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Some results on the complement of a new graph associated to a commutative ring

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Abstract: The rings considered in this article are commutative with identity which admit at least one nonzero proper ideal. Let R be a ring. We denote the collection of all ideals of R by $\mathbb{I}(R)$ and $\mathbb{I}(R)\backslash\{(0)\}$ by $\mathbb{I}(R)^*$. Alilou et al. [A. Alilou, J. Amjadi and S.M. Sheikholeslami, *A new graph associated to a commutative ring*, Discrete Math. Algorithm. Appl. **8** (2016) Article ID: 1650029 (13 pages)] introduced and investigated a new graph associated to R, denoted by Ω_R^* which is an undirected graph whose vertex set is $\mathbb{I}(R)^*\backslash\{R\}$ and distinct vertices I, J are joined by an edge in this graph if and only if either $(Ann_R I)J = (0)$ or $(Ann_R J)I = (0)$. Several interesting theorems were proved on Ω_R^* in the aforementioned paper and they illustrate the interplay between the graph-theoretic properties of Ω_R^* and the ring-theoretic properties of R. The aim of this article is to investigate some properties of $(\Omega_R^*)^c$, the complement of the new graph Ω_R^* associated to R.

Keywords: Annihilating ideal of a ring, maximal N-prime of (0), special principal ideal ring, connected graph, diameter, girth

AMS Subject classification: 13A15, 05C25

1. Introduction

The rings considered in this article are commutative with identity which admit at least one nonzero proper ideal. Let R be a ring. An ideal I of R such that $I \notin \{(0), R\}$ is referred to as a nontrival ideal. Inspired by the work of I. Beck in [9], during the last two decades, several researchers have associated a graph with certain subsets of a ring and explored the interplay between the ring-theoretic properties of a ring

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with the graph-theoretic properties of the graph associated with it (see for example, [3, 4, 6, 7]). Recall from [10] that an ideal I of R is said to be an *annihilating ideal* if there exists $r \in R\backslash\{0\}$ such that $Ir = (0)$. As in [10], we denote the set of all annihilating ideals of R by $\mathbb{A}(R)$ and $\mathbb{A}(R)\backslash\{(0)\}$ by $\mathbb{A}(R)^*$. Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. As the ideals of a ring also play an important role in studying its structure, M. Behboodi and Z. Rakeei in [10] introduced and investigated an undirected graph called the *annihilating-ideal graph* of R, denoted by $\mathbb{AG}(R)$, whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I, J are joined by an edge in $\mathbb{AG}(R)$ if and only if $IJ = (0)$. In [10, 11], M. Behboodi and Z. Rakeei explored the influence of certain graph-theoretic parameters of $\mathbb{AG}(R)$ on the ring structure of R. The annihilating-ideal graph of a commutative ring and other related graphs have been studied by several researchers (see for example, [1, 2, 14, 18, 19]). Motivated by the work done on the annihilating-ideal graph of a commutative ring, in [2], Alilou, Amjadi and Sheikholeslami introduced and studied a new graph associated to a commutative ring R, denoted by Ω_R^*, which is an undirected graph whose vertex set is the set of all nontrivial ideals of R and distinct vertices I, J are joined by an edge in this graph if and only if either $(Ann_RI)J = (0)$ or $(Ann_RJ)I = (0)$ (that is, if and only if either $Ann_RI \subseteq Ann_RJ$ or $Ann_RJ \subseteq Ann_RI$), where for an ideal I of R, the *annihilator of I in* R, denoted by Ann_RI is defined as $Ann_RI = \{r \in R : Ir = (0)\}$. Let R be a ring such that R is not a field. Several interesting and inspiring theorems were proved on Ω_R^* in [2] (see for example, Theorems 4, 10, and 20).

Let $G = (V, E)$ be a simple graph. Recall from ([8], Definition 1.1.13) that the *complement* of G, denoted by G^c, is a graph whose vertex set is V and distinct vertices u, v are joined by an edge in G^c if and only if there is no edge in G joining u and v. Let R be a ring with at least one nontrivial ideal. The aim of this article is to investigate some properties of $(\Omega_R^*)^c$. It is useful to mention here that distinct nontrivial ideals A, B of R are joined by an edge in $(\Omega_R^*)^c$ if and only if $Ann_RA \not\subseteq Ann_RB$ and $Ann_RB \not\subseteq Ann_RA$.

It is useful to recall the following definitions and results from commutative ring theory. Let R be a ring and let I be a proper ideal of R. Recall from [15] that a prime ideal $\mathfrak{p}$ of R is said to be a *maximal N-prime* of I if $\mathfrak{p}$ is maximal with respect to the property of being contained in $Z_R(R/I) = \{r \in R : rx \in I \text{ for some } x \in R\backslash I\}$. Let $x \in Z(R)$. Let $S = R\backslash Z(R)$. Note that S is a multiplicatively closed subset of R and $Rx \cap S = \emptyset$. It follows from Zorn's lemma and ([16], Theorem 1) that there exists a maximal N-prime $\mathfrak{p}$ of (0) in R such that $x \in \mathfrak{p}$. Hence, if $\{\mathfrak{p}_\alpha\}_{\alpha\in\Lambda}$ is the set of maximal N-primes of (0) in R, then it follows that $Z(R) = \cup_{\alpha\in\Lambda}\mathfrak{p}_\alpha$. Observe that R has only one maximal N-prime of (0) if and only if $Z(R)$ is an ideal of R. We use $nil(R)$ to denote the nilradical of a ring R. A ring R is said to be *reduced* if $nil(R) = (0)$. Recall from ([12], Exercise 16, p.111) that a ring R is said to be *von Neumann regular* if given $x \in R$, there exists $y \in R$ such that $x = x^2y$. For a ring R, we denote the Krull dimension of R by $dimR$. It is known that R is von Neumann regular if and only if R is reduced and $dimR = 0$ ([12], Exercise 16, p.111). We denote the cardinality of a set A using the notation $|A|$.

Next, we recall the following definitions from graph theory. The graphs considered in this article are undirected and simple. Let $G = (V, E)$ be a graph. Let $a, b \in V$, $a \neq b$. Recall that the *distance* between a and b, denoted by $d(a, b)$ is defined as the length of a shortest path between a and b in G if such a path exists; otherwise $d(a, b) = \infty$. We define $d(a, a) = 0$. G is said to be *connected* if for any distinct $a, b \in V$, there exists a path in G between a and b. Recall from ([8], Definition 4.2.1) that the *diameter* of a connected graph $G = (V, E)$ denoted by $diam(G)$ is defined as $diam(G) = \max\{d(a, b) : a, b \in V\}$. Let $a \in V$. The *eccentricity* of a, denoted by $e(a)$ is defined as $e(a) = \max\{d(a, b) : b \in V\}$. G is said to be *bipartite* if the vertex set V can be partitioned into two nonempty subsets V_1 and V_2 such that each edge of G has one end in V_1 and the other in V_2. A simple bipartite graph with vertex partition V_1 and V_2 is said to be *complete* if each element of V_1 is adjacent to every element of V_2. A complete bipartite graph with vertex partition V_1 and V_2 is said to be a *star* if either $|V_1| = 1$ or $|V_2| = 1$. Recall from ([8], p. 159) that the *girth* of G, denoted by girth(G) is defined as the length of a shortest cycle in G. If a graph G does not contain any cycle, then we define $girth(G) = \infty$.

Let R be a ring which admits at least one nontrivial ideal. In Section 2 of this article, we discuss regarding the connectedness of $(\Omega_R^*)^c$. Let R be a reduced ring with at least two nontrivial ideals. It is shown that $(\Omega_R^*)^c$ is connected if and only if $\mathbb{AG}(R)$ is a spanning subgraph of $(\Omega_R^*)^c$ and it is observed in such a case that $diam((\Omega_R^*)^c) \leq 3$ (see Proposition 1). It is noted in Remark 1 that if R is reduced and if $(\Omega_R^*)^c$ is connected, then R must have at least two maximal N-primes of (0). Let R be a reduced ring which admits only a finite number $n \geq 2$ of maximal N-primes of (0). In Proposition 2, it is proved that for such a ring R, $(\Omega_R^*)^c$ is connected if and only if $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings, where F_i is a field for each $i \in \{1, 2, \ldots, n\}$. Moreover, for such a ring R, $diam((\Omega_R^*)^c)$ is shown to be equal to 1 or 2 (see Proposition 4). For a von Neumann regular ring R, it is proved in Proposition 3 that $(\Omega_R^*)^c$ is connected if and only if R is Noetherian. Let R be an Artinian ring which is not local. If R is not reduced, then it is verified in Remark 3 that $(\Omega_R^*)^c$ is connected and $diam((\Omega_R^*)^c) = 3$. Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. In [18], we associated and investigated some properties of an undirected graph denoted by $\Omega(R)$ whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I, J are joined by an edge in $\Omega(R)$ if and only if $I + J \in \mathbb{A}(R)$. In [19], we studied the interplay between the graph-theoretic properties of $(\Omega(R))^c$ and the ring-theoretic properties of R. It is useful to recall here that distinct nonzero annihilating ideals I, J are adjacent in $(\Omega(R))^c$ if and only if $I + J \notin \mathbb{A}(R)$. Let H be the subgraph of $(\Omega_R^*)^c$ induced on $\mathbb{A}(R)^*$. It is observed in Lemma 5 that $(\Omega(R))^c$ is a spanning subgraph of H. Let R be a ring such that $|\mathbb{A}(R)^*| \geq 2$. In Theorem 2, classification of rings R such that $(\Omega_R^*)^c$ is a path of order 4 is obtained. In Proposition 6, classification of rings R such that H is complete bipartite is given. It is proved in Proposition 7 that H is complete if and only if $R \cong F_1 \times F_2$ as rings, where F_1 and F_2 are fields. With $|\mathbb{A}(R)^*| \geq 3$, in Proposition 8, necessary and sufficient conditions on R are determined in order that H be a star graph.

