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Sufficient conditions for maximally edge-connected and super-edge-connected graphs *

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Abstract: Let G be a connected graph with minimum degree δ and edge-connectivity λ. A graph is maximally edge-connected if $\lambda = \delta$, and it is super-edge-connected if every minimum edge-cut is trivial; that is, if every minimum edge-cut consists of edges incident with a vertex of minimum degree. In this paper, we show that a connected graph or a connected triangle-free graph is maximally edge-connected or super-edge-connected if the number of edges is large enough.

Keywords: Edge-connectivity; Maximally edge-connected graphs; Super-edge-connected graphs

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1. Terminology and introduction

Let G be a finite and simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* and *size* of G are defined by $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$, respectively. If $N(v) = N_G(v)$ is the neighborhood of

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the vertex $v \in V(G)$, then we denote by $d(v) = |N(v)|$ the *degree* of v and by $\delta = \delta(G)$ the *minimum degree* of the graph G. For a subset $X \subset V(G)$, use $G[X]$ to denote the subgraph of G induced by X. For two subsets X and Y of $V(G)$ let $[X, Y]$ be the set of edges with one endpoint in X and the other one in Y. We write K_n for the *complete graph* of order n and $K_{p,q}$ for the *complete bipartite graph* whose partition sets have cardinality p and q, respectively. An *edge-cut* of a connected graph G is a set of edges whose removal disconnects G. The *edge connectivity* $\lambda = \lambda(G)$ of a connected graph G is defined as the minimum cardinality of an edge-cut over all edge-cuts of G. An edge-cut S is a *minimum edge-cut* or a λ-*cut* if $|S| = \lambda(G)$. The inequality $\lambda(G) \leq \delta(G)$ is immediate. We call a connected graph *maximally edge-connected*, if $\lambda(G) = \delta(G)$. In 1981, Bauer et al. [1] proposed the concept of super-edge connectedness. A graph is called *super-edge-connected* or *super-*λ if every minimum edge-cut is trivial; that is, if every minimum edge-cut consists of edges incident with a vertex of minimum degree. Thus every super-edge-connected graph is also maximally edge-connected.

Sufficient conditions for graphs to be maximally edge-connected or super-edge-connected were given by several authors, see for example the survey paper by Hellwig and Volkmann [3]. The starting point was an article by Chartrand [2] in 1966. He observed that if δ is large enough, then the graph is maximally edge-connected. A similar condition for super-edge-connectivity was given by Kelmans [4] six years later. Over the years, these results have been strengthened many times and in many ways. However, there seems to be no result in the literature showing that graphs of sufficiently large size are maximally edge-connected. In this paper we fill this gap.

We show that a connected graph or a connected triangle-free graph G is maximally edge-connected or super-λ if the number of edges is large enough. For example, if $m \geq \binom{n-\delta-1}{2} + \binom{\delta+1}{2} + \delta - 1$, then G is maximally edge-connected, unless G is a graph obtained from $K_{\delta+1} \cup K_{n-\delta-1}$ by adding $\delta - 1$ edges between $K_{\delta+1}$ and $K_{n-\delta-1}$.

2. Maximally edge-connected graphs

Theorem 1. Let G be a connected graph of order $n \geq 2$, size m, minimum degree δ and edge-connectivity λ. If

$$m \geq \binom{n-\delta-1}{2} + \binom{\delta+1}{2} + \delta - 1, \tag{1}$$

then $\lambda = \delta$, unless G is a graph obtained from $K_{\delta+1} \cup K_{n-\delta-1}$ by adding $\delta - 1$ edges between $K_{\delta+1}$ and $K_{n-\delta-1}$.

Proof. Suppose to the contrary that $\lambda \leq \delta - 1$. Let S be an arbitrary λ-cut, and let X and Y denote the vertex sets of the two components of $G - S$ with $|X| \leq |Y|$. Each vertex in X is adjacent to at most $|X| - 1$ vertices of X, and exactly λ edges join vertices of X to vertices of Y. Thus

$$\delta|X| \leq \sum_{x \in X} d(x) \leq |X|(|X| - 1) + \lambda,$$

and so $(|X| - 1)(|X| - \delta) \geq \delta - \lambda > 0$, which means that $|X| \geq \delta + 1$. Thus $|Y| \geq |X| \geq \delta + 1$. Therefore we have $\delta + 1 \leq |X| \leq |Y| \leq n - \delta - 1$. Since there are no edges between X and Y in $G - S$, we obtain

$$m - \lambda = |E(G - S)| \leq \binom{n}{2} - |X| \cdot |Y|.$$

This bound together with $\delta + 1 \leq |X| \leq |Y| \leq n - \delta - 1$, $n \geq 2|X|$ and $|X| + |Y| = n$ lead to

$$m \leq \binom{n}{2} - (\delta + 1)(n - \delta - 1) + (\delta - 1) = \binom{n - \delta - 1}{2} + \binom{\delta + 1}{2} + \delta - 1.$$

Combining this with (1), we have $m = \binom{n-\delta-1}{2} + \binom{\delta+1}{2} + \delta - 1$, and the two inequalities in the proof above must be equalities. Therefore $\lambda = \delta - 1$, $|X| = \delta + 1$, $|Y| = n - \delta - 1$, $G[X] = K_{\delta+1}$ and $G[Y] = K_{n-\delta-1}$. This completes the proof. □

For the next result, we use the famous theorem of Mantel [5] and Turán [6].

