

## Roman domination excellent graphs: trees

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**Abstract:** A Roman dominating function (RDF) on a graph  $G = (V, E)$  is a labeling  $f : V \rightarrow \{0, 1, 2\}$  such that every vertex with label 0 has a neighbor with label 2. The weight of  $f$  is the value  $f(V) = \sum_{v \in V} f(v)$ . The Roman domination number,  $\gamma_R(G)$ , of  $G$  is the minimum weight of an RDF on  $G$ . An RDF of minimum weight is called a  $\gamma_R$ -function. A graph  $G$  is said to be  $\gamma_R$ -excellent if for each vertex  $x \in V$  there is a  $\gamma_R$ -function  $h_x$  on  $G$  with  $h_x(x) \neq 0$ . We present a constructive characterization of  $\gamma_R$ -excellent trees using labelings. A graph  $G$  is said to be in class  $UVR$  if  $\gamma(G-v) = \gamma(G)$  for each  $v \in V$ , where  $\gamma(G)$  is the domination number of  $G$ . We show that each tree in  $UVR$  is  $\gamma_R$ -excellent.

**Keywords:** Roman domination number, excellent tree, coalescence

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### 1. Introduction and preliminaries

For basic notation and graph theory terminology not explicitly defined here, we in general follow Haynes et al. [9]. Specifically, let  $G$  be a simple graph with vertex set  $V(G)$  and edge set  $E(G)$ . A *spanning subgraph* for  $G$  is a subgraph of  $G$  which contains every vertex of  $G$ . In a graph  $G$ , for a subset  $S \subseteq V(G)$  the *subgraph induced* by  $S$  is the graph  $\langle S \rangle$  with vertex set  $S$  and edge set  $\{xy \in E(G) \mid x, y \in S\}$ . The complement  $\overline{G}$  of  $G$  is the graph whose vertex set is  $V(G)$  and whose edges are the pairs of nonadjacent vertices of  $G$ . We write  $K_n$  for the *complete graph* of order  $n$  and  $P_n$  for the *path* on  $n$  vertices. Let  $C_m$  denote the *cycle* of length  $m$ . For any vertex  $x$  of a graph  $G$ ,  $N_G(x)$  denotes the set of all neighbors of  $x$  in  $G$ ,  $N_G[x] = N_G(x) \cup \{x\}$  and the degree of  $x$  is  $deg_G(x) = |N_G(x)|$ . The *minimum* and *maximum* degrees of a graph  $G$  are denoted by  $\delta(G)$  and  $\Delta(G)$ , respectively. For a subset  $S$  of vertices, let

$N_G[S] = \cup_{v \in S} N_G[v]$ . The *external private neighborhood*  $epn(v, S)$  of  $v \in S$  is defined by  $epn(v, S) = \{u \in V(G) - S \mid N_G(u) \cap S = \{v\}\}$ . A *leaf* is a vertex of degree one and a *support vertex* is a vertex adjacent to a leaf. If  $F$  and  $H$  are disjoint graphs,  $v_F \in V(F)$  and  $v_H \in V(H)$ , then the *coalescence*  $(F \cdot H)(v_F, v_H : v)$  of  $F$  and  $H$  via  $v_F$  and  $v_H$ , is the graph obtained from the union of  $F$  and  $H$  by identifying  $v_F$  and  $v_H$  in a vertex labeled  $v$ . If  $F$  and  $H$  are graphs with exactly one vertex in common, say  $x$ , then the *coalescence*  $(F \cdot H)(x)$  of  $F$  and  $H$  via  $x$  is the union of  $F$  and  $H$ .

Let  $Y$  be a finite set of integers which has positive as well as non-positive elements. Denote by  $P(Y)$  the collection of all subsets of  $Y$ . Given a graph  $G$ , for a  $Y$ -valued function  $f : V(G) \rightarrow Y$  and a subset  $S$  of  $V(G)$  we define  $f(S) = \sum_{v \in S} f(v)$ . The *weight* of  $f$  is  $f(V(G))$ . A  *$Y$ -valued Roman dominating function* on a graph  $G$  is a function  $f : V(G) \rightarrow Y$  satisfying the conditions: (a)  $f(N_G[v]) \geq 1$  for each  $v \in V(G)$ , and (b) if  $v \in V(G)$  and  $f(v) \leq 0$ , then there is  $u_v \in N_G(v)$  with  $f(u_v) = \max\{k \mid k \in Y\}$ . For a  $Y$ -valued Roman dominating function  $f$  on a graph  $G$ , where  $Y = \{r_1, r_2, \dots, r_k\}$  and  $r_1 < r_2 < \dots < r_k$ , let  $V_{r_i}^f = \{v \in V(G) \mid f(v) = r_i\}$  for  $i = 1, \dots, k$ . Since these  $k$  sets determine  $f$ , we can equivalently write  $f = (V_{r_1}^f; V_{r_2}^f; \dots; V_{r_k}^f)$ . If  $f$  is  $Y$ -valued Roman dominating function on a graph  $G$  and  $H$  is a subgraph of  $G$ , then we denote the restriction of  $f$  on  $H$  by  $f|_H$ . The  *$Y$ -Roman domination number* of a graph  $G$ , denoted  $\gamma_{YR}(G)$ , is defined to be the minimum weight of a  $Y$ -valued dominating function on  $G$ . As examples, let us mention: (a) the domination number  $\gamma(G) \equiv \gamma_{\{0,1\}R}(G)$ , (b) the minus domination number [6], where  $Y = \{-1, 0, 1\}$ , (c) the signed domination number [5], where  $Y = \{-1, 1\}$ , (d) the Roman domination number  $\gamma_R(G) \equiv \gamma_{\{0,1,2\}R}(G)$  [4], and (e) the signed Roman domination number [1], where  $Y = \{-1, 1, 2\}$ . A  $Y$ -valued Roman dominating function  $f$  on  $G$  with weight  $\gamma_{YR}(G)$  is called a  *$\gamma_{YR}$ -function* on  $G$ .

Now we introduce a new partition of a vertex set of a graph, which plays a key role in the paper. In determining this partition, all  $\gamma_{YR}$ -functions of a graph are necessary. For each  $X \in P(Y)$  we define the set  $V^X(G)$  as consisting of all  $v \in V(G)$  with  $\{f(v) \mid f \text{ is a } \gamma_{YR}\text{-function on } G\} = X$ . Then all members of the family  $(V^X(G))_{X \in P(Y)}$  clearly form a partition of  $V(G)$ . We call this partition the  *$\gamma_{YR}$ -partition of  $G$* .