Let R be a ring which admits at least one nontrivial ideal. Section 3 of this article contains a discussion on the girth of $(\Omega_R^*)^c$. Let R be a reduced ring which is not an integral domain. If R has a unique maximal N-prime of (0), then it is verified in

Proposition 9 that $girth((\Omega_R^*)^c) = 3$. If R has exactly two maximal N-primes of (0), then it is proved in Proposition 10 that $girth((\Omega_R^*)^c) \in \{3, 4, \infty\}$. Let R be a ring (which can possibly be non-reduced) such that R has at least three maximal N-primes of (0). It is noted in Proposition 11 that $girth((\Omega_R^*)^c) = 3$. Let R be a non-reduced ring which has at most two maximal N-primes of (0). We are not able to determine $girth((\Omega_R^*)^c)$. Some examples are provided to illustrate the results obtained in this section.

Let R be a ring. We denote the set of all maximal ideals of R using the notation $Max(R)$. A ring R is said to be *quasilocal* (respectively, *semiquasilocal*) if R has a unique maximal ideal (respectively, R has only a finite number of maximal ideals). A Noetherian quasilocal (respectively, semiquasilocal) ring is referred to as a *local* (respectively, *semilocal*) ring. Let A, B be sets. We use $A \subset B$ to denote proper inclusion.

2. On the connectedness of $(\Omega_R^*)^c$

As mentioned in the introduction, the rings considered in this article are commutative with identity which admit at least one nontrivial ideal. First, we determine some necessary conditions on the ring R in order that $(\Omega_R^*)^c$ be connected.

Lemma 1. *Let R be a ring and I be a nontrivial ideal of R such that $I \notin \mathbb{A}(R)^*$. Then I is an isolated vertex of $(\Omega_R^*)^c$.*

Proof. It is already noted in the introduction that nontrivial ideals A, B of R are adjacent in $(\Omega_R^*)^c$ if and only if $Ann_R A \not\subseteq Ann_R B$ and $Ann_R B \not\subseteq Ann_R A$. As $I \notin \mathbb{A}(R)^*$, we obtain that $Ann_R I = (0)$. Let J be any nontrivial ideal of R with $J \neq I$. Then $Ann_R I = (0) \subseteq Ann_R J$. Hence, I and J are not adjacent in $(\Omega_R^*)^c$. This proves that I is an isolated vertex of $(\Omega_R^*)^c$. □

Corollary 1. *Let R be a ring such that R admits at least two nontrivial ideals. If $(\Omega_R^*)^c$ is connected, then any nontrivial ideal of R is an annihilating ideal of R.*

Proof. As R has at least two nontrivial ideals and $(\Omega_R^*)^c$ is connected, it follows that no nontrivial ideal of R is an isolated vertex of $(\Omega_R^*)^c$. Hence, we obtain from Lemma 1 that each nontrivial ideal of R is an annihilating ideal of R. □

Let R be a reduced ring with at least two nontrivial ideals. We prove in Proposition 1 that $(\Omega_R^*)^c$ is connected if and only if each nontrivial ideal of R is an annihilating ideal of R. We use Lemma 2 in the proof of Proposition 1.

Lemma 2. *Let R be a reduced ring. Let $I, J \in \mathbb{A}(R)^*$ be such that $IJ = (0)$. Then I and J are adjacent in $(\Omega_R^*)^c$.*

Proof. We claim that $Ann_R I \not\subseteq Ann_R J$ and $Ann_R J \not\subseteq Ann_R I$. Suppose that $Ann_R I \subseteq Ann_R J$. Then from $IJ = (0)$, it follows that $J \subseteq Ann_R I \subseteq Ann_R J$. This implies that $J^2 = (0)$. This is impossible since R is reduced and $J \neq (0)$. Therefore, $Ann_R I \not\subseteq Ann_R J$. Similarly, it can be shown that $Ann_R J \not\subseteq Ann_R I$. This proves that I and J are adjacent in $(\Omega_R^*)^c$. □

Proposition 1. *Let R be a reduced ring which admits at least two nontrivial ideals. Then the following statements are equivalent:*
(i) $(\Omega_R^*)^c$ is connected.
(ii) Each nontrivial ideal of R is an annihilating ideal of R.

Proof. $(i) \Rightarrow (ii)$ This follows immediately from Corollary 1. It is useful to note that this part of this Proposition does not need the hypothesis that R is reduced.
$(ii) \Rightarrow (i)$ Note that the vertex set of $(\Omega_R^*)^c$ equals $\mathbb{A}(R)^*$. It follows from Lemma 2 that $\mathbb{AG}(R)$ is a spanning subgraph of $(\Omega_R^*)^c$. It is well-known that $\mathbb{AG}(R)$ is connected and $diam(\mathbb{AG}(R)) \leq 3$ ([10], Theorem 2.1). Therefore, we obtain that $(\Omega_R^*)^c$ is connected and $diam((\Omega_R^*)^c) \leq 3$. □

Remark 1. Let R be a reduced ring which admits $\mathfrak{p}$ as its unique maximal N-prime of (0). Then $(\Omega_R^*)^c$ is not connected.

Proof. Note that $\mathfrak{p} = Z(R)$. Let $x \in \mathfrak{p}$, $x \neq 0$. Note that there exists $y \in R\backslash\{0\}$ such that $xy = 0$. As R is reduced and $x \neq 0$, whereas $xy = 0$, it follows that $Rx \neq Ry$. Thus R has at least two nontrivial ideals. We assert that $\mathfrak{p} \notin \mathbb{A}(R)$. Suppose that $\mathfrak{p} \in \mathbb{A}(R)$. Then there exists $a \in R\backslash\{0\}$ such that $\mathfrak{p}a = (0)$. This implies that $a \in Z(R) = \mathfrak{p}$. Hence, $a^2 = 0$. This is impossible since R is reduced and $a \neq 0$. Therefore, $\mathfrak{p} \notin \mathbb{A}(R)$. Hence, we obtain from Corollary 1 that $(\Omega_R^*)^c$ is not connected. □

Let R be a reduced ring which admits only a finite number $n \geq 2$ of maximal N-primes of (0). In Proposition 2, we classify such rings R in order that $(\Omega_R^*)^c$ be connected. We use Lemmas 3 and 4 in the proof of Proposition 2 and some other results of this article.

Lemma 3. *Let R be a reduced ring which admits only a finite number $n \geq 2$ of maximal N-primes of (0). Let $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ be the set of all maximal N-primes of (0) in R. If $(\Omega_R^*)^c$ is connected, then $\cap_{i=1}^n \mathfrak{p}_i = (0)$.*

Proof. Assume that $(\Omega_R^*)^c$ is connected. It follows from Corollary 1 that each nontrivial ideal of R is an annihilating ideal of R. Hence, for each $i \in \{1, 2, \ldots, n\}$, there exists $a_i \in R\backslash\{0\}$ such that $\mathfrak{p}_i a_i = (0)$. It follows from ([9], Lemma 3.6) that $a_i a_j = 0$ for all distinct $i, j \in \{1, 2, \ldots, n\}$. Let $i \in \{1, 2, \ldots, n\}$. Note that $a_i \notin \mathfrak{p}_i$. Hence, we obtain that $a_i \in \cap_{j \in \{1,2,\ldots,n\}\backslash\{i\}} \mathfrak{p}_j$. Since $Z(R) = \cup_{i=1}^n \mathfrak{p}_i$, it follows that

$\sum_{i=1}^{n} a_i \notin Z(R)$. Let us denote $\sum_{i=1}^{n} a_i$ by a. Let $x \in \cap_{i=1}^{n} \mathfrak{p}_i$. It follows from $ax = 0$ and $a \notin Z(R)$ that $x = 0$. This proves that $\cap_{i=1}^{n} \mathfrak{p}_i = (0)$. □

Lemma 4. *Let R be a ring. If $(\Omega_R^*)^c$ is connected, then each maximal N-prime $\mathfrak{p}$ of (0) in R is a maximal ideal of R.*

Proof. Assume that $(\Omega_R^*)^c$ is connected. Let $\mathfrak{p}$ be a maximal N-prime of (0) in R. Let $\mathfrak{m}$ be a maximal ideal of R such that $\mathfrak{p} \subseteq \mathfrak{m}$. If $\mathfrak{p} = \mathfrak{m}$, then the proof is complete. Suppose that $\mathfrak{p} \neq \mathfrak{m}$. Since $(\Omega_R^*)^c$ is connected, there exists a path in $(\Omega_R^*)^c$ between $\mathfrak{p}$ and $\mathfrak{m}$. Hence, $\mathfrak{m}$ is not an isolated vertex of $(\Omega_R^*)^c$. This implies by Lemma 1 that $\mathfrak{m} \in \mathbb{A}(R)^*$. So, there exists $x \in R \backslash \{0\}$ such that $\mathfrak{m}x = (0)$. Hence, $\mathfrak{m} \subseteq Z(R)$ and so, $\mathfrak{p} \subset \mathfrak{m} \subseteq Z(R)$. This is impossible since $\mathfrak{p}$ is maximal with respect to the property of being contained in $Z(R)$. Therefore, $\mathfrak{p}$ is a maximal ideal of R. □

Proposition 2. *Let R be a reduced ring . Suppose that R has only a finite number $n \geq 2$ of maximal N-primes of (0). Then the following statements are equivalent:*
(i) $(\Omega_R^*)^c$ is connected.
(ii) $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings, where F_i is a field for each $i \in \{1, 2, \ldots, n\}$.

Proof. $(i) \Rightarrow (ii)$ Let $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ denote the set of all maximal N-primes of (0) in R. We know from Lemma 3 that $\cap_{i=1}^{n} \mathfrak{p}_i = (0)$. It follows from Lemma 4 that $\mathfrak{p}_i$ is a maximal ideal of R for each $i \in \{1, 2, \ldots, n\}$. Observe that $\mathfrak{p}_i + \mathfrak{p}_j = R$ for all distinct $i, j \in \{1, 2, \ldots, n\}$ and $\cap_{i=1}^{n} \mathfrak{p}_i = (0)$. Therefore, we obtain from the Chinese remainder theorem ([5], Proposition 1.10 (ii) and (iii)) that the mapping $f : R \to R/\mathfrak{p}_1 \times R/\mathfrak{p}_2 \times \cdots \times R/\mathfrak{p}_n$ defined by $f(r) = (r + \mathfrak{p}_1, r + \mathfrak{p}_2, \ldots, r + \mathfrak{p}_n)$ is an isomorphism of rings. Let $i \in \{1, 2, \ldots, n\}$. Since $\mathfrak{p}_i$ is a maximal ideal of R, it follows that $R/\mathfrak{p}_i$ is a field. Let us denote $R/\mathfrak{p}_i$ by F_i. Then F_i is a field for each $i \in \{1, 2, \ldots, n\}$ and $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings.