Theorem 2. *For any triangle-free graph G of order n, we have $|E(G)| \leq \lfloor \frac{1}{4} n^2 \rfloor$, with equality if and only if $G = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.*

Theorem 3. Let G be a connected triangle-free graph of order $n \geq 2$, size m, minimum degree δ and edge-connectivity λ. If

$$m \geq \left\lfloor \frac{n^2}{4} \right\rfloor - \delta(n - 2\delta) + \delta - 1, \tag{2}$$

then $\lambda = \delta$, unless G is a triangle-free graph obtained from $K_{\delta,\delta} \cup K_{\lfloor n/2 \rfloor - \delta, \lceil n/2 \rceil - \delta}$ by adding $\delta - 1$ edges between $K_{\delta,\delta}$ and $K_{\lfloor n/2 \rfloor - \delta, \lceil n/2 \rceil - \delta}$.

Proof. Suppose to the contrary that $\lambda \leq \delta - 1$. Let S be an arbitrary λ-cut, and let X and Y denote the vertex sets of the two components of $G - S$ with $|X| \leq |Y|$. Using Theorem 2, we conclude that

$$|E(G[X])| \leq \left\lfloor \frac{|X|^2}{4} \right\rfloor \text{ and } |E(G[Y])| \leq \left\lfloor \frac{|Y|^2}{4} \right\rfloor, \tag{3}$$

with equalities if and only if $G[X] = K_{\lfloor |X|/2 \rfloor, \lceil |X|/2 \rceil}$ and $G[Y] = K_{\lfloor |Y|/2 \rfloor, \lceil |Y|/2 \rceil}$. Note that $2|E(G[X])| = \sum_{x \in X} d(x) - \lambda$. Thus

$$\delta |X| \leq \sum_{x \in X} d(x) \leq \frac{|X|^2}{2} + \lambda,$$

and so $|X|^2 - 2\delta|X| + 2\lambda \geq 0$. It follows that

$$(|X| - 1)(|X| - (2\delta - 1)) \geq 2\delta - 2\lambda - 1 \geq 1 > 0,$$

which means that $|X| \geq 2\delta$. Thus $|Y| \geq |X| \geq 2\delta$. Therefore we arrive at

$$2\delta \leq |X| \leq \frac{n}{2} \leq |Y| \leq n - 2\delta.$$

This together with $|X| + |Y| = n$ and (3) lead to

$$\begin{aligned}
m &= |E(G[X])| + |E(G[Y])| + \lambda \\
&\leq \left\lfloor \frac{|X|^2}{4} \right\rfloor + \left\lfloor \frac{|Y|^2}{4} \right\rfloor + \delta - 1 = \left\lfloor \frac{|X|^2}{4} \right\rfloor + \left\lfloor \frac{(n - |X|)^2}{4} \right\rfloor + \delta - 1 \\
&\leq \left\lfloor \frac{|X|^2}{4} + \frac{(n - |X|)^2}{4} \right\rfloor + \delta - 1 \\
&= \left\lfloor \frac{n^2}{4} + \frac{1}{2}(|X|^2 - n|X|) \right\rfloor + \delta - 1 \qquad (\text{as } 2\delta \leq |X| \leq \frac{n}{2}) \\
&\leq \left\lfloor \frac{n^2}{4} - \delta(n - 2\delta) \right\rfloor + \delta - 1 \\
&= \left\lfloor \frac{n^2}{4} \right\rfloor - \delta(n - 2\delta) + \delta - 1.
\end{aligned}$$

Combining this with (2), we have $m = \lfloor \frac{n^2}{4} \rfloor - \delta(n - 2\delta) + \delta - 1$, and so $\lambda = \delta - 1$, $|X| = 2\delta$, $|Y| = n - 2\delta$, $|E(G[X])| = \delta^2$ and $|E(G[Y])| = \lfloor \frac{n^2}{4} \rfloor - \delta n + \delta^2$. Therefore, $G[X] = K_{\delta,\delta}$, $G[Y] = K_{\lfloor n/2 \rfloor - \delta, \lceil n/2 \rceil - \delta}$ and $|[X, Y]| = \delta - 1$. This completes the proof. □

3. Super edge-connected graphs

Theorem 4. *Let G be a connected graph of order $n \geq 4$, size m, minimum degree δ and edge-connectivity λ. If*

$$m \geq \binom{n}{2} - \delta(n-\delta) + \delta, \tag{4}$$

then G is super-λ, unless $\delta \geq 2$ and G is a graph obtained from $K_\delta \cup K_{n-\delta}$ by adding δ edges between K_δ and $K_{n-\delta}$ such that the minimum degree of G is δ.

