. *If G is a regular hypo-$\mathcal{E}D$ graph having a γ-critical vertex, then* $|V(G)| = (\Delta(G) + 1)(\gamma(G) - 1) + 1 = (\delta(G) + 1)(\gamma(G) - 1) + 1$.

Theorem 11. *Let G be a connected graph with $(\Delta(G) + 1)(\gamma(G) - 1) + 1 \geq 4$ vertices. If G is a vc-graph, then G is both a hypo-$\mathcal{E}D$ graph and a hypo-$\mathcal{U}D$ regular graph.*

Proof. Let G be a vc-graph. By Theorem 2, G is regular. Let $x \in V(G)$ and $D = \{x_1, ..., x_k\}$ a γ-set of $G - x$. Then

$$|V(G-x)| \leq \Sigma_{r=1}^{k}|N[x_r]| = \Sigma_{r=1}^{k}(\Delta(G)+1) = (\gamma(G)-1)(\Delta(G)+1) = |V(G-x)|.$$

Hence, $N[x_1], N[x_2], ..., N[x_k]$ form a partition of $V(G-x)$ and we can conclude that D is an EDS of $G - x$. Thus G is a hypo-$\mathcal{E}D$ graph. Now by Theorem 9, G is a hypo-$\mathcal{U}D$ graph. □

Proposition 6. *Let $G = C(n; \{1, 2, ..., k\})$, where $n \geq 4$ and $1 \leq k < \lfloor n/2 \rfloor$. Then G is a hypo-$\mathcal{U}D$ graph if and only if $2k+1$ divides $n-1$. If $2k+1$ divides $n-1$, then $n = |V(G)| = (\Delta(G) + 1)(\gamma(G) - 1) + 1$, and G is a hypo-$\mathcal{E}D$ graph.*

Proof. Note that G is a $2k$-regular. First let $2k + 1$ divides $n - 1$. By Theorem 5 we have that $n = |V(G)| = (\Delta(G) + 1)(\gamma(G) - 1) + 1$ and G is a vc-graph. Now G is both a hypo-$\mathcal{E}D$ graph and a hypo-$\mathcal{U}D$ graph, because Theorem 11.
If G is a hypo-$\mathcal{U}D$ graph, then by Theorem 6, G is a vc-graph. But then Theorem 5 implies that $2k + 1$ divides $n - 1$. □

4. Open problems and questions

We conclude the paper by listing some interesting problems and directions for further research. Let $\mathcal{P} \in \{\mathcal{ED}, \mathcal{UD}\}$.

- *Find all ordered pairs (n, k) of integers such that there is a hypo-$\mathcal{P}$ graph G of order n and the domination number k.*

If $\mathcal{P} = \mathcal{UD}$, then by Proposition 2, $1 \leq \gamma(G) \leq \lfloor 2n/5 \rfloor + 1$. Furthermore, (a) if $k = 1$, then $n = 2$, (b) if $k = 2$, then $n \geq 4$ is even, and (c) if $k = \lfloor 2n/5 \rfloor + 1$, then $(n, k) \in \{(2, 1), (4, 2), (7, 3)\}$. Note that in [20] a characterization is given for the connected n-order graphs G for which $\gamma(G) = 2n/5$.
If $\mathcal{P} = \mathcal{ED}$, then $2 \leq \gamma(G) \leq n/2$ (Proposition 4) and moreover if $\gamma(G) = n/2$, then $n = 4$. A characterization of n-vertex connected graphs G whose domination number satisfies $\gamma(G) = (n - 1)/2$ is obtained in [4].