$(ii) \Rightarrow (i)$ Let us denote the ring $F_1 \times F_2 \times \cdots \times F_n$ by T. Note that T is reduced and each nontrivial ideal of T is of the form Te for some nontrivial idempotent e of T. Hence, each nontrivial ideal of T is an annihilating ideal of T. Therefore, we obtain from $(ii) \Rightarrow (i)$ of Proposition 1 that $(\Omega_T^*)^c$ is connected. Since $R \cong T$ as rings, it follows that $(\Omega_R^*)^c$ is connected. □

Let R be von Neumann regular and let $x \in R$. Note that there exists $y \in R$ such that $x = x^2 y$. Observe that $e = xy$ is an idempotent element of R. It is not hard to verify that $u = x + 1 - e$ is a unit in R and $x = ue$. Let R be a von Neumann regular ring with at least two maximal ideals. In Proposition 3, we classify von Neumann regular rings R in order that $(\Omega_R^*)^c$ be connected.

Proposition 3. *Let R be a von Neumann regular ring which is not a field. Then the following statements are equivalent:*
(i) $(\Omega_R^*)^c$ is connected.

(ii) $R \cong F_1 \times F_2 \times \cdots \times F_n$ *as rings for some* $n \geq 2$, *where* F_i *is a field for each* $i \in \{1, 2, \ldots, n\}$.

Proof. $(i) \Rightarrow (ii)$ Since $(\Omega_R^*)^c$ is connected, we obtain from Corollary 1 that if I is any nontrivial ideal of R, then $I \in \mathbb{A}(R)^*$. Let $\mathfrak{p}$ be any prime ideal of R. As $\mathfrak{p} \in \mathbb{A}(R)^*$, there exists $x \in R\backslash\{0\}$ such that $\mathfrak{p}x = (0)$. Since $dim R = 0$, it follows that $\mathfrak{p}$ is a maximal ideal of R. Hence, $\mathfrak{p} = ((0) :_R x)$. Note that $x = ue$, where u is a unit of R and e is a nontrivial idempotent element of R. Therefore, $\mathfrak{p} = R(1 - e)$. This proves that any prime ideal of R is finitely generated and hence by Cohen's theorem ([5], Exercise 1, p.84), we obtain that R is Noetherian. Therefore, it follows from ([12], Exercise 22, p.112) that $R \cong F_1 \times \cdots \times F_n$ as rings, where F_i is a field for each $i \in \{1, \ldots, n\}$. As R is not a field, it is clear that $n \geq 2$.

$(ii) \Rightarrow (i)$ This follows immediately from $(ii) \Rightarrow (i)$ of Proposition 2. $\square$

Let $n \geq 2$ and let $R = F_1 \times F_2 \times \cdots \times F_n$, where F_i is a field for each $i \in \{1, 2, \ldots, n\}$. In Proposition 4, we determine $diam((\Omega_R^*)^c)$.

Proposition 4. *Let* $n \geq 2$ *and let* $R = F_1 \times F_2 \times \cdots \times F_n$, *where* F_i *is a field for each* $i \in \{1, 2, \ldots, n\}$. *Then the following hold.*
(i) $diam((\Omega_R^*)^c) = 1$ *if* $n = 2$.
(ii) $diam((\Omega_R^*)^c) = 2$ *if* $n \geq 3$.

Proof. (i) Assume that $n = 2$. Note that the set of all nontrivial ideals of R equals $\{\mathfrak{m}_1 = (0) \times F_2, \mathfrak{m}_2 = F_1 \times (0)\}$. Since R is reduced and $\mathfrak{m}_1\mathfrak{m}_2 = (0) \times (0)$, it follows from Lemma 2 that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are adjacent in $(\Omega_R^*)^c$. Therefore, $diam((\Omega_R^*)^c) = 1$.

(ii) Assume that $n \geq 3$. Let I, J be any two distinct nontrivial ideals of R. Observe that $I = Re$ and $J = Rf$ for some nontrivial idempotent elements e, f of R. Suppose that I and J are not adjacent in $(\Omega_R^*)^c$. Then either $Ann_R I \subseteq Ann_R J$ or $Ann_R J \subseteq Ann_R I$. Without loss of generality, we can assume that $Ann_R I \subseteq Ann_R J$. Hence, $R(1-e) \subseteq R(1-f)$. This implies that $(1-e)f = 0$. Observe that $IR(1-e) = (0)$ and $JR(1-e) = (0)$. Since R is reduced, it follows from Lemma 2 that $I - R(1-e) - J$ is a path of length 2 in $(\Omega_R^*)^c$. This proves that between any two nontrivial ideals I, J of R, there exists a path of length at most two between I and J in $(\Omega_R^*)^c$. Therefore, $diam((\Omega_R^*)^c) \leq 2$. We next verify that $diam((\Omega_R^*)^c) = 2$. Indeed, we show that $e(I) = 2$ for any nontrivial ideal I of R. Observe that either $I \in Max(R)$ or $I \notin Max(R)$. Suppose that $I \in Max(R)$. Let $\mathfrak{m} \in Max(R)$ be such that $I \neq \mathfrak{m}$. Since $n \geq 3$, it follows that $I \cap \mathfrak{m}$ is a nontrivial ideal of R. As $I \cap \mathfrak{m} \subset I$, it follows that $Ann_R I \subseteq Ann_R(I \cap \mathfrak{m})$ and so, I and $I \cap \mathfrak{m}$ are not adjacent in $(\Omega_R^*)^c$. Hence, $d(I, I \cap \mathfrak{m}) = 2$ in $(\Omega_R^*)^c$. Suppose that $I \notin Max(R)$. Let $\mathfrak{n} \in Max(R)$ be such that $I \subset \mathfrak{n}$. Note that I and $\mathfrak{n}$ are not adjacent in $(\Omega_R^*)^c$ and so, $d(I, \mathfrak{n}) = 2$ in $(\Omega_R^*)^c$. This proves that $e(I) = 2$ for any nontrivial ideal I of R and therefore, $diam((\Omega_R^*)^c) = 2$. $\square$

Let R be a non-reduced ring. We next discuss the connectedness of $(\Omega_R^*)^c$. First, we consider non-reduced rings R such that R has only one maximal N-prime of (0). Recall that a principal ideal ring R is said to be a *special principal ideal ring* (SPIR) if R has only one prime ideal. If $\mathfrak{m}$ is the only prime ideal of a SPIR R, then $\mathfrak{m}$ is necessarily nilpotent. If R is a SPIR with $\mathfrak{m}$ as its only prime ideal, then we denote it by saying that $(R, \mathfrak{m})$ is a SPIR. Suppose that $\mathfrak{m} \neq (0)$. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. Then it follows from $(iii) \Rightarrow (i)$ of ([5], Proposition 8.8) that $\{\mathfrak{m}^i : i \in \{1, \ldots, n-1\}\}$ is the set of all nontrivial ideals of R.

Proposition 5. *Let R be a non-reduced ring which admits $\mathfrak{p}$ as its unique maximal N-prime of (0) Then the following statements are equivalent:*
(i) $(\Omega_R^*)^c$ is connected.
(ii) $(R, \mathfrak{p})$ is a SPIR with $\mathfrak{p}^2 = (0)$.

Proof. $(i) \Rightarrow (ii)$ Let $x \in R \backslash \{0\}$ be such that $x^2 = 0$. We claim that $\mathfrak{p} = Rx$. Suppose that $\mathfrak{p} \neq Rx$. Since, $(\Omega_R^*)^c$ is connected, there exists a path in $(\Omega_R^*)^c$ between Rx and $\mathfrak{p}$. Hence, there exists a nontrivial I of R such that I and $\mathfrak{p}$ are adjacent in $(\Omega_R^*)^c$. This implies by Lemma 1 that $I \in \mathbb{A}(R)^*$. So, there exists $r \in R\backslash\{0\}$ such that $Ir = (0)$. Hence, $I \subseteq Z(R) = \mathfrak{p}$. Therefore, $Ann_R\mathfrak{p} \subseteq Ann_R I$ and so, I and $\mathfrak{p}$ are not adjacent in $(\Omega_R^*)^c$. This is a contradiction. Therefore, $\mathfrak{p} = Rx$. We know from Lemma 4 that $\mathfrak{p} \in Max(R)$. It follows from $\mathfrak{p}^2 = (0)$, that $\mathfrak{p}$ is the unique maximal ideal of R and it is the only nontrivial ideal of R. Hence, $(R, \mathfrak{p})$ is a SPIR with $\mathfrak{p}^2 = (0)$.
$(ii) \Rightarrow (i)$ Note that $\mathfrak{p}$ is the only nontrivial ideal of R. Hence, $(\Omega_R^*)^c$ is a graph whose vertex set is $\{\mathfrak{p}\}$ and so, it is connected. □

Let R be a non-reduced ring such that R admits only a finite number $n \geq 2$ of maximal N-primes of (0). In Theorem 7, we provide a sufficient condition for $(\Omega_R^*)^c$ to be connected. We need some preliminary results that are needed for proving Theorem 7.

Lemma 5. *Let R be a ring and let $I_1, I_2 \in \mathbb{A}(R)^*$ be such that $I_1 + I_2 \notin \mathbb{A}(R)$. Then I_1 and I_2 are adjacent in $(\Omega_R^*)^c$.*

Proof. Since $I_1 + I_2 \notin \mathbb{A}(R)$, we obtain that $Ann_R I_1 \cap Ann_R I_2 = (0)$. As $Ann_R I_i \neq (0)$ for each $i \in \{1, 2\}$, it follows that $Ann_R I_1 \not\subseteq Ann_R I_2$ and $Ann_R I_2 \not\subseteq Ann_R I_1$. This proves that I_1 and I_2 are adjacent in $(\Omega_R^*)^c$. □

For a ring R, we denote the Jacobson radical of R by $J(R)$.