Proof. Suppose to the contrary that G is not super-λ. Let S be an arbitrary λ-cut such that each of the two components of $G-S$ has at least two vertices. Let X and Y denote the vertex sets of the two components of $G-S$ with $2 \leq |X| \leq |Y|$. Each vertex in X is adjacent to at most $|X|-1$ vertices of X, and exactly λ edges join vertices of X to vertices of Y. Thus

$$\delta|X| \leq \sum_{x \in X} d(x) \leq |X|(|X|-1) + \lambda \leq |X|(|X|-1) + \delta,$$

and so $(|X|-1)(|X|-\delta) \geq 0$, which means that $|X| \geq \delta$. Thus $|Y| \geq |X| \geq \delta$. Therefore we have $\delta \leq |X| \leq |Y| \leq n-\delta$. Since there are no edges between X and Y in $G-S$, we obtain

$$m - \lambda = |E(G-S)| \leq \binom{n}{2} - |X| \cdot |Y|.$$

This bound together with $\delta \leq |X| \leq |Y| \leq n-\delta$ and $|X|+|Y| = n$ lead to

$$m \leq \binom{n}{2} - \delta(n-\delta) + \delta.$$

Combining this with (4), we have $m = \binom{n}{2} - \delta(n-\delta)+\delta$, and the two inequalities in the proof above must be equalities. Therefore, $\lambda = \delta$, $|X| = \delta \geq 2$, $|Y| = n-\delta$, $G[X] = K_\delta$ and $G[Y] = K_{n-\delta}$. This completes the proof. $\square$

Theorem 5. Let G be a connected triangle-free graph of order n, size m, minimum degree $\delta \geq 3$ and edge-connectivity λ. If

$$m \geq \left\lfloor \frac{(n+1)^2}{4} \right\rfloor - \delta(n - 2\delta + 1), \tag{5}$$

then G is super-λ, unless G is a triangle-free graph obtained from $K_{\delta,\delta-1} \cup K_{\lfloor \frac{n+1}{2} \rfloor - \delta, \lceil \frac{n+1}{2} \rceil - \delta}$ by adding δ edges between $K_{\delta,\delta-1}$ and $K_{\lfloor \frac{n+1}{2} \rfloor - \delta, \lceil \frac{n+1}{2} \rceil - \delta}$ such that the minimum degree of G is δ.

Proof. Suppose to the contrary that G is not super-λ. Let S be an arbitrary λ-cut such that each of the two components of $G-S$ has at least two vertices. Let X and Y denote the vertex sets of the two components of $G-S$ with $2 \le |X| \le |Y|$. Using Theorem 2, we conclude that

$$|E(G[X])| \le \left\lfloor \frac{|X|^2}{4} \right\rfloor \text{ and } |E(G[Y])| \le \left\lfloor \frac{|Y|^2}{4} \right\rfloor, \tag{6}$$

with equalities if and only if $G[X] = K_{\lfloor |X|/2 \rfloor, \lceil |X|/2 \rceil}$ and $G[Y] = K_{\lfloor |Y|/2 \rfloor, \lceil |Y|/2 \rceil}$. Note that we have seen in the proof of Theorem 3 that

$$\delta |X| \le \sum_{x \in X} d(x) \le \frac{|X|^2}{2} + \lambda \le \frac{|X|^2}{2} + \delta,$$

and so $|X|^2 - 2\delta |X| + 2\delta \ge 0$, that is

$$(|X|-1)(|X|-(2\delta-1)) \ge -1.$$

If $|X| = 2$, then $\delta \le 2$. Nevertheless, the hypothesis is $\delta \ge 3$, which means $|X| \ge 3$, and so $|X| \ge (2\delta-1) - \frac{1}{|X|-1} \ge (2\delta-1) - \frac{1}{2}$. Hence we have $|X| \ge 2\delta - 1$ and so $|Y| \ge |X| \ge 2\delta - 1$. Therefore $2\delta - 1 \le |X| \le \frac{n}{2} \le |Y| \le n - 2\delta + 1$. This together with $|X| + |Y| = n$ and (6) lead to