Fricke et al. [7] in 2002 began the study of graphs, which are excellent with respect to various graph parameters. Let us concentrate here on the parameter  $\gamma_{YR}$ . A vertex  $v \in V(G)$  is said to be (a)  *$\gamma_{YR}$ -good*, if  $h(v) \geq 1$  for some  $\gamma_{YR}$ -function  $h$  on  $G$ , and (b)  *$\gamma_{YR}$ -bad* otherwise. A graph  $G$  is said to be  *$\gamma_{YR}$ -excellent* if all vertices of  $G$  are  $\gamma_{YR}$ -good. Any vertex-transitive graph is  $\gamma_{YR}$ -excellent. Note that when  $\gamma_{YR} \equiv \gamma$ , the set of all  $\gamma$ -good and the set of all  $\gamma$ -bad vertices of a graph  $G$  form the  $\gamma$ -partition of  $G$ . For further results on this topic see e.g. [2, 10–15].

In this paper we begin an investigation of  $\gamma_{YR}$ -excellent graphs in the case when  $Y = \{0, 1, 2\}$ . In what follows we shall write  $\gamma_R$  instead of  $\gamma_{\{0,1,2\}R}$ , and we shall abbreviate a  $\{0, 1, 2\}$ -valued Roman dominating function to an *RD-function*. Let us describe all members of the  $\gamma_R$ -partition of any graph  $G$  (we write  $V^i(G)$ ,  $V^{ij}(G)$  and  $V^{ijk}(G)$  instead of  $V^{\{i\}}(G)$ ,  $V^{\{i,j\}}(G)$  and  $V^{\{i,j,k\}}(G)$ , respectively).

- (i)  $V^i(G) = \{x \in V(G) \mid f(x) = i \text{ for each } \gamma_R\text{-function } f \text{ on } G\}$ ,  $i = 1, 2, 3$ ;

- (ii)  $V^{012}(G) = \{x \in V(G) \mid \text{there are } \gamma_R\text{-functions } f_x, g_x, h_x \text{ on } G \text{ with}$   
 $f_x(x) = 0, g_x(x) = 1 \text{ and } h_x(x) = 2\}$ ;
- (iii)  $V^{ij}(G) = \{x \in V(G) - V^{012}(G) \mid \text{there are } \gamma_R\text{-functions } f_x \text{ and } g_x \text{ on } G$   
with  $f_x(x) = i$  and  $g_x(x) = j\}, 0 \leq i < j \leq 2$ .

Clearly a graph  $G$  is  $\gamma_R$ -excellent if and only if  $V^0(G) = \emptyset$ .

It is often of interest to know how the value of a graph parameter is affected when a small change is made in a graph. In this connection, Hansberg, Jafari Rad and Volkmann studied in [8] changing and unchanging of the Roman domination number of a graph when a vertex is deleted, or an edge is added.

**Lemma 1.** ([8]) *Let  $v$  be a vertex of a graph  $G$ . Then  $\gamma_R(G - v) < \gamma_R(G)$  if and only if there is a  $\gamma_R$ -function  $f = (V_0^f; V_1^f; V_2^f)$  on  $G$  such that  $v \in V_1^f$ . If  $\gamma_R(G - v) < \gamma_R(G)$  then  $\gamma_R(G - v) = \gamma_R(G) - 1$ .*

Lemma 1 implies that  $V^1(G), V^{01}(G), V^{12}(G), V^{012}(G)$  form a partition of  $V^-(G) = \{x \in V(G) \mid \gamma_R(G - x) + 1 = \gamma_R(G)\}$ .

**Lemma 2.** ([8]) *Let  $x$  and  $y$  be non-adjacent vertices of a graph  $G$ . Then  $\gamma_R(G) \geq \gamma_R(G + xy) \geq \gamma_R(G) - 1$ . Moreover,  $\gamma_R(G + xy) = \gamma_R(G) - 1$  if and only if there is a  $\gamma_R$ -function  $f$  on  $G$  such that  $\{f(x), f(y)\} = \{1, 2\}$ .*

The same authors defined the following two classes of graphs:

- (i)  $\mathcal{R}_{CVR}$  is the class of graphs  $G$  such that  $\gamma_R(G - v) < \gamma_R(G)$  for all  $v \in V(G)$ .
- (ii)  $\mathcal{R}_{CEA}$  is the class of graphs  $G$  such that  $\gamma_R(G + e) < \gamma_R(G)$  for all  $e \in E(\overline{G})$ .

**Remark 1.** By Lemmas 1 and 2 it easy follows that:

- (i) each graph in  $\mathcal{R}_{CVR} \cup \mathcal{R}_{CEA}$  is  $\gamma_R$ -excellent,
- (ii) if  $G$  is a  $\gamma_R$ -excellent graph,  $e \in E(\overline{G})$  and  $\gamma_R(G) = \gamma_R(G + e)$ , then  $G + e$  is  $\gamma_R$ -excellent,
- (iii) each graph (in particular each  $\gamma_R$ -excellent graph) is a spanning subgraph of a graph in  $\mathcal{R}_{CEA}$  with the same Roman domination number.

Denote by  $\mathbf{G}_{n,k}$  the family of all mutually non-isomorphic  $n$ -order  $\gamma_R$ -excellent connected graphs having the Roman domination number equal to  $k$ . With the family  $\mathbf{G}_{n,k}$ , we associate the poset  $\mathbb{R}\mathbb{E}_{n,k} = (\mathbf{G}_{n,k}, \prec)$  with the order  $\prec$  given by  $H_1 \prec H_2$  if and only if  $H_2$  has a spanning subgraph which is isomorphic to  $H_1$  (see [16] for terminology on posets). Remark 1 shows that all maximal elements of  $\mathbb{R}\mathbb{E}_{n,k}$  are in  $\mathcal{R}_{CEA}$ . Here we concentrate on the set of all minimal elements of  $\mathbb{R}\mathbb{E}_{n,k}$ . Clearly a graph  $H \in \mathbf{G}_{n,k}$  is a minimal element of  $\mathbb{R}\mathbb{E}_{n,k}$  if and only if for each  $e \in E(H)$  at

least one of the following holds: (a)  $H - e$  is not connected, (b)  $\gamma_R(H) \neq \gamma_R(H - e)$ , and (c)  $H - e$  is not  $\gamma_R$ -excellent. All trees in  $\mathbf{G}_{n,k}$  are obviously minimal elements of  $\mathbb{R}\mathbb{E}_{n,k}$ .