Lemma 6. *Let R be a ring such that each nontrivial ideal of R is an annihilating ideal of R. Let $W = \{I : I \in \mathbb{A}(R)^*$ such that $I \not\subseteq J(R)\}$. Then the subgraph H of $(\Omega_R^*)^c$ induced on W is connected and moreover, $diam(H) \leq 2$.*

Proof. Let $I_1, I_2 \in W$ be such that $I_1 \neq I_2$. Suppose that I_1 and I_2 are not adjacent in H. Then either $Ann_R I_1 \subseteq Ann_R I_2$ or $Ann_R I_2 \subseteq Ann_R I_1$. Without loss of generality, we can assume that $Ann_R I_1 \subseteq Ann_R I_2$. Since $I_2 \in W$, there exists a maximal ideal $\mathfrak{m}$ of R such that $I_2 \not\subseteq \mathfrak{m}$. We assert that $I_1 \not\subseteq \mathfrak{m}$. Suppose that $I_1 \subseteq \mathfrak{m}$. Then we obtain that $Ann_R \mathfrak{m} \subseteq Ann_R I_1$. Since $Ann_R I_1 \subseteq Ann_R I_2$, we get that $Ann_R \mathfrak{m} \subseteq Ann_R I_2$. This is impossible since $I_2 + \mathfrak{m} = R$. This proves that $I_1 \not\subseteq \mathfrak{m}$. Therefore, $I_1 + \mathfrak{m} = I_2 + \mathfrak{m} = R$. It is clear that $\mathfrak{m} \in W$ and it follows from Lemma 5 that $I_1 - \mathfrak{m} - I_2$ is a path of length 2 between I_1 and I_2 in H. This shows that H is connected and moreover, $diam(H) \leq 2$. □

Lemma 7. *Let $n \geq 2$ and let $(R_i, \mathfrak{m}_i)$ be a quasilocal ring for each $i \in \{1, 2, \dots, n\}$. Suppose that each proper ideal of R_i is an annihilating ideal of R_i for each $i \in \{1, 2, \dots, n\}$. Let $R = R_1 \times R_2 \times \cdots \times R_n$. Then $(\Omega_R^*)^c$ is connected and moreover, $diam((\Omega_R^*)^c) \leq 3$.*

Proof. Let $i \in \{1, 2, \dots, n\}$. Let $\mathfrak{M}_i = I_1 \times I_2 \times \cdots \times I_n$, where $I_i = \mathfrak{m}_i$ and $I_j = R_j$ for all $j \in \{1, 2, \dots, n\} \backslash \{i\}$. It is clear that R is semiquasilocal with $\{\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_n\}$ as its set of all maximal ideals. As each proper ideal of R_i is an annihilating ideal of R_i for each $i \in \{1, 2, \dots, n\}$, we obtain that each proper ideal of R is an annihilating ideal of R. Note that $J(R) = \mathfrak{m}_1 \times \mathfrak{m}_2 \times \cdots \times \mathfrak{m}_n$. Let A, B be nontrivial ideals of R with $A \neq B$. We now verify that there exists a path of length at most three between A and B in $(\Omega_R^*)^c$. We can assume that A and B are not adjacent in $(\Omega_R^*)^c$. We consider the following cases.

Case 1. $A \not\subseteq J(R)$ and $B \not\subseteq J(R)$.
In this case, we know from Lemma 6 that there exists a path of length at most two between A and B in $(\Omega_R^*)^c$.

Case 2. $A \subseteq J(R)$ whereas $B \not\subseteq J(R)$.
Note that A is of the form $A = A_1 \times A_2 \times \cdots \times A_n$, where A_i is an ideal of R_i with $A_i \subseteq \mathfrak{m}_i$ for each $i \in \{1, 2, \dots, n\}$. Note that $A_i \neq (0)$ for at least one $i \in \{1, 2, \dots, n\}$. Fix $i \in \{1, 2, \dots, n\}$ such that $A_i \neq (0)$. Observe that $Ann_{R_i} A_i$ is a nontrivial ideal of R_i . It can happen that $A_j = (0)$ for each $j \in \{1, 2, \dots, n\} \backslash \{i\}$. Fix $j \in \{1, 2, \dots, n\}$ with $j \neq i$. Let C be an ideal of R defined by $C = C_1 \times C_2 \times \cdots \times C_n$ with $C_j = R_j$ and $C_k = (0)$ for all $k \in \{1, 2, \dots, n\} \backslash \{j\}$. It is clear that $C \not\subseteq J(R)$, $Ann_R A \not\subseteq Ann_R C$, and $Ann_R C \not\subseteq Ann_R A$. Therefore, A and C are adjacent in $(\Omega_R^*)^c$. Suppose that $A_t \neq (0)$ for some $t \in \{1, 2, \dots, n\}$ with $t \neq i$. In such a case, define the ideal D of R by $D = D_1 \times D_2 \times \cdots \times D_n$ with $D_i = R_i$ and $D_k = (0)$ for all $k \in \{1, 2, \dots, n\} \backslash \{i\}$. It is clear that $D \not\subseteq J(R)$, $Ann_R A \not\subseteq Ann_R D$, and $Ann_R D \not\subseteq Ann_R A$. Hence, A and D are adjacent in $(\Omega_R^*)^c$. We know from Lemma 6 that there exists a path of length at most two between C and B in $(\Omega_R^*)^c$ and there exists a path of length at most two between D and B in $(\Omega_R^*)^c$. This proves that there exists a path of length at most three between A and B in $(\Omega_R^*)^c$.

Case 3. $A \subseteq J(R)$ and $B \subseteq J(R)$.
Note that A is of the form $A = A_1 \times A_2 \times \cdots \times A_n$ and B is of the form $B = B_1 \times B_2 \times \cdots \times B_n$, where for each $i \in \{1, 2, \dots, n\}$, A_i, B_i are proper ideals of R_i. We are

assuming that A and B are not adjacent in $(\Omega_R^*)^c$. Hence, either $Ann_R A \subseteq Ann_R B$ or $Ann_R B \subseteq Ann_R A$. Without loss of generality, we can assume that $Ann_R A \subseteq Ann_R B$. Then $Ann_{R_i} A_i \subseteq Ann_{R_i} B_i$ for each $i \in \{1, 2, \ldots, n\}$. Note that $B_i \neq (0)$ for at least one $i \in \{1, 2, \ldots, n\}$. It follows from $Ann_{R_i} A_i \subseteq Ann_{R_i} B_i$ that $A_i \neq (0)$. It can happen that there are distinct $i, t \in \{1, 2, \ldots, n\}$ such that $B_i \neq (0)$ and $B_t \neq (0)$. In such a case, $A_i \neq (0)$ and $A_t \neq (0)$. In such a situation, we know from the proof of Case 2 of this lemma that both A and B are adjacent to $D = D_1 \times D_2 \times \cdots \times D_n$ in $(\Omega_R^*)^c$, where $D_i = R_i$ and $D_k = (0)$ for all k$\in \{1, 2, \ldots, n\} \backslash \{i\}$. Hence, $A - D - B$ is a path of length two between A and B in $(\Omega_R^*)^c$. Suppose that there exists a unique $i \in \{1, 2, \ldots, n\}$ such that $B_i \neq (0)$. Then $A_i \neq (0)$. It can happen that $A_j = (0)$ for all $j \in \{1, 2, \ldots, n\} \backslash \{i\}$. Fix $j \in \{1, 2, \ldots, n\} \backslash \{i\}$. Observe that it follows from the proof of Case 2 of this lemma that both A and B are adjacent to $C = C_1 \times C_2 \times \cdots \times C_n$ in $(\Omega_R^*)^c$, where $C_j = R_j$ and $C_k = (0)$ for all $k \in \{1, 2, \ldots, n\} \backslash \{j\}$. Hence, $A - C - B$ is a path of length two between A and B in $(\Omega_R^*)^c$. Suppose that there exists $j \in \{1, 2, \ldots, n\} \backslash \{i\}$ such that $A_j \neq (0)$. With C, D as above, it is clear that $A - D - C - B$ is a path of length three between A and B in $(\Omega_R^*)^c$.

Thus for any distinct nontrivial ideals A, B of R, there exists a path of length at most three between A and B in $(\Omega_R^*)^c$. This proves that $(\Omega_R^*)^c$ is connected and $diam((\Omega_R^*)^c) \leq 3$. □

Theorem 1. *Let R be a non-reduced ring which has only a finite number $n \geq 2$ of maximal N-primes of (0). Let $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ denote the set of all maximal N-primes of (0) in R. If $\cap_{i=1}^n \mathfrak{p}_i$ is nilpotent, then the following statements are equivalent:*
(i) $(\Omega_R^*)^c$ is connected.
(ii) Each nontrivial ideal of R is an annihilating ideal of R and R is semiquasilocal with $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ as its set of all maximal ideals.

Proof. $(i) \Rightarrow (ii)$ Assume that $(\Omega_R^*)^c$ is connected. It follows from Corollary 1 that any nontrivial ideal of R is an annihilating ideal of R. We know from Lemma 4 that $\mathfrak{p}_i \in Max(R)$ for each $i \in \{1, 2, \ldots, n\}$. Note that $Z(R) = \cup_{i=1}^n \mathfrak{p}_i$. Let $\mathfrak{m} \in Max(R)$. As $\mathfrak{m} \in \mathbb{A}(R)$, we get that $\mathfrak{m} \subseteq Z(R) = \cup_{i=1}^n \mathfrak{p}_i$. Therefore, we obtain from Prime avoidance lemma ([5], Proposition 1.11 (i)) that $\mathfrak{m} \subseteq \mathfrak{p}_i$ for some $i \in \{1, 2, \ldots, n\}$ and so, $\mathfrak{m} = \mathfrak{p}_i$. This shows that R is semiquasilocal with $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ as its set of all maximal ideals.