- *If G is a hypo-$\mathcal{P}$ graph of order n and the domination number k, what is the maximum/minimum number of edges in G?*

- *Find all hypo-$\mathcal{ED}$ trees and all hypo-$\mathcal{ED}$ unicyclic graphs.*

- *Characterize the hypo-$\mathcal{ED}$ graphs G with $\gamma(G) = 2$.*

- *Characterize the hypo-$\mathcal{P}$ graphs G for which $\overline{G}$ is also a hypo-$\mathcal{P}$ graph. In particular, characterize/find all self complementary hypo-$\mathcal{P}$ graphs.*

If both G and $\overline{G}$ are hypo-$\mathcal{P}$ graphs, then by Theorem 6 and Proposition 4, it follows that both G and $\overline{G}$ must be connected. Note that C_5 and the *bull* (the graph obtained from $K_3 \circ K_1$ by removing exactly one leaf) are self-complementary hypo-$\mathcal{ED}$ graphs.

- *Characterize the hypo-$\mathcal{ED}$ graphs G such that $\overline{G}$ has an EDS.*

If a graph G is K_{2n} minus a perfect matching, $n \geq 2$, then we already know that G is a hypo-$\mathcal{ED}$ graph. Since $\overline{G}$ is a union of n copies of K_2, $\overline{G}$ has an EDS.

- *Characterize the hypo-$\mathcal{UD}$ graphs G such that $\overline{G}$ has a unique γ-set.*

Brigham et al. [7] defined a graph G to be *domination bicritical* if $\gamma(G - S) < \gamma(G)$ for any set $S \subseteq V(G)$ of 2 vertices.

- *Does there exist a bicritical hypo-$\mathcal{UD}$ graph?*

- *Does there exist a hypo-$\mathcal{UD}$ graph with a cut-vertex?*

A graph G is γ-*EA-critical* if $\gamma(G + e) < \gamma(G)$ for each edge $e \in E(\overline{G})$. Clearly if G is K_{2n} minus a perfect matching, $n \geq 2$, then G is a hypo-$\mathcal{P}$ γ- EA-critical graph.

- *Find results on the hypo-$\mathcal{P}$ γ-EA-critical graphs.*

- *Is it true that $b(G) = \delta(G) + 1$ for each hypo-$\mathcal{UD}$ graph?*

Let G be a graph and let $\mathcal{L}$ and $\mathcal{R}$ be arbitrary graph-properties. We define a dominating set D of a graph G to be a *dominating* $(\mathcal{L}, \mathcal{R})$*-set* of G if D is a $\mathcal{L}$-set of G and $V(G) - D$ is a $\mathcal{R}$-set of G. We define the *domination number with respect to the ordered pair* $(\mathcal{L}, \mathcal{R})$ *of graph-properties*, denoted by $\gamma_{(\mathcal{L},\mathcal{R})}(G)$, to be the smallest cardinality of a dominating $(\mathcal{L}, \mathcal{R})$-set of G. A dominating $(\mathcal{L}, \mathcal{R})$-set of G with cardinality $\gamma_{(\mathcal{L},\mathcal{R})}(G)$ is called a $\gamma_{(\mathcal{L},\mathcal{R})}$*-set* of G. Clearly $\gamma_{(\mathcal{I},\mathcal{I})} \equiv \gamma$. Among the many examples of such numbers one can find in the literature are the independent/total/connected/acyclic/paired/restrained/total-restrained/outer-connected domination numbers. For details, see e.g. [14, 15]. We define a graph G to be a *hypo-unique* $(\mathcal{L}, \mathcal{R})$*-domination graph* if G has at least two $\gamma_{(\mathcal{L},\mathcal{R})}$-sets, but $G - v$ has a unique minimum dominating $(\mathcal{L}, \mathcal{R})$-set for each $v \in V(G)$.

- *Find results on the hypo-unique* $(\mathcal{L}, \mathcal{R})$*-domination graphs.*

References

[1] M. Araya and G. Wiener, *On cubic planar hypohamiltonian and hypotraceable graphs*, Electron. J. Combin. **18** (2011), no. 1, #P85.

[2] A. Bange, D. Barkauskas and P. Slater, *Disjoint dominating sets in trees*, Sandia Laboratories Report SAND 78-1087J (1987).

[3] ———, *Effcient dominating sets in graphs*, Applications of Mathematics (R. Ringeisen and F. Roberts, eds.), SIAM, Philadelphia, PA, 1988, pp. 189–199.