The remainder of this paper is organized as follows. In Section 2, we formulate our main result, namely, a constructive characterization of  $\gamma_R$ -excellent trees. We present a proof of this result in Sections 3 and 4. Applications of our main result are given in Sections 5 and 6. We conclude in Section 7 with some open problems.

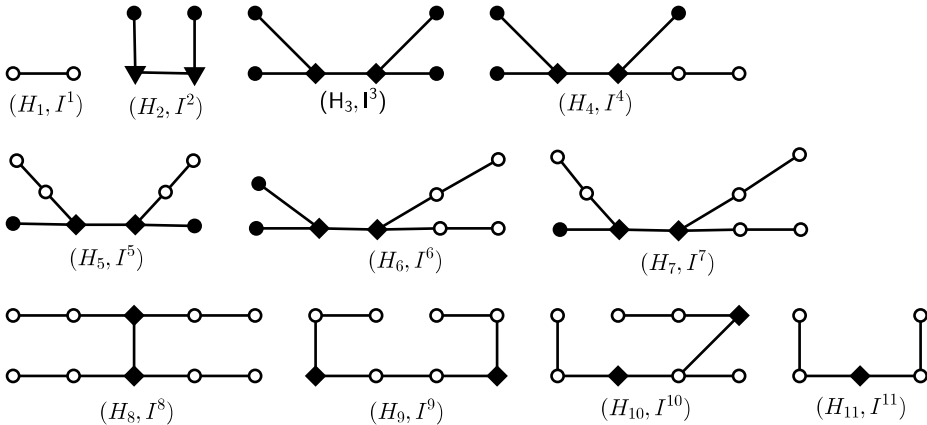
We end this section with the following useful result.

**Lemma 3.** ([4]) *Let  $f = (V_0^f; V_1^f; V_2^f)$  be any  $\gamma_R$ -function on a graph  $G$ . Then each component of a graph  $\langle V_1^f \rangle$  has order at most 2 and no edge of  $G$  joins  $V_1^f$  and  $V_2^f$ .*

In most cases Lemmas 1, 2 and 3 will be used in the sequel without specific reference.

## 2. The main result

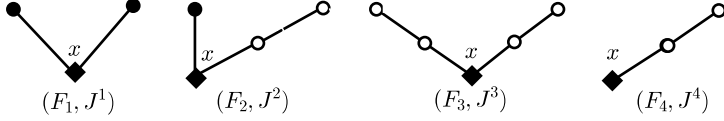
In this section, we present a constructive characterization of  $\gamma_R$ -excellent trees using labelings. We define a *labeling* of a tree  $T$  as a function  $S : V(T) \rightarrow \{A, B, C, D\}$ . A labeled tree is denoted by a pair  $(T, S)$ . The label of a vertex  $v$  is also called its *status*, denoted  $sta_T(v : S)$  or  $sta_T(v)$  if the labeling  $S$  is clear from context. We denote the sets of vertices of status  $A, B, C$  and  $D$  by  $S_A(T), S_B(T), S_C(T)$  and  $S_D(T)$ , respectively. In all figures in this paper we use  $\bullet$  for a vertex of status  $A$ ,  $\blacktriangledown$  for a vertex of status  $B$ ,  $\blacklozenge$  for a vertex of status  $C$ , and  $\circ$  for a vertex of status  $D$ . If  $H$  is a subgraph of  $T$ , then we denote the restriction of  $S$  on  $H$  by  $S|_H$ .



**Figure 1.** All trees with  $|L_B \cup L_C| \leq 2$ .

To state a characterization of  $\gamma_R$ -excellent trees, we introduce four types of operations. Let  $\mathcal{T}$  be the family of labeled trees  $(T, S)$  that can be obtained from a

sequence of labeled trees  $\tau : (T^1, S^1), \dots, (T^j, S^j)$ , ( $j \geq 1$ ), such that  $(T^1, S^1)$  is in  $\{(H_1, I^1), \dots, (H_5, I^5)\}$  (see Figure 1) and  $(T, S) = (T^j, S^j)$ , and, if  $j \geq 2$ ,  $(T^{i+1}, S^{i+1})$  can be obtained recursively from  $(T^i, S^i)$  by one of the operations  $O_1, O_2, O_3$  and  $O_4$  listed below; in this case  $\tau$  is said to be a  $\mathcal{T}$ -sequence of  $T$ . When the context is clear we shall write  $T \in \mathcal{T}$  instead of  $(T, S) \in \mathcal{T}$ .



**Figure 2.**  $(F, J)$ -graphs

**Operation  $O_1$ .** The labeled tree  $(T^{i+1}, S^{i+1})$  is obtained from  $(T^i, S^i)$  and  $(F, J) \in \{(F_1, J^1), (F_2, J^2), (F_3, J^3)\}$  (see Figure 2) by adding the edge  $ux$ , where  $u \in V(T_i)$ ,  $x \in V(F)$  and  $sta_{T^i}(u) = sta_F(x) = C$ .

**Operation  $O_2$ .** The labeled tree  $(T^{i+1}, S^{i+1})$  is obtained from  $(T^i, S^i)$  and  $(F_4, J^4)$  (see Figure 2) by adding the edge  $ux$ , where  $u \in V(T^i)$ ,  $x \in V(F_4)$ ,  $sta_{T^i}(u) = D$ , and  $sta_{F_4}(x) = C$ .

**Operation  $O_3$ .** The labeled tree  $(T^{i+1}, S^{i+1})$  is obtained from  $(T^i, S^i)$  and  $(H_k, I^k)$ ,  $k \in \{2, 3, \dots, 7\}$  (see Figure 1), in such a way that  $T^{i+1} = (T^i \cdot H_k)(u, v : u)$ , where  $sta_{T^i}(u) = sta_{H_k}(v) = A$ , and  $sta_{T^{i+1}}(u) = A$ .

**Operation  $O_4$ .** The labeled tree  $(T^{i+1}, S^{i+1})$  is obtained from  $(T^i, S^i)$  and  $(H_k, I^k)$ ,  $k \in \{3, 4, 6\}$  (see Figure 1), in such a way that  $T^{i+1} = (T^i \cdot H_k)(u, v : u)$ , where  $sta_{T^i}(u) = D$ ,  $sta_{H_k}(v) = A$ , and  $sta_{T^{i+1}}(u) = D$ .