$(ii) \Rightarrow (i)$ Note that for each $i \in \{1, 2, \ldots, n\}$, there exists $a_i \in R \backslash \{0\}$ such that $\mathfrak{p}_i = ((0) :_R a_i)$. Note that $J(R) = \cap_{i=1}^n \mathfrak{p}_i$ and as $J(R)$ is nilpotent, there exists $k \geq 1$ such that $(J(R))^k = (0)$. Since R is not reduced, it follows that $k \geq 2$. Observe that for all distinct $i, j \in \{1, 2, \ldots, n\}$, ${\mathfrak{p}_i}^k + {\mathfrak{p}_j}^k = R$ and $\cap_{i=1}^n \mathfrak{p}_i^k = \prod_{i=1}^n \mathfrak{p}_i^k = (0)$. Therefore, we obtain from the Chinese remainder theorem ([5], Proposition 1.10 (ii) and (iii)) that the mapping $f : R \to R/\mathfrak{p}_1^k \times R/\mathfrak{p}_2^k \times \cdots \times R/\mathfrak{p}_n^k$ given by $f(r) = (r + \mathfrak{p}_i^k, r + \mathfrak{p}_2^k, \ldots, r + \mathfrak{p}_n^k)$ is an isomorphism of rings. Let $i \in \{1, 2, \ldots, n\}$ and let us denote the ring $R/\mathfrak{p}_i^k$ by R_i. Note that R_i is quasilocal with $\mathfrak{m}_i = \mathfrak{p}_i/\mathfrak{p}_i^k$ as its unique maximal ideal. It is clear that $\mathfrak{m}_i^k =$ zero ideal of R_i. Let us denote the ring $R_1 \times R_2 \times \cdots \times R_n$ by T. We know from Lemma 7 that $(\Omega_T^*)^c$ is connected and

$diam((\Omega_T^*)^c) \leq 3$. Since $R \cong T$ as rings, we obtain that $(\Omega_R^*)^c$ is connected and moreover, $diam((\Omega_R^*)^c) \leq 3$. □

Let R, $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ be as in the statement of Theorem 1. If $\cap_{i=1}^n \mathfrak{p}_i$ is nilpotent and if $(\Omega_R^*)^c$ is connected, then it is shown in the proof of $(ii) \Rightarrow (i)$ of Theorem 1 that $diam((\Omega_R^*)^c) \leq 3$. As an immediate consequence of Remark 2, we deduce in Corollary 2 that $diam((\Omega_R^*)^c) = 3$.

Remark 2. Let R be a ring. Let $\mathfrak{p}$ be a prime ideal of R such that $\mathfrak{p} = ((0) :_R x)$ for some $x \in \mathfrak{p}$. Suppose that $\mathfrak{p} \neq Rx$. Then the following hold.
(i) Rx and $\mathfrak{p}$ are not adjacent in $(\Omega_R^*)^c$.
(ii) There is no path of length 2 between Rx and $\mathfrak{p}$ in $(\Omega_R^*)^c$.

Proof. (i) Since $Rx \subset \mathfrak{p}$, it is clear that Rx and $\mathfrak{p}$ are not adjacent in $(\Omega_R^*)^c$.
(ii) Suppose that there exists a path of length 2 between Rx and $\mathfrak{p}$ in $(\Omega_R^*)^c$. Let $Rx - I - \mathfrak{p}$ be a path of length 2 in $(\Omega_R^*)^c$ between Rx and $\mathfrak{p}$. Since Rx and I are adjacent in $(\Omega_R^*)^c$, it follows that $Ann_R Rx \not\subseteq Ann_R I$ and $Ann_R I \not\subseteq Ann_R Rx$. Note that $Ann_R Rx = \mathfrak{p}$. Thus $Ann_R I \not\subseteq \mathfrak{p}$. It follows from $I Ann_R I = (0) \subseteq \mathfrak{p}$ and the hypothesis that $\mathfrak{p}$ is a prime ideal of R that $I \subseteq \mathfrak{p}$. Hence, I and $\mathfrak{p}$ are not adjacent in $(\Omega_R^*)^c$. This is a contradiction. Therefore, there exists no path of length 2 between Rx and $\mathfrak{p}$ in $(\Omega_R^*)^c$. □

Corollary 2. *Let R be a non-reduced ring. Suppose that R has only a finite number $n \geq 2$ of maximal N-primes of (0). Let $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ be the set of all maximal N-primes of (0) in R. If $(\Omega_R^*)^c$ is connected and if $\cap_{i=1}^n \mathfrak{p}_i$ is nilpotent, then $diam((\Omega_R^*)^c) = 3$.*

Proof. Assume that $(\Omega_R^*)^c$ is connected. We know from $(i) \Rightarrow (ii)$ of Theorem 1 that for each $i \in \{1, 2, \ldots, n\}$, there exists $a_i \in R\backslash\{0\}$ such that $\mathfrak{p}_i = ((0) :_R a_i)$. Moreover, R is semiquasilocal with $Max(R) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$. Under the assumption that $\cap_{i=1}^n \mathfrak{p}_i$ is nilpotent, it is shown in the proof of $(ii) \Rightarrow (i)$ of Theorem 1 that $diam((\Omega_R^*)^c) \leq 3$. Let $i \in \{1, 2, \ldots, n\}$. Now, $\mathfrak{p}_i a_i = (0)$. As $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for each $j \in \{1, 2, \ldots, n\}\backslash\{i\}$, it follows that $a_i \in \mathfrak{p}_j$ for all $j \in \{1, 2, \ldots, n\}\backslash\{i\}$. We claim that $a_k \in \cap_{i=1}^n \mathfrak{p}_i$ for some $k \in \{1, 2, \ldots n\}$. That is, equivalently $a_k \in \mathfrak{p}_k$ for some $k \in \{1, 2, \ldots, n\}$. Suppose that $a_k \notin \mathfrak{p}_k$ for each $k \in \{1, 2, \ldots, n\}$. Note that for each $k \in \{1, 2, \ldots, n\}$, $a_k \in \mathfrak{p}_i$ for all $i \in \{1, 2, \ldots, n\}\backslash\{k\}$. Let us denote the element $\sum_{i=1}^n a_i$ by a. Since $Z(R) = \cup_{i=1}^n \mathfrak{p}_i$, it follows that $a \notin Z(R)$. As R is not reduced, we obtain that $\cap_{i=1}^n \mathfrak{p}_i \neq (0)$. Let $x \in \cap_{i=1}^n \mathfrak{p}_i$, $x \neq 0$. From $ax = 0$ and $a \notin Z(R)$, we get that $x = 0$. This is a contradiction. Therefore, $a_k \in \cap_{i=1}^n \mathfrak{p}_i$ for some $k \in \{1, 2, \ldots, n\}$. Note that $\mathfrak{p}_k = ((0)_R : a_k)$, $a_k \in \mathfrak{p}_k$, and it is clear that $Ra_k \neq \mathfrak{p}_k$. Now, it follows from Remark 2 that $d(Ra_k, \mathfrak{p}_k) \geq 3$ in $(\Omega_R^*)^c$. As is already noted that $diam((\Omega_R^*)^c) \leq 3$, we obtain that $diam((\Omega_R^*)^c) = 3$. □

Remark 3. Let R be a non-reduced Artinian ring. If R is local with $\mathfrak{m}$ as its unique maximal ideal, then as $\mathfrak{m}$ is the unique maximal N-prime of (0) in R, it follows from Proposition 5 that $(\Omega_R^*)^c$ is connected if and only if $(R, \mathfrak{m})$ is a SPIR with $\mathfrak{m}^2 = (0)$. Suppose that R is not local. Since R is Artinian, we know from ([5], Proposition 8.3) that R has only a finite number of maximal ideals. Let $\{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n\}$ be the set of all maximal ideals of R. Note that $J(R) = \cap_{i=1}^n \mathfrak{m}_i$. It follows from ([5], Corollary 8.2 and Proposition 8.4) that there exists $k \in \mathbb{N}$ such that $(J(R))^k = (0)$. It is clear that $k \geq 2$. Let $i \in \{1, 2, \ldots, n\}$. Let us denote $\{1, 2, \ldots, n\} \backslash \{i\}$ by A_i. Observe that $\prod_{j \in A_i} \mathfrak{m}_j^k \neq (0)$. It is convenient to denote $\prod_{j \in A_i} \mathfrak{m}_j^k$ by I_i. Note that $\mathfrak{m}_i^k I_i = (0)$. We can choose $t \geq 1$ least with the property that $\mathfrak{m}_i^t I_i = (0)$. Let $x_i \in \mathfrak{m}_i^{t-1} I_i \backslash \{0\}$. Then it is clear that $\mathfrak{m}_i = ((0) :_R x_i)$. This shows that $\{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n\}$ is the set of all maximal N-primes of (0) in R and each proper ideal of R is an annihilating ideal of R. Now, it follows from $(ii) \Rightarrow (i)$ of Theorem 1 that $(\Omega_R^*)^c$ is connected. Moreover, we obtain from Corollary 2 that $diam((\Omega_R^*)^c) = 3$.

Let $n \geq 2$ and let $R, \mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ be as in the statement of Theorem 1. It is shown in Theorem 1 that $(ii) \Rightarrow (i)$ of Theorem 1 holds under the assumption that $\cap_{i=1}^n \mathfrak{p}_i$ is nilpotent. We provide an example in Example 1 to illustrate that the above assumption is not necessary.

Example 1. Let $S = K[[X, Y]]$ be the power series ring in two variables X, Y over a field K. Let $I = SX^2 + SXY$. Let $T = S/I$. Let $R = T \times T$. Then $(\Omega_R^*)^c$ is connected and moreover, $diam((\Omega_R^*)^c) = 3$.