$$\begin{aligned}
m &= |E(G[X])| + |E(G[Y])| + \lambda \\
&\le \left\lfloor \frac{|X|^2}{4} \right\rfloor + \left\lfloor \frac{|Y|^2}{4} \right\rfloor + \delta = \left\lfloor \frac{|X|^2}{4} \right\rfloor + \left\lfloor \frac{(n-|X|)^2}{4} \right\rfloor + \delta \\
&\le \left\lfloor \frac{|X|^2}{4} + \frac{(n-|X|)^2}{4} \right\rfloor + \delta \\
&= \left\lfloor \frac{n^2}{4} + \frac{1}{2}(|X|^2 - n|X|) \right\rfloor + \delta \quad (\text{as } 2\delta - 1 \le |X| \le \frac{n}{2}) \\
&\le \left\lfloor \frac{n^2}{4} - \frac{1}{2}(2\delta-1)(n-2\delta+1) \right\rfloor + \delta \\
&= \left\lfloor \frac{(n+1)^2+1}{4} \right\rfloor - \delta(n-2\delta+1) \\
&= \left\lfloor \frac{(n+1)^2}{4} \right\rfloor - \delta(n-2\delta+1).
\end{aligned}$$

Combining this with (5), we have $m = \lfloor \frac{(n+1)^2}{4} \rfloor - \delta(n-2\delta+1)$, and so $\lambda = \delta$, $|X| = 2\delta - 1$, $|Y| = n - 2\delta + 1$, $|E(G[X])| = \delta(\delta-1)$ and $|E(G[Y])| = \lfloor \frac{(n+1)^2}{4} \rfloor - \delta(n+1) + \delta^2$. Therefore $G[X] = K_{\delta,\delta}$, $G[Y] = K_{\lfloor \frac{n+1}{2} \rfloor - \delta, \lceil \frac{n+1}{2} \rceil - \delta}$ and $|[X,Y]| = \delta$. □

Theorem 6. Let G be a connected triangle-free graph of order n, size m, minimum degree $\delta \leq 2$ and edge-connectivity λ. If

$$m \geq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + \delta + 1, \tag{7}$$

then G is super-λ, unless G is a triangle-free graph obtained from $K_2 \cup K_{\lfloor \frac{n-2}{2} \rfloor, \lceil \frac{n-2}{2} \rceil}$ by adding δ edge(s) between K_2 and $K_{\lfloor \frac{n-2}{2} \rfloor, \lceil \frac{n-2}{2} \rceil}$ such that the minimum degree of G is δ.

Proof. In the proof of Theorem 5, $|X|^2 - 2\delta|X| + 2\delta \geq 0$. If $\delta \leq 2$, then $|X| \geq 2$ is enough to guarantee $|X|^2 - 2\delta|X| + 2\delta \geq 0$. Thus, with the same proceeding of the proof of Theorem 5, it is easy to get

$$\begin{aligned} m &\leq \left\lfloor \frac{n^2}{4} + \frac{1}{2}(|X|^2 - n|X|) \right\rfloor + \lambda \quad (\text{as } 2 \leq |X| \leq \frac{n}{2}) \\ &\leq \left\lfloor \frac{n^2}{4} - \frac{1}{2}(2n-4) \right\rfloor + \delta \\ &= \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + \delta + 1. \end{aligned}$$

Combining this with (7), we have $m = \lfloor \frac{(n-2)^2}{4} \rfloor + \delta + 1$, and so $\lambda = \delta$, $|X| = 2$, $|Y| = n-2$, $|E(G[X])| = 1$ and $|E(G[Y])| = \lfloor \frac{(n-2)^2}{4} \rfloor$. Therefore $G[X] = K_2$, $G[Y] = K_{\lfloor \frac{n-2}{2} \rfloor, \lceil \frac{n-2}{2} \rceil}$ and $|[X,Y]| = \delta$. □

References

[1] D. Bauer, F. Boesch, C. Suffel, and R. Tindell, *Connectivity extremal problems and the design of reliable probabilistic networks*, The theory and application of graphs, Wiley, New York (1981), 45–54.

[2] G. Chartrand, *A graph-theoretic approach to a communications problem*, SIAM J. Appl. Math. **14** (1966), no. 4, 778–781.

[3] A. Hellwig and L. Volkmann, *Maximally edge-connected and vertex-connected graphs and digraphs: A survey*, Discrete Math. **308** (2008), no. 15, 3265–3296.

[4] A.K. Kelmans, *Asymptotic formulas for the probability of k-connectedness of random graphs*, Theory Probab. Appl. **17** (1972), no. 2, 243–254.

[5] W. Mantel, *Problem 28*, Wiskundige Opgaven **10** (1907), 60–61.

[6] P. Turán, *On an extremal problem in graph theory (hungarian)*, Mat. Fiz. Lapok **48** (1941), 436–452.