Remark that if  $y \in V(T^i)$  and  $i \leq k \leq j$ , then  $sta_{T^i}(y) = sta_{T^k}(y)$ . Now we are prepared to state the main result.

**Theorem 1.** *Let  $T$  be a tree of order at least 2. Then  $T$  is  $\gamma_R$ -excellent if and only if there is a labeling  $S : V(T) \rightarrow \{A, B, C, D\}$  such that  $(T, S)$  is in  $\mathcal{T}$ . Moreover, if  $(T, S) \in \mathcal{T}$  then*

$$(\mathcal{P}_1) \quad S_B(T) = \{x \in V^{02}(T) \mid \deg(x) = 2 \text{ and } |N(x) \cap V^{02}(T)| = 1\}, \quad S_A(T) = V^{01}(T), \\ S_D(T) = V^{012}(T), \text{ and } S_C(T) = V^{02}(T) - S_B(T).$$

### 3. Preparation for the proof of Theorem 1

#### 3.1. Coalescence

We shall concentrate on the coalescence of two graphs via a vertex in  $V^{01}$  and derive the properties which will be needed for the proof of our main result.

**Proposition 1.** *Let  $G = (G_1 \cdot G_2)(x)$  be a connected graph and  $x \in V^{01}(G)$ . Then the following holds.*

- (i) *If  $f$  is a  $\gamma_R$ -function on  $G$  and  $f(x) = 1$ , then  $f|_{G_i}$  is a  $\gamma_R$ -function on  $G_i$ , and  $f|_{G_i-x}$  is a  $\gamma_R$ -function on  $G_i - x$ ,  $i = 1, 2$ .*
- (ii)  $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$ .
- (iii) *If  $h$  is a  $\gamma_R$ -function on  $G$  and  $h(x) = 0$ , then exactly one of the following holds:*
  - (iii.1)  $h|_{G_1}$  is a  $\gamma_R$ -function on  $G_1$ ,  $h|_{G_2-x}$  is a  $\gamma_R$ -function on  $G_2 - x$ , and  $h|_{G_2}$  is no RD-function on  $G_2$ ;
  - (iii.2)  $h|_{G_1-x}$  is a  $\gamma_R$ -function on  $G_1 - x$ ,  $h|_{G_1}$  is no RD-function on  $G_1$ , and  $h|_{G_2}$  is a  $\gamma_R$ -function on  $G_2$ .
- (iv) *Either  $\{x\} = V^{01}(G_1) \cap V^{01}(G_2)$  or  $\{x\} = V^{01}(G_i) \cap V^1(G_j)$ , where  $\{i, j\} = \{1, 2\}$ .*

**Proof.** (i) and (ii): Since  $f(x) = 1$ ,  $f|_{G_i}$  is an RD-function on  $G_i$ , and  $f|_{G_i-x}$  is an RD-function on  $G_i - x$ ,  $i = 1, 2$ . Assume  $g_1$  is a  $\gamma_R$ -function on  $G_1$  with  $g_1(V(G_1)) < f|_{G_1}(V(G_1))$ . Define an RD-function  $f'$  as follows:  $f'(u) = g_1(u)$  for all  $u \in V(G_1)$  and  $f'(u) = f(u)$  when  $u \in V(G_2 - x)$ . Then  $f'(V(G)) = g_1(V(G_1)) + f|_{G_2-x}(V(G_2 - x)) < f(V(G))$ , a contradiction. Thus,  $f|_{G_i}$  is a  $\gamma_R$ -function on  $G_i$ ,  $i = 1, 2$ . Now, Lemma 1 implies that  $f|_{G_i-x}$  is a  $\gamma_R$ -function on  $G_i - x$ ,  $i = 1, 2$ . Hence  $\gamma_R(G) = f|_{G_1}(V(G_1)) + f|_{G_2}(V(G_2)) - f(x) = \gamma_R(G_1) + \gamma_R(G_2) - 1$ .

(iii) First note that  $h(x) = 0$  implies  $h|_{G_i}$  is an RD-function on  $G_i$  for some  $i \in \{1, 2\}$ , say  $i = 1$ . If  $h|_{G_2}$  is an RD-function on  $G_2$  then  $\gamma_R(G) = h(V(G)) \geq \gamma_R(G_1) + \gamma_R(G_2)$ , a contradiction with (ii). Thus,  $h|_{G_2-x}$  is an RD-function on  $G_2 - x$ . Now we have  $\gamma_R(G_1) + \gamma_R(G_2) - 1 = \gamma_R(G) = h(V(G)) = h|_{G_1}(V(G_1)) + h|_{G_2-x}(V(G_2 - x)) \geq \gamma_R(G_1) + (\gamma_R(G_2) - 1)$ . Hence  $h|_{G_1}$  is a  $\gamma_R$ -function on  $G_1$  and  $h|_{G_2-x}$  is a  $\gamma_R$ -function on  $G_2 - x$ .

(iv) Let  $f_1$  be a  $\gamma_R$ -function on  $G_1$ . Assume first that  $f_1(x) = 2$ . Define an RD-function  $g$  on  $G$  as follows:  $g(u) = f_1(u)$  when  $u \in V(G_1)$  and  $g(u) = f(u)$  when  $u \in V(G_2 - x)$ , where  $f$  is defined as in (i). The weight of  $g$  is  $\gamma_R(G_1) + (\gamma_R(G_2) + 1) - 2 = \gamma_R(G)$ . But  $g(x) = 2$  and  $x \in V^{01}(G)$ , a contradiction. Thus  $f_1(x) \neq 2$ . Now by (i) we have  $x \in V^1(G_i) \cup V^{01}(G_i)$ ,  $i = 1, 2$ , and by (iii),  $x \in V^{01}(G_j)$  for some  $j \in \{1, 2\}$ .  $\square$

**Proposition 2.** *Let  $G = (G_1 \cdot G_2)(x)$ , where  $G_1$  and  $G_2$  are connected graphs and  $\{x\} = V^{01}(G_1) \cap V^{01}(G_2)$ .*

- (i) *If  $f_i$  is a  $\gamma_R$ -function on  $G_i$  with  $f_i(x) = 1$ ,  $i = 1, 2$ , then the function  $f : V(G) \rightarrow \{0, 1, 2\}$  with  $f|_{G_i} = f_i$ ,  $i = 1, 2$ , is a  $\gamma_R$ -function on  $G$ .*
- (ii)  $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$ .
- (iii) *Let  $V_R = \{V^0, V^1, V^2, V^{01}, V^{02}, V^{12}, V^{012}\}$ . Then for any  $A \in V_R$ ,  $A(G_1) \cup A(G_2) = A(G)$ .*

**Proof.** (i) and (ii): Note that  $f$  is an RD-function on  $G$  and  $\gamma_R(G) \leq f(V(G)) = f_1(V(G_1)) + f_2(V(G_2)) - f(x) = \gamma_R(G_1) + \gamma_R(G_2) - 1$ . Now let  $h$  be any  $\gamma_R$ -function on  $G$ .