Proof. Observe that S is local with $\mathfrak{m} = SX + SY$ as its unique maximal ideal. Note that $\mathfrak{m} = (I :_S X)$. It is clear that T is local with $\mathfrak{m}/I$ as its unique maximal ideal. Observe that $(\mathfrak{m}/I)(X + I) = (0 + I)$ and $X \notin I$. Hence, each proper ideal of T is an annihilating ideal of T. As $R = T \times T$, we obtain from Lemma 7 that $(\Omega_R^*)^c$ is connected and $diam((\Omega_R^*)^c) \leq 3$. Note that $Z(T) = \mathfrak{m}/I$. Therefore, $\{\mathfrak{p}_1 = \mathfrak{m}/I \times T, \mathfrak{p}_2 = T \times \mathfrak{m}/I\}$ is the set of all maximal N-primes of the zero ideal in R. As $(Y + I)^k \neq 0 + I$ for any $k \geq 1$, it follows that $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \mathfrak{m}/I \times \mathfrak{m}/I$ is not nilpotent. Observe that $\mathfrak{p}_1 = ((0 + I, 0 + I) :_R (X + I, 0 + I))$, $\mathfrak{p}_1 \neq R(X + I, 0 + I)$. Therefore, we obtain from Remark 2 that $d(R(X + I, 0 + I), \mathfrak{p}_1) \geq 3$ in $(\Omega_R^*)^c$ and so, $diam((\Omega_R^*)^c) = 3$. □

In Theorem 2, we classify rings R such that $(\Omega_R^*)^c$ is a path of order 4.

Theorem 2. *Let R be a ring. Then the following statements are equivalent:*
(i) $(\Omega_R^*)^c$ is a path of order 4.
(ii) $R \cong F \times S$ as rings, where F is a field and $(S, \mathfrak{m})$ is a SPIR with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^2 = (0)$.

Proof. $(i) \Rightarrow (ii)$ It follows from (i) that $(\Omega_R^*)^c$ is connected and R has exactly four nontrivial ideals. Therefore, R is necessarily Artinian. If R is local, then we obtain from $(i) \Rightarrow (ii)$ of Proposition 5 that R has only one nontrivial ideal. Hence, R must have at least two maximal ideals. Let n be the number of maximal ideals of R. If $n \geq 3$, then R admits at least six nontrivial ideals. This is impossible. Hence,

$n = 2$. Let $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ denote the set of all maximal ideals of R. If $\mathfrak{m}_1 \cap \mathfrak{m}_2 = (0)$, then R is isomorphic to the direct product of two fields. In such a case, R has exactly two nontrivial ideals. This is a contradiction. Therefore, $\mathfrak{m}_1 \cap \mathfrak{m}_2 \neq (0)$. Since R has exactly four nontrivial ideals, it follows that either $\mathfrak{m}_1 = \mathfrak{m}_1^2$ or $\mathfrak{m}_2 = \mathfrak{m}_2^2$. Without loss of generality, we can assume that $\mathfrak{m}_1 = \mathfrak{m}_1^2$. Note that $J(R) = \mathfrak{m}_1 \cap \mathfrak{m}_2$. As $J(R) \neq (0)$, it follows from Nakayama's lemma ([5], Proposition 2.6) that $J(R) \neq (J(R))^2$. Hence, it follows that $\mathfrak{m}_2 \neq \mathfrak{m}_2^2$. Therefore, $\{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_2^2, \mathfrak{m}_1 \cap \mathfrak{m}_2\}$ is the set of all nontrivial ideals of R. Moreover, $(J(R))^2 = (0)$. Note that $\mathfrak{m}_1 + \mathfrak{m}_2^2 = R$ and $\mathfrak{m}_1 \cap \mathfrak{m}_2^2 = (0)$. Hence, we obtain from the Chinese remainder theorem ([5], Proposition 1.10 (ii) and (iii)) that the mapping $f : R \to R/\mathfrak{m}_1 \times R/\mathfrak{m}_2^2$ given by $f(r) = (r + \mathfrak{m}_1, r + \mathfrak{m}_2^2)$ is an isomorphism of rings. Let us denote $R/\mathfrak{m}_1$ by F and $R/\mathfrak{m}_2^2$ by S. Observe that for any $x \in \mathfrak{m}_2 \backslash (\mathfrak{m}_1 \cup \mathfrak{m}_2^2)$, $\mathfrak{m}_2 = Rx$. Let us denote $\mathfrak{m}_2/\mathfrak{m}_2^2$ by $\mathfrak{m}$. It is clear that $(S, \mathfrak{m})$ is a local ring with $\mathfrak{m} = S(x + \mathfrak{m}_2^2) \neq (0 + \mathfrak{m}_2^2)$ and $\mathfrak{m}^2 = (0 + \mathfrak{m}_2^2)$. Note that F is a field and $(S, \mathfrak{m})$ is a SPIR with $\mathfrak{m} \neq$ zero ideal but $\mathfrak{m}^2 =$ zero ideal and $R \cong F \times S$ as rings.
$(ii) \Rightarrow (i)$ Assume that $R \cong F \times S$ as rings, where F is a field and $(S, \mathfrak{m})$ is a SPIR with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^2 = (0)$. It is not hard to show that $(\Omega_R^*)^c$ is the path of order 4 given by $(0) \times \mathfrak{m} - F \times (0) - (0) \times S - F \times \mathfrak{m}$. □

Let R be a ring such that $|\mathbb{A}(R)^*| \geq 2$. Let H be the subgraph of $(\Omega_R^*)^c$ induced on $\mathbb{A}(R)^*$. In Proposition 6, we classify rings R such that H is a complete bipartite graph. We use Lemma 8 in the proof of Proposition 6.

Lemma 8. *Let R be a reduced ring such that R has exactly two minimal prime ideals. Let H be the subgraph of $(\Omega_R^*)^c$ induced on $\mathbb{A}(R)^*$. Then $H = \mathbb{AG}(R) = (\Omega(R))^c$.*

Proof. Note that the vertex set of $H =$ the vertex set of $\mathbb{AG}(R) =$ the vertex set of $(\Omega(R))^c = \mathbb{A}(R)^*$. Let $I, J \in \mathbb{A}(R)^*$ be such that $I \neq J$. If I and J are adjacent in $\mathbb{AG}(R)$, then we know from Lemma 2 that I and J are adjacent in $(\Omega_R^*)^c$. Suppose that I and J are adjacent in $(\Omega_R^*)^c$. We assert that I and J are adjacent in $\mathbb{AG}(R)$, that is, $IJ = (0)$. Suppose that $IJ \neq (0)$. Let $\{\mathfrak{p}_1, \mathfrak{p}_2\}$ denote the set of all minimal prime ideals of R. Note that $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$ and $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$. Moreover, if $A \in \mathbb{A}(R)^*$, then $A \subseteq Z(R)$ and so, either $A \subseteq \mathfrak{p}_1$ or $A \subseteq \mathfrak{p}_2$. From $IJ \neq (0)$, it follows that either $IJ \not\subseteq \mathfrak{p}_1$ or $IJ \not\subseteq \mathfrak{p}_2$. Without loss of generality, we can assume that $IJ \not\subseteq \mathfrak{p}_1$. Then $I \not\subseteq \mathfrak{p}_1$ and $J \not\subseteq \mathfrak{p}_1$. Hence, $Ann_R I \subseteq \mathfrak{p}_1$ and $Ann_R J \subseteq \mathfrak{p}_1$. Observe that $I \subseteq \mathfrak{p}_2$ and $J \subseteq \mathfrak{p}_2$. Hence, $I\mathfrak{p}_1 = J\mathfrak{p}_1 = (0)$ and so, $\mathfrak{p}_1 \subseteq Ann_R I \cap Ann_R J$. Therefore, $Ann_R I = Ann_R J = \mathfrak{p}_1$. This is in contradiction to the assumption that I and J are adjacent in $(\Omega_R^*)^c$. Hence, $IJ = (0)$ and so, I and J are adjacent in $\mathbb{AG}(R)$. Therefore, we obtain that $H = \mathbb{AG}(R)$.
We next verify that $H = (\Omega(R))^c$. Let $I, J \in \mathbb{A}(R)^*$ be such that $I \neq J$. Suppose that I and J are adjacent in $(\Omega(R))^c$. This implies that $I + J \notin \mathbb{A}(R)$. Hence, we obtain from Lemma 5 that I and J are adjacent in $(\Omega_R^*)^c$. Suppose that I and J are adjacent in $(\Omega_R^*)^c$. Then it is shown in the previous paragraph that $IJ = (0)$. Without loss of generality, we can assume that $I \subseteq \mathfrak{p}_1$ and $J \subseteq \mathfrak{p}_2$. Note that $I \not\subseteq \mathfrak{p}_2$

and $J \not\subseteq \mathfrak{p}_1$. Hence, we get that $I + J \not\subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2$. Since any annihilating ideal of R is contained in $Z(R)$ and $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$, it follows that $I + J \notin \mathbb{A}(R)$. Therefore, I and J are adjacent in $(\Omega(R))^c$. This proves that $H = (\Omega(R))^c$. □

Proposition 6. *Let R be a ring with $|\mathbb{A}(R)^*| \geq 2$. Let H be the subgraph of $(\Omega_R^*)^c$ induced on $\mathbb{A}(R)^*$. Then the following statements are equivalent:*
(i) H is a complete bipartite graph.
(ii) R is reduced and has exactly two minimal prime ideals.

Proof. $(i) \Rightarrow (ii)$ We adapt an argument found in the proof of $(i) \Rightarrow (ii)$ of ([19], Proposition 2.10). Let H be a complete bipartite graph with vertex partition V_1 and V_2. Note that V_1 and V_2 are nonempty, $V_1 \cap V_2 = \emptyset$, and $\mathbb{A}(R)^* = V_1 \cup V_2$. Let us denote $\cup_{I \in V_1} I$ by A and $\cup_{J \in V_2} J$ by B. We claim that A and B are ideals of R. Let $a_1, a_2 \in A$. Then there exist $I_1, I_2 \in V_1$ such that $a_1 \in I_1$ and $a_2 \in I_2$. If $I_1 = I_2$, then it is clear that $a_1 + a_2 \in I_1 \subseteq A$. If $I_1 \neq I_2$, then I_1 and I_2 are not adjacent in $(\Omega_R^*)^c$. Hence, it follows from Lemma 5 that $I_1 + I_2 \in \mathbb{A}(R)$. If $I_1 + I_2 \in V_2$, then we obtain that I_1 and $I_1 + I_2$ are adjacent in $(\Omega_R^*)^c$. This is impossible. Therefore, $I_1 + I_2 \in V_1$. Hence, we get that $a_1 + a_2 \in I_1 + I_2 \subseteq A$. Let $r \in R$ and $a \in A$. Note that there exists $I \in V_1$ such that $a \in I$. Hence, $ra \in I \subseteq A$. This proves that A is an ideal of R. Similarly, it can be shown that B is an ideal of R. Now, it can be shown as in the proof of $(i) \Rightarrow (ii)$ of ([19], Proposition 2.10) that both A and B are maximal N-primes of (0) in R and $A \cap B = (0)$. It is now clear that R is a reduced ring and $\{A, B\}$ is the set of all minimal prime ideals of R.