[4] Xu Baogen, E. Cockayne, S.T. Hedetniemi, and Z. Shangchao, *Extremal graphs for inequalities involving domination parameters*, Discrete Math. **216** (2000), no. 1–3, 1–10.

[5] D. Bauer, F. Harary, J. Nieminen, and C. Suffel, *Domination alteration sets in graphs*, Discrete Math. **47** (1983), 153–161.

[6] R. Brigham, P. Chinn, and R. Dutton, *Vertex domination-critical graphs*, Networks **18** (1988), no. 3, 173–179.

[7] R. Brigham, T. Haynes, M. Henning, and D. Rall, *Bicritical domination*, Discrete Math. **305** (2005), no. 1–3, 18–32.

[8] A. Coetzer, *Criticality of the lower domination parameters of a graph*, Master's thesis, University of Stellenbosch, 3 2007.

[9] J. Fink, M. Jacobson, L. Kinch, and J. Roberts, *On graphs having domination number half their order*, Period. Math. Hungar. **16** (1985), no. 4, 287–293.

[10] M. Fischermann, *Domination parameters and their unique realizations*, Ph.D. thesis, Techn. Hochsch Aachen, 2 2002.

[11] J. Fulman, D. Hanson, and G. MacGillivray, *Vertex domination-critical graphs*, Networks **25** (1995), no. 3, 41–43.

[12] W. Goddard and M. Henning, *Independent domination in graphs: A survey and recent results*, Discr. Math. **313** (2013), no. 7, 839–854.

[13] P.G.P. Grobler, *Critical concepts in domination, independence and irredundance of graphs*, Ph.D. thesis, University of Sauth Africa, 11 1988.

[14] T. Haynes, S.T. Hedetniemi, and P. Slater, *Domination in graphs: Advanced topics*, Marcel Dekker New York, 1998.

[15] ———, *Fundamentals of domination in graphs*, Marcel Dekker New York, 1998.

[16] M. Henning and A. Yeo, *Total domination in graphs*, Springer, 2013.

[17] F.-T. Hu and J.-M. Xu, *On the complexity of the bondage and reinforcement problems*, J. of Complexity **28** (2012), no. 2, 192–201.

[18] S. Kapoor, *Hypo-eulerian and hypo-traversable graphs*, Elemente der Mathematik **28** (1973), 111–116.

[19] S. Klavzar, I. Peterin, and I.G. Yero, *Graphs that are simultaneously efficient open domination and efficient closed domination graphs*, http://arxiv.org/pdf/ 1511.01916v1.pdf, 2015.

[20] W. McCuaig and B. Shepherd, *Domination in graphs with minimum degree two*, J. Graph Theory **13** (1989), no. 6, 749–762.

[21] J. Mitchem, *Hypo-properties in graphs*, The Many Facets of Graph Theory (G. Chartrand and S.F. Kapoor, eds.), Springer-Verlag, Berlin, 1969, pp. 223–230.

[22] O. Ore, *Theory of graphs*, Amer. Math. Soc, Providence, RI, 1962.

[23] C. Payan and N. Xuong, *Domination-balanced graphs*, J. Graph Theory **6** (1982), no. 1, 23–32.

[24] M.D. Plummer, *Graph factors and factorization: 1985–2003: A survey*, Discrete Math. **307** (2007), no. 7–8, 791–821.

[25] U. Teschner, *New results about the bondage number of a graph*, Discrete Math. **171** (1997), no. 1–3, 249–259.

[26] K. Wagner, *Fastplättbare graphen*, J. Combin. Theory **3** (1967), 326–365.

[27] J.-M. Xu, *On bondage numbers of graphs: a survey with some comments*, Inter. Journ. Comb. **2013** (2013), 34p.

[28] C.T. Zamfirescu, *On hypohamiltonian and almost hypohamiltonian graphs*, J. Graph Theory **79** (2015), no. 1, 63–81.