*Case 1:*  $h(x) \geq 1$ . Then  $h|_{G_i}$  is an RD-function on  $G_i$ ,  $i = 1, 2$ . If  $h(x) = 2$  then since  $x \in V^{01}(G_1) \cap V^{01}(G_2)$ ,  $h|_{G_i}$  is no  $\gamma_R$ -function on  $G_i$ ,  $i = 1, 2$ . Hence  $\gamma_R(G) \geq (\gamma_R(G_1) + 1) + (\gamma_R(G_2) + 1) - h(x) = \gamma_R(G_1) + \gamma_R(G_2)$ , a contradiction. If  $h(x) = 1$  then  $\gamma_R(G) = h(V(G)) = h(V(G_1)) + h(V(G_2)) - h(x) \geq \gamma_R(G_1) + \gamma_R(G_2) - 1$ . Thus  $h(x) = 1$ ,  $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$  and  $f$  is a  $\gamma_R$ -function on  $G$ .

*Case 2:*  $h(x) = 0$ . Then at least one of  $h|_{G_1}$  and  $h|_{G_2}$  is an RD-function, say the first. If  $h|_{G_2}$  is an RD-function on  $G_2$  then  $h(V(G)) \geq \gamma_R(G_1) + \gamma_R(G_2)$ , a contradiction. Hence  $h|_{G_2-x}$  is a  $\gamma_R$ -function on  $G_2 - x$ . But then  $\gamma_R(G) = h(V(G)) \geq \gamma_R(G_1) + \gamma_R(G_2 - x) \geq \gamma_R(G_1) + \gamma_R(G_2) - 1 \geq \gamma_R(G)$ .

Thus, (i) and (ii) hold.

(iii): Let  $g_1$  be a  $\gamma_R$ -function on  $G_1$  with  $g_1(x) = 0$ , and  $g_2$  a  $\gamma_R$ -function on  $G_2 - x$ . Then the RD-function  $g$  on  $G$  for which  $g|_{G_1} = g_1$  and  $g|_{G_2-x} = g_2$  has weight  $g_1(V(G_1)) + g_2(V(G_2 - x)) = \gamma_R(G_1) + \gamma_R(G_2 - x) = \gamma_R(G_1) + \gamma_R(G_2) - 1 = \gamma_R(G)$ . Hence by (i),  $x \in V^{01}(G) \cup V^{012}(G)$ . However, by Case 1 it follows that  $h(x) \neq 2$  for any  $\gamma_R$ -function  $h$  on  $G$ . Thus  $x \in V^{01}(G)$ .

Let  $y \in V(G_1 - x)$ ,  $l_1$  a  $\gamma_R$ -function on  $G_1$ , and  $h$  a  $\gamma_R$ -function on  $G$ . We shall prove that the following holds.

**Claim 4.1** There are a  $\gamma_R$ -function  $l$  on  $G$ , and a  $\gamma_R$ -function  $h_1$  on  $G_1$  such that  $l(y) = l_1(y)$  and  $h_1(y) = h(y)$ .

Define an RD-function  $l$  on  $G$  as  $l|_{G_1} = l_1$  and  $l|_{G_2-x} = l_2$ , where  $l_2$  is a  $\gamma_R$ -function on  $G_2 - x$ . Since  $l(V(G)) = \gamma_R(G_1) + \gamma_R(G_2 - x) = \gamma_R(G)$ ,  $l$  is a  $\gamma_R$ -function on  $G$  and  $l(y) = l_1(y)$ .

Assume now that there is no  $\gamma_R$ -function  $h_1$  on  $G_1$  with  $h_1(y) = h(y)$ . Proposition 1 implies that,  $h|_{G_1-x}$  is a  $\gamma_R$ -function on  $G_1 - x$ . But then the function  $h' : V(G_1) \rightarrow \{0, 1, 2\}$  defined as  $h'(u) = 1$  when  $u = x$  and  $h'(u) = h|_{G_1}(u)$  otherwise, is a  $\gamma_R$ -function on  $G_1$  with  $h'(y) = h|_{G_1}(y)$ , a contradiction.

By Claim 4.1 and since  $x \in V^{01}(G)$ ,  $A(G_1) = A(G) \cap V(G_1)$  for any  $A \in V_R$ . By symmetry,  $A(G_2) = A(G) \cap V(G_2)$ . Therefore  $A(G_1) \cup A(G_2) = A(G)$  for any  $A \in V_R$ .  $\square$

**Lemma 4.** Let  $G = (G_1 \cdot G_2)(x)$ , where  $G_1$  and  $G_2$  are connected graphs and  $\{x\} = V^{012}(G_1) \cap V^{01}(G_2)$ . Then  $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$  and  $x \in V^{012}(G)$ .

**Proof.** Let  $f_i$  be a  $\gamma_R$ -function on  $G_i$  with  $f_i(x) = 1$ ,  $i = 1, 2$ . Then the function  $f$  defined as  $f|_{G_i} = f_i$  is an RD-function on  $G_i$ ,  $i = 1, 2$ . Hence  $\gamma_R(G) \leq f(V(G)) = \gamma_R(G_1) + \gamma_R(G_2) - 1$ . Let now  $h$  be any  $\gamma_R$ -function on  $G$ .

**Case 1:**  $h(x) = 2$ .

Since  $x \in V^{012}(G_1) \cap V^{01}(G_2)$ ,  $h|_{G_1}$  is a  $\gamma_R$ -function on  $G_1$  and  $h|_{G_2}$  is an RD-function on  $G_2$  of weight more than  $\gamma_R(G_2)$ . Hence  $\gamma_R(G) = h(V(G)) \geq \gamma_R(G_1) + (\gamma_R(G_2) + 1) - h(x)$ . Thus  $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$ .