$(ii) \Rightarrow (i)$ Assume that R is reduced and has exactly two minimal prime ideals. Let $\{\mathfrak{p}_1, \mathfrak{p}_2\}$ denote the set of all minimal prime ideals of R. Note that $\mathbb{AG}(R)$ is a complete bipartite graph with vertex partition $V_1 = \{I \in \mathbb{A}(R)^* : I \subseteq \mathfrak{p}_1\}$ and $V_2 = \{J \in \mathbb{A}(R)^* : J \subseteq \mathfrak{p}_2\}$. We know from Lemma 8 that $H = \mathbb{AG}(R)$. Therefore, H is a complete bipartite graph. □

Proposition 7. *Let R be a ring with $|\mathbb{A}(R)^*| \geq 2$. Let H be the subgraph of $(\Omega_R^*)^c$ induced on $\mathbb{A}(R)^*$. Then the following statements are equivalent:*
(i) H is complete.
(ii) $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$.
(iii) $(\Omega_R^*)^c$ is complete.

Proof. $(i) \Rightarrow (ii)$ Let $I \in \mathbb{A}(R)^*$. Let J be any nonzero ideal of R such that $J \subseteq I$. Then it is clear that $J \in \mathbb{A}(R)^*$ and $Ann_R I \subseteq Ann_R J$. If $I \neq J$, then I and J are not adjacent in $(\Omega_R^*)^c$. This contradicts the assumption that H is complete. Therefore, $J = I$. This shows that each $I \in \mathbb{A}(R)^*$ is a minimal ideal of R. Hence, we obtain from ([10], Theorem 1.1) that R is Artinian. It is already noted in Remark 3 that if I is any proper ideal of R, then $I \in \mathbb{A}(R)$. We know from ([5], Proposition 8.3) that R has only a finite number of maximal ideals. Let $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ denote the set of all maximal ideals of R. If R is local, then $\mathfrak{m}_1$ is the only element of $\mathbb{A}(R)^*$ and this is in contradiction to the hypothesis that $|\mathbb{A}(R)^*| \geq 2$. Hence, we obtain that $n \geq 2$. As

$\mathfrak{m}_1$ is a minimal ideal of R and $\mathfrak{m}_1 \cap \mathfrak{m}_2 \subset \mathfrak{m}_1$, it follows that $\mathfrak{m}_1 \cap \mathfrak{m}_2 = (0)$. Since $\mathfrak{m}_1 + \mathfrak{m}_2 = R$, we obtain from the Chinese remainder theorem ([5], Proposition 1.10 (ii) and (iii)) that the mapping $f : R \to R/\mathfrak{m}_1 \times R/\mathfrak{m}_2$ defined by $f(r) = (r + \mathfrak{m}_1, r + \mathfrak{m}_2)$ is an isomorphism of rings. Let us denote $R/\mathfrak{m}_i$ by F_i for each $i \in \{1, 2\}$. Then F_i is a field for each $i \in \{1, 2\}$ and $R \cong F_1 \times F_2$ as rings.
$(ii) \Rightarrow (iii)$ Let us denote the ring $F_1 \times F_2$ by T, where F_1 and F_2 are fields. It is already noted in Proposition 4 (i) that $(\Omega_T^*)^c$ is complete. Since $R \cong T$ as rings, it follows that $(\Omega_R^*)^c$ is complete.
$(iii) \Rightarrow (i)$ It follows from Corollary 1 that if I is any nontrivial ideal of R, then $I \in \mathbb{A}(R)^*$. Hence, we obtain that $H = (\Omega_R^*)^c$ and so, H is complete. □

Proposition 8. *Let R be a ring with $|\mathbb{A}(R)^*| \geq 3$. Let H be the subgraph of $(\Omega_R^*)^c$ induced on $\mathbb{A}(R)^*$. Then the following statements are equivalent:*
(i) H is a star graph.
(ii) $R \cong D \times F$ as rings, where F is a field and D is an integral domain but not a field.

Proof. $(i) \Rightarrow (ii)$ Let H be a star graph with vertex partition V_1 and V_2 such that $|V_1| = 1$. Let $V_1 = \{I\}$. Since H is a complete bipartite graph with vertex partition V_1 and V_2, it follows from the proof of $(i) \Rightarrow (ii)$ of Proposition 6 that R is reduced and has exactly two minimal prime ideals A and B , where $A = I$ and $B = \cup_{J \in V_2} J$. We know from Lemma 8 that $H = (\Omega(R))^c$. Hence, $(\Omega(R))^c$ is star and so, we obtain from $(i) \Rightarrow (ii)$ of ([19], Proposition 2.12) that $R \cong D \times F$ as rings, where F is a field and D is an integral domain but not a field.
$(ii) \Rightarrow (i)$ Let us denote $D \times F$ by T, where F is a field and D is an integral domain but not a field. From $(ii) \Rightarrow (i)$ of ([19], Proposition 2.12), we obtain that $(\Omega(T))^c$ is a star graph. As $R \cong T$ as rings, we get that $(\Omega(R))^c$ is a star graph. Since R is reduced and has exactly two minimal prime ideals, it follows from Lemma 8 that $H = (\Omega(R))^c$. Therefore, H is a star graph. □

3. On the girth of $(\Omega_R^*)^c$

Let R be a ring with $|\mathbb{A}(R)^*| \geq 2$. In this section, we discuss regarding $girth((\Omega_R^*)^c)$. Let H be the subgraph of $(\Omega_R^*)^c$ induced on $\mathbb{A}(R)^*$. We know from Lemma 1 that if I is an ideal of R such that $I \notin \mathbb{A}(R)$, then I is an isolated vertex of $(\Omega_R^*)^c$. Hence, it follows that $girth((\Omega_R^*)^c) =$ girth of H. Moreover, we know from Lemma 5 that $(\Omega(R))^c$ is a subgraph of H. Hence, in this section, we use results that were proved on the $girth((\Omega(R))^c)$ in ([19], Section 3).

Proposition 9. *Let R be a reduced ring with $|\mathbb{A}(R)^*| \geq 2$. If R has a unique maximal N-prime of (0), then $girth((\Omega_R^*)^c) = 3$.*

Proof. First, we claim that R has an infinite number of minimal prime ideals. Suppose that R has only a finite number $n \geq 2$ of minimal prime ideals. Let

$\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ denote the set of all minimal prime ideals of R. Note that $\cap_{i=1}^{n}\mathfrak{p}_i = (0)$ and $Z(R) = \cup_{i=1}^{n}\mathfrak{p}_i$. This implies that $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ is the set of all maximal N-primes of (0) in R. This is in contradiction to the assumption that R has a unique maximal N-prime of (0). Therefore, R has an infinite number of minimal prime ideals. Now, it follows from ([19], Proposition 3.8) that $girth((\Omega(R))^c) = 3$. Since $(\Omega(R))^c$ is a subgraph of $(\Omega_R^*)^c$, we obtain that $girth((\Omega_R^*)^c) = 3$. □

Proposition 10. *Let R be a reduced ring such that R has exactly two maximal N-primes of (0). Then $girth((\Omega_R^*)^c) \in \{3, 4, \infty\}$.*

Proof. If R has at least three minimal prime ideals, then we know from ([19], Proposition 3.8) that $girth((\Omega(R))^c) = 3$ and so, $girth((\Omega_R^*)^c) = 3$. Suppose that R has exactly two minimal prime ideals. Let H be the subgraph of $(\Omega_R^*)^c$ induced on $\mathbb{A}(R)^*$. We know from $(ii) \Rightarrow (i)$ of Proposition 6 that H is a complete bipartite graph. Therefore, $girth((\Omega_R^*)^c) = girth(H) \in \{4, \infty\}$. This proves that $girth((\Omega_R^*)^c) \in \{3, 4, \infty\}$. □

We next present some examples to illustrate Propositions 9 and 10. Example 2 given below is found in ([13], Example, page 16).

Example 2. Let K be a field and $\{X_i\}_{i=1}^{\infty}$ be a set of independent indeterminates over K. Let $D = \cup_{n=1}^{\infty}K[[X_1, \ldots, X_n]]$, where $K[[X_1, \ldots, X_n]]$ is the power series ring in n variables $X_1, \ldots, X_n$ over K. Let I be the ideal of D generated by $\{X_iX_j : i, j \in \mathbb{N}, i \neq j\}$. Let $R = D/I$. Then R is a reduced ring, R has a unique maximal N-prime of (0), and $girth((\Omega_R^*)^c) = 3$.

Proof. Let $i \in \mathbb{N}$. It is convenient to denote $X_i + I$ by x_i. The following facts about the ring R have been mentioned in ([13], Example, page 16).

(1) R is quasilocal with $\mathfrak{m}$ = the ideal of R generated by $\{x_i : i \in \mathbb{N}\}$ as its unique maximal ideal.

(2) Let $i \in \mathbb{N}$ and $\mathfrak{p}_i$ be the ideal of R generated by $\{x_j : j \in \mathbb{N}, j \neq i\}$. Then $\{\mathfrak{p}_i : i \in \mathbb{N}\}$ is the set of all minimal prime ideals of R.

It was shown in ([17], Example 3.4 (i)) that $\mathfrak{m} = Z(R)$. Hence, R has $\mathfrak{m}$ as its unique maximal N-prime of (0). It follows from $\cap_{i=1}^{\infty}\mathfrak{p}_i = (0)$ that R is reduced. It follows from Proposition 9 that $girth((\Omega_R^*)^c) = 3$. We verify here that $(\Omega(R))^c$ admits an infinite clique. Since R is reduced, it is clear that $\mathfrak{m} \notin \mathbb{A}(R)$. Observe that for each $i \in \mathbb{N}$, $\mathfrak{p}_i = ((0 + I) :_R x_i)$ and so, $\mathfrak{p}_i \in \mathbb{A}(R)^*$. Note that for all distinct $i, j \in \mathbb{N}$, $\mathfrak{p}_i + \mathfrak{p}_j = \mathfrak{m} \notin \mathbb{A}(R)$. Hence, the subgraph of $(\Omega(R))^c$ induced on $\{\mathfrak{p}_i : i \in \mathbb{N}\}$ is an infinite clique. Since $(\Omega(R))^c$ is a subgraph of $(\Omega_R^*)^c$, it follows that $(\Omega_R^*)^c$ admits an infinite clique. □

Example 3. Let R be as in Example 2 and let $T = R \times R$. Then T is a reduced ring, T has exactly two maximal N-primes of its zero ideal, and $girth((\Omega_T^*)^c) = 3$.

Proof. We know from Example 2 that R is reduced. Hence, it follows that T is reduced. Also, it is noted in the verification of Example 2 that $Z(R) = \mathfrak{m}$, where $\mathfrak{m}$ is the unique maximal ideal of R. Observe that T has exactly two maximal N-primes of $(0,0)$ and they are given by $\mathfrak{P}_1 = \mathfrak{m} \times R$ and $\mathfrak{P}_2 = R \times \mathfrak{m}$. It is observed in the proof of Example 2 that the subgraph of $(\Omega(R))^c$ induced on $\{\mathfrak{p}_i : i \in \mathbb{N}\}$ is an infinite clique. Hence, we obtain that the subgraph of $(\Omega(T))^c$ induced by $\{\mathfrak{p}_i \times R : i \in \mathbb{N}\}$ is an infinite clique. Therefore, $girth((\Omega(T))^c) = girth((\Omega_T^*)^c) = 3$. □

Example 4. If $R = \mathbb{Z} \times \mathbb{Z}$, then $girth((\Omega_R^*)^c) = 4$.

Proof. It is clear that R is reduced and $\{\mathfrak{p}_1 = (0) \times \mathbb{Z}, \mathfrak{p}_2 = \mathbb{Z} \times (0)\}$ is the set of all minimal prime ideals of R. We know from Lemma 8 that $H = \mathbb{AG}(R)$, where H is the subgraph of $(\Omega_R^*)^c$ induced by $\mathbb{A}(R)^*$. Observe that $\mathbb{AG}(R)$ is a complete bipartite graph with vertex partition $V_1 = \{A \in \mathbb{A}(R)^* : A \subseteq \mathfrak{p}_1\}$ and $V_2 = \{B \in \mathbb{A}(R)^* : B \subseteq \mathfrak{p}_2\}$. As V_i contains at least two elements for each $i \in \{1,2\}$, it follows that $girth((\Omega_R^*)^c) = girth(H) = 4$. □

Example 5. If $R = \mathbb{Z} \times \mathbb{Q}$, then $girth((\Omega_R^*)^c) = \infty$.

Proof. Let H be the subgraph of $(\Omega_R^*)^c$ induced by $\mathbb{A}(R)^*$. We know from $(ii) \Rightarrow (i)$ of Proposition 8 that H is a star graph. Hence, $girth((\Omega_R^*)^c) = girth(H) = \infty$. □

Example 6. Let $R = F_1 \times F_2$, where F_1, F_2 are fields. Then $girth((\Omega_R^*)^c) = \infty$.

Proof. It is already noted in the proof of Proposition 4 (i) that $(\Omega_R^*)^c$ is a complete graph on two vertices. Therefore, $girth((\Omega_R^*)^c) = \infty$. □

Let R be a ring which is possibly non-reduced. We next discuss regarding $girth((\Omega_R^*)^c)$. We are not able to determine $girth((\Omega_R^*)^c)$ in the case when R has at most two maximal N-primes of (0). However, we present some remarks and examples of rings R describing the nature of cycles of $(\Omega_R^*)^c$.

Remark 4. Recall that a ring R is a *chained ring* if the ideals of R are comparable under the inclusion relation. Thus if R is a chained ring , then $Z(R)$ is an ideal of R and hence, R has a unique maximal N-prime of (0). Let R be a chained ring with at least one nontrivial ideal. Then $(\Omega_R^*)^c$ has no edges and so, $girth((\Omega_R^*)^c) = \infty$.

Example 7. Let $T = K[[X]]$ be the power series ring in one variable X over a field K and let $R = T/X^5T$. Then $girth((\Omega_R^*)^c) = \infty$.

Proof. It is well-known that T is a discrete valuation ring and $\{X^nT : n \in \mathbb{N}\}$ is the set of all nontrivial ideals of T. Observe that R is a chained ring and the set of all nontrivial ideals of R equals $\{X^iT/X^5T : i \in \{1,2,3,4\}\}$. It follows from Remark 4 that $girth((\Omega_R^*)^c) = \infty$. $\square$

We next provide an example of a quasilocal ring $(R, \mathfrak{p})$ in Example 8 such that $\mathfrak{p}$ is the unique maximal N-prime of (0) and $girth((\Omega_R^*)^c) = 3$. The ring R given in Example 8 is from ([16], Exercises 6 and 7, pages 62-63).

Example 8. Let $S = K[X,Y]$ be the polynomial ring in two variables X, Y over a field K. Let $\mathfrak{m} = SX + SY$. Let $T = S_{\mathfrak{m}}$. Let $\mathcal{P}$ be the set of all pairwise nonassociate prime elements of the unique factorization domain T. Let $W = \bigoplus_{p\in\mathcal{P}}(T/Tp)$ be the direct sum of the T- modules T/Tp, where p varies over $\mathcal{P}$. Let $R = T \oplus W$ be the ring obtained on using Nagata's principle of idealization. Then $girth((\Omega_R^*)^c) = 3$.

Proof. Since T is local with $\mathfrak{m}T$ as its unique maximal ideal, it follows that R is quasilocal with $\mathfrak{p} = \mathfrak{m}T \oplus W$ as its unique maximal ideal. It was shown in ([18], Example 2.8) that $\mathfrak{p}$ is the unique maximal N-prime ideal of the zero ideal in R. It was verified in ([19], Remark 3.2 (ii)) that $girth((\Omega(R))^c) = 3$. Indeed, it was shown in ([19], Remark 3.2 (ii)) that $(\Omega(R))^c$ contains an infinite clique. Since $(\Omega(R))^c$ is a subgraph of $(\Omega_R^*)^c$, we get that $girth((\Omega_R^*)^c) = 3$. $\square$

Example 9. Let $R = F \times S$, where F is a field and $(S, \mathfrak{m})$ is a SPIR with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^2 = (0)$. Then R has exactly two maximal N-primes of $(0,0)$ and $girth((\Omega_R^*)^c) = \infty$.

Proof. It is clear that $\{\mathfrak{p}_1 = (0) \times S, \mathfrak{p}_2 = F \times \mathfrak{m}\}$ is the set of all maximal N-primes of $(0,0)$ in R. We know from $(ii) \Rightarrow (i)$ of Theorem 2 that $(\Omega_R^*)^c$ is a path of order 4. Hence, $girth((\Omega_R^*)^c) = \infty$. $\square$

Remark 5. Let R_1, R_2 be rings such that $Z(R_i) \in \mathbb{A}(R_i)^*$ for each $i \in \{1,2\}$. Let $R = R_1 \times R_2$. Then R has exactly two maximal N-primes of $(0,0)$ and $girth((\Omega_R^*)^c) = 3$.

Proof. Observe that $\{\mathfrak{p}_1 = Z(R_1) \times R_2, \mathfrak{p}_2 = R_1 \times Z(R_2)\}$ is the set of all maximal N-primes of $(0,0)$ in R. Let $I_1 = R_1 \times (0), I_2 = (0) \times R_2$, and $I_3 = Z(R_1) \times Z(R_2)$. As $I_1 + I_2 = R \notin \mathbb{A}(R)$, it follows from Lemma 5 that I_1 and I_2 are adjacent in $(\Omega_R^*)^c$. By assumption, $Ann_{R_i}Z(R_i)$ is a nontrivial ideal of R_i for each $i \in \{1,2\}$. Observe that $Ann_R I_1 = (0) \times R_2 \not\subseteq Ann_R I_3 = Ann_{R_1}Z(R_1) \times Ann_{R_2}Z(R_2)$ and $Ann_R I_3 \not\subseteq Ann_R I_1$. Similarly, $Ann_R I_2 = R_1 \times (0) \not\subseteq Ann_R I_3$ and $Ann_R I_3 \not\subseteq Ann_R I_2$. Hence, we obtain that I_3 is adjacent to both I_1 and I_2 in $(\Omega_R^*)^c$. From the above discussion, it is clear that $I_1 - I_2 - I_3 - I_1$ is a cycle of length 3 in $(\Omega_R^*)^c$. Therefore, we get that $girth((\Omega_R^*)^c) = 3$. $\square$

Example 10. Let R be as in Example 7 and let $S = R \times R$. Then $girth((\Omega_S^*)^c) = 3$.

Proof. Observe that $Z(R) = XT/X^5T \in \mathbb{A}(R)^*$. Hence, on applying Remark 5 with $R_1 = R_2 = R$, we obtain that $girth((\Omega_S^*)^c) = 3$. $\square$

Proposition 11. *If R is a ring which admits at least three maximal N-primes of* (0)*, then* $girth((\Omega_R^*)^c) = 3$.

Proof. We know from ([19], Corollary 3.11) that $girth((\Omega(R))^c) = 3$. Since $(\Omega(R))^c$ is a subgraph of $(\Omega_R^*)^c$, it follows that $girth((\Omega_R^*)^c) = 3$. □

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