**Case 2:**  $h(x) = 1$ .

Then obviously  $h|_{G_1}$  and  $h|_{G_2}$  are  $\gamma_R$ -functions. Hence  $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$ .

**Case 3:**  $h(x) = 0$ .

Hence at least one of  $h|_{G_1}$  and  $h|_{G_2}$  is a  $\gamma_R$ -function. If both  $h|_{G_1}$  and  $h|_{G_2}$  are  $\gamma_R$ -functions, then  $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2)$ , a contradiction. Hence either  $h|_{G_1}$  and  $h|_{G_2-x}$  are  $\gamma_R$ -functions, or  $h|_{G_1-x}$  and  $h|_{G_2}$  are  $\gamma_R$ -functions. Since  $\{x\} = V^{012}(G_1) \cap V^{01}(G_2)$ , in both cases we have  $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$ .

Thus,  $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$  and  $x \in V^{012}(G)$ .  $\square$

### 3.2. Three lemmas for trees

**Lemma 5.** *Let  $T$  be a  $\gamma_R$ -excellent tree of order at least 2. Then  $V(T) = V^{01}(T) \cup V^{012}(T) \cup V^{02}(T)$ .*

**Proof.** Let  $x \in V(T)$ ,  $y \in N(x)$  and  $f$  a  $\gamma_R$ -function on  $T$ . Suppose  $x \in V^1(T)$ . If  $f(y) = 1$ , then the RD-function  $g$  on  $T$  defined as  $g(x) = 2$ ,  $g(y) = 0$  and  $g(u) = f(u)$  for all  $u \in V(T) - \{x, y\}$  is a  $\gamma_R$ -function on  $T$ , a contradiction. But then  $N(x) \subseteq V^0(T)$ , which is impossible.

Suppose now  $x \in V^2(T) \cup V^{12}(T)$ . Hence  $x$  is not a leaf. Choose a  $\gamma_R$ -function  $h$  on  $T$  such that (a)  $h(x) = 2$ , and (b)  $k = |epn[x, V_2^h]|$  to be as small as possible. Let  $epn[x, V_2^h] = \{y_1, y_2, \dots, y_k\}$  and denote by  $T_i$  the connected component of  $T - x$ , which contains  $y_i$ . Hence  $h(y_i) = 0$  for all  $i \leq k$ . Since  $T$  is  $\gamma_R$ -excellent, there is a  $\gamma_R$ -function  $f_k$  on  $T$  with  $f_k(y_k) \neq 0$ . Since  $x \in V^2(T) \cup V^{12}(T)$ ,  $f_k(x) \neq 0$ . If  $f_k(y_k) = 1$  then  $f_k(x) = 1$ , which easily implies  $x \in V^{012}(T)$ , a contradiction. Hence  $f_k(y_k) = f_k(x) = 2$ . Define a  $\gamma_R$ -function  $l$  on  $T$  as  $l|_{T_k} = f_k|_{T_k}$  and  $l(u) = h(u)$  for all  $u \in V(T) - V(T_k)$ . But  $|epn[x, V_2^l]| < k$ , a contradiction with the choice of  $h$ . Thus  $V^1(T) \cup V^2(T) \cup V^{12}(T)$  is empty, and the required follows.  $\square$

**Lemma 6.** *Let  $T$  be a tree and  $V^-(T)$  is not empty. Then each component of  $\langle V^-(T) \rangle$  is either  $K_1$  or  $K_2$ .*

**Proof.** Assume that  $P : x_1, x_2, x_3$  is a path in  $T$  and  $x_1, x_2, x_3 \in V^-(T)$ . Then there is a  $\gamma_R$ -function  $f_i$  on  $T$  with  $f_i(x_i) = 1$ ,  $i = 1, 2, 3$  (by Lemma 1). Denote by  $T_j$  the connected component of  $T - x_2x_3$  that contains  $x_j$ ,  $j = 1, 3$ . Then  $f_2|_{T_j}$  and  $f_j|_{T_j}$  are  $\gamma_R$ -functions on  $T_j$ ,  $j = 1, 3$ . Now define a  $\gamma_R$ -function  $h$  on  $T$  such that  $h|_{T_j} = f_j|_{T_j}$ ,  $j = 1, 3$ , and  $h(u) = f_2(u)$  when  $u \in V(T) - (V(T_1) \cup V(T_3))$ . But  $h(x_1) = h(x_2) = h(x_3) = 1$ , a contradiction.  $\square$

**Lemma 7.** *Let  $T$  be a  $\gamma_R$ -excellent tree of order at least 2.*



- (i) If  $x \in V^{012}(T)$ , then  $x$  is adjacent to exactly one vertex in  $V^-(T)$ , say  $y_1$ , and  $y_1 \in V^{012}(T)$ .
- (ii) Let  $x \in V^{02}(T)$ . If  $\deg(x) \geq 3$  then  $x$  has exactly 2 neighbors in  $V^-(T)$ . If  $\deg(x) = 2$  then either  $N_T(x) \subseteq V^{012}(T)$  or there is a path  $u, x, y, z$  in  $T$  such that  $u, z \in V^{01}(T)$ ,  $y \in V^{02}(T)$  and  $\deg(y) = 2$ .
- (iii)  $V^{01}(T)$  is either empty or independent.

**Proof.** Let  $x \in V^{012}(T) \cup V^{02}(T)$  and  $N(x) = \{y_1, y_2, \dots, y_r\}$ . If  $x$  is a leaf, then clearly  $x, y_1 \in V^{012}(T)$ . So, let  $r \geq 2$ . Denote by  $T_i$  the connected component of  $T - x$  which contains  $y_i$ ,  $i \geq 1$ . Choose a  $\gamma_R$ -function  $h$  on  $T$  such that (a)  $h(x) = 2$ , and (b)  $k = |\text{epn}[x, V_2^h]|$  to be as small as possible. Let without loss of generality  $\text{epn}[x, V_2^h] = \{y_1, y_2, \dots, y_k\}$ . By the definition of  $h$  it immediately follows that (c)  $h|_{T_j}$  is a  $\gamma_R$ -function on  $T_j$  for all  $j \geq k + 1$ , (d) for each  $i \in \{1, \dots, k\}$ ,  $h|_{T_i}$  is no RD-function on  $T_i$ , and (e)  $h|_{T_i - y_i}$  is a  $\gamma_R$ -function on  $T_i - y_i$ ,  $i \in \{1, \dots, k\}$ . Hence  $\gamma_R(T_i) \leq \gamma_R(T_i - y_i) + 1$  for all  $i \in \{1, \dots, k\}$ . Assume that the equality does not hold for some  $i \leq k$ . Define an RD-function  $h_i$  on  $T$  as follows:  $h_i(u) = h(u)$  when  $u \in V(T) - V(T_i)$  and  $h_i|_{T_i} = h'_i$ , where  $h'_i$  is some  $\gamma_R$ -function on  $T_i$ . But then either  $h_i$  has weight less than  $\gamma_R(T)$  or  $h_i$  is a  $\gamma_R$ -function on  $T$  with  $\text{epn}[x, V_2^{h_i}] = \text{epn}[x, V_2^h] - \{y_i\}$ . In both cases we have a contradiction. Thus  $\gamma_R(T_i) = \gamma_R(T_i - y_i) + 1$  for all  $i \in \{1, \dots, k\}$ . Therefore  $\gamma_R(T) = h(V(T)) = 2 + \sum_{i=1}^k (\gamma_R(T_i) - 1) + \sum_{j=k+1}^r \gamma_R(T_j) = 2 - k + \sum_{i=1}^r \gamma_R(T_i) = 2 - k + \gamma_R(T - x)$ . Thus  $\gamma_R(T) = 2 - k + \gamma_R(T - x)$ .

(i) Since  $\gamma_R(T - x) + 1 = \gamma_R(T)$ ,  $k = 1$ . We already know that  $h|_{T_j}$  is a  $\gamma_R$ -function on  $T_j$ ,  $j \geq 2$ . Assume that  $y_j \in V^{012}(T) \cup V^{01}(T)$  for some  $j \geq 2$ . Then there is a  $\gamma_R$ -function  $l$  on  $T$  with  $l(y_j) = 1$ . Clearly  $l|_{T_j}$  is a  $\gamma_R$ -function on  $T_j$ . Now define a  $\gamma_R$ -function  $h''$  on  $T$  as follows:  $h''(u) = h(u)$  when  $u \in V(T) - V(T_j)$  and  $h''|_{T_j} = l|_{T_j}$ . But then  $h''(x) = 2, h''(y_j) = 1$  and  $xy_j \in E(G)$ , which is impossible. Thus,  $y_2, y_3, \dots, y_r \in V^{02}(T)$ . Define now  $\gamma_R$ -functions  $h_1$  and  $h_2$  on  $T$  as follows:  $h_1(u) = h_2(u) = h(u)$  for all  $u \in V(T) - \{x, y_1\}$ ,  $h_1(x) = h_1(y_1) = 1, h_2(x) = 0$  and  $h_2(y_1) = 2$ . Thus  $y_1 \in V^{012}(T)$ .

(ii) Since  $\gamma_R(T - x) = \gamma_R(T)$ ,  $k = 2$ . Recall that  $h|_{T_j}$  is a  $\gamma_R$ -function on  $T_j$ ,  $j \geq 3$ , and  $\gamma_R(T_i - y_i) = \gamma_R(T_i) - 1$  for  $i = 1, 2$ . Hence there is a  $\gamma_R$ -function  $f_i$  on  $T_i$  with  $f_i(y_i) = 1, i = 1, 2$ .

Suppose first that  $r \geq 3$ . As in the proof of (i), we obtain  $y_3, \dots, y_r \in V^{02}(T)$ . Hence there is a  $\gamma_R$ -function  $g$  on  $T$  such that  $g(y_3) = 2$ . By the choice of  $h$ ,  $g(x) = 0$ . Then  $g|_{T_i}$  is a  $\gamma_R$ -function on  $T_i$ ,  $i = 1, 2$ . Define now a  $\gamma_R$ -function  $g'$  on  $T$  as  $g'|_{T_i} = f_i, i = 1, 2$ , and  $g'(u) = g(u)$  when  $u \in V(T) - (V(T_1) \cup V(T_2))$ . Since  $g'(y_1) = g'(y_2) = 1, y_1, y_2 \in V^-(T)$ .

So, let  $r = 2$  and let  $f$  be a  $\gamma_R$ -function on  $T$  with  $f(x) = 0$ . Then there is  $y_s$  such that  $f(y_s) = 2$ , say  $s = 2$ . Hence  $y_2 \in V^{02}(T) \cup V^{012}(T)$  and  $f|_{T_1}$  is a  $\gamma_R$ -function on  $T_1$ . Define the  $\gamma_R$ -function  $l$  on  $T$  as  $l|_{T_1} = f_1$  and  $l(u) = f(u)$  when  $u \in V(T) - V(T_1)$ . Since  $l(y_1) = 1, y_1 \in V^{01}(T) \cup V^{012}(T)$ .

Assume first that  $y_1 \in V^{012}(T)$ . Then there is a  $\gamma_R$ -function  $f'$  on  $T$  with  $f'(y_1) = 2$ . Since  $x \in V^{02}(T)$  and  $\deg(x) = 2$ ,  $f'(x) = 0$ . Hence  $f'|_{T_2}$  is a  $\gamma_R$ -function on  $T_2$ . But then we can choose  $f'$  so that  $f'|_{T_2} = f_2$ . Thus  $y_2 \in V^{012}(T)$ .

So let  $y_1 \in V^{01}(T)$  and suppose  $y_2 \in V^{012}(T)$ . Then there is a  $\gamma_R$ -function  $f''$  on  $T$  with  $f''(y_2) = 1$ . Since  $x \in V^{02}(T)$ ,  $f''(x) = 0$  and  $f''(y_1) = 2$ , a contradiction. Thus, if  $y_1 \in V^{01}(T)$  then  $y_2 \in V^{02}(T)$ .

Finally, let us consider a path  $y_1, x, y_2, z$  in  $T$ , where  $y_1 \in V^{01}(T)$ ,  $x, y_2 \in V^{02}(T)$  and  $\deg(x) = 2$ . Assume to the contrary that  $N(y_2) = \{z_1, z_2, \dots, z_s = x\}$  with  $s \geq 3$ . Denote by  $T_{z_p}$  the connected component of  $T - y_2$  that contains  $z_p$ ,  $p = 1, 2, \dots, s$ . By applying results proved above for  $x \in V^{02}(T)$  with  $\deg(x) \geq 3$  to  $y_2$ , we obtain that (a)  $y_2$  has exactly 2 neighbors in  $V^-(T)$ , say, without loss of generality,  $z_1, z_2 \in V^-(T)$ , and (b)  $\gamma_R(T_{z_i} - z_i) = \gamma_R(T_{z_i}) - 1$ , where  $i = 1, 2$ . Recall now that:  $h(x) = 2$ ,  $h|_{T_i}$  is no RD-function on  $T_i$  and  $h|_{T_i - y_i}$  is a  $\gamma_R$ -function on  $T_i - y_i$ ,  $i = 1, 2$ . Hence  $h(y_1) = h(y_2) = 0$  and  $h|_{T_{z_j}}$  is a  $\gamma_R$ -function on  $T_{z_j}$ ,  $j \leq s - 1$ . Since  $\gamma_R(T_{z_i} - z_i) = \gamma_R(T_{z_i}) - 1$ ,  $i = 1, 2$ , additionally we can choose  $h$  so that  $h(z_1) = h(z_2) = 1$ . But then the function  $h_1$  defined as  $h_1(u) = h(u)$  when  $u \in V(T) - \{y_1, x, y_2, z_1, z_2\}$  and  $h_1(y_1) = h_1(x) = 1$ ,  $h_1(y_2) = 2$ ,  $h_1(z_1) = h_1(z_2) = 0$  is a  $\gamma_R$ -function on  $T$ . Now  $h_1(x) = 1$ ,  $h_1(y_2) = 2$  and  $xy_2 \in E(G)$  lead to a contradiction. Thus,  $N(y_2) = \{x, z\}$ . Suppose  $z \notin V^{01}(T)$ . Then there is a  $\gamma_R$ -function  $h_4$  on  $T$  with  $h_4(z) = 2$ . If  $h_4(y_2) = 2$ , then  $h_4(x) = 0$  and the function  $h_5$  on  $T$  defined as  $h_5(x) = h_5(y_2) = 1$  and  $h_5(u) = h_4(u)$  otherwise, is a  $\gamma_R$ -function on  $T$ , a contradiction. Hence  $h_4(y_2) = 0$  and since  $y_1 \in V^{01}(T)$ ,  $h_4(x) = 2$  and  $h_4(y_1) = 0$ . But then the function  $h_6$  on  $T$  defined as  $h_6(x) = h_6(y_1) = 1$  and  $h_6(u) = h_4(u)$  otherwise, is a  $\gamma_R$ -function on  $T$ , a contradiction. Therefore  $z \in V^{01}(T)$ , and we are done.

(iii) Assume that  $u_1, u_2 \in V^{01}(T)$  are adjacent. Let  $T_{u_i}$  be the component of  $T - u_1u_2$  that contains  $u_i$ ,  $i = 1, 2$ . Let  $g_i$  be a  $\gamma_R$ -function on  $T$  with  $g_i(u_i) = 1$ ,  $i = 1, 2$ . Hence  $g_i(T_{u_j})$  is a  $\gamma_R$ -function on  $T_{u_j}$ ,  $i, j = 1, 2$ . Thus  $\gamma_R(T) = \gamma_R(T_{u_1}) + \gamma_R(T_{u_2})$ . Define now a  $\gamma_R$ -function  $g_3$  on  $T$  as  $g_3|_{T_i} = g_i|_{T_i}$ ,  $i = 1, 2$ . But then a function  $g_4$  defined as  $g_4(u) = g_3(u)$  when  $u \in V(T) - \{u_1, u_2\}$ ,  $g_4(u_1) = 2$  and  $g_4(u_2) = 0$  is a  $\gamma_R$ -function on  $T$ , contradicting  $u_1 \in V^{01}(T)$ . Thus  $V^{01}(T)$  is independent.  $\square$

## 4. Proof of the main result

**Proof of Theorem 1.** Let  $T$  be a  $\gamma_R$ -excellent tree. First, we shall prove the following statement.

( $\mathcal{P}_2$ ) There is a labeling  $L : V(T) \rightarrow \{A, B, C, D\}$  such that (a)  $L_A(T)$  is either empty or independent, (b) each component of  $\langle L_B(T) \rangle$  and  $\langle L_D(T) \rangle$  is isomorphic to  $K_2$ , (c) each element of  $L_B(T)$  has degree 2 and it is adjacent to exactly one vertex in  $L_A(T)$ , (d) each vertex  $v$  in  $L_C(T)$  has exactly 2 neighbors in  $L_A(T) \cup L_D(T)$ , and if  $\deg(v) = 2$  then both neighbors of  $v$  are in  $L_D(T)$ .

By Lemma 5 we know that  $V(T) = V^{01}(T) \cup V^{012}(T) \cup V^{02}(T)$ . Define a labeling  $L : V(T) \rightarrow \{A, B, C, D\}$  by  $L_A(T) = V^{01}(T)$ ,  $L_D(T) = V^{012}(T)$ ,  $L_B(T) = \{x \in$

$V^{02}(T) \mid \deg(x) = 2 \text{ and } |N(x) \cap V^{02}(T)| = 1\}$ , and  $L_C(T) = V^{02}(T) - L_B(T)$ . The validity of  $(\mathcal{P}_2)$  immediately follows by Lemma 7.

Denote by  $\mathcal{T}_1$  the family of all labeled, as in  $(\mathcal{P}_2)$ , trees  $T$ . We shall show that if  $(T, L) \in \mathcal{T}_1$  then  $(T, L) \in \mathcal{T}$ .

(I) *Proof of  $(T, L) \in \mathcal{T}_1 \Rightarrow (T, L) \in \mathcal{T}$ .*

Let  $(T, L) \in \mathcal{T}_1$ . The following claim is immediate.

**Claim 1.1**

- (i) Each leaf of  $T$  is in  $L_A(T) \cup L_D(T)$ .
- (ii) If  $v$  is a support vertex of  $T$ , then  $v$  is adjacent to at most 2 leaves.
- (iii) If  $u_1$  and  $u_2$  are leaves adjacent to the same support vertex, then  $u_1, u_2 \in L_A(T)$ .

We now proceed by induction on  $k = |L_B \cup L_C|$ . The base case,  $k \leq 2$ , is an immediate consequence of the following easy claim, the proof of which is omitted.

**Claim 1.2** (see Fig.1)

- (i) If  $k = 0$  then  $(T, L) = (H_1, I^1)$ .
- (ii) If  $k = 1$  then  $(T, L)$  is obtained from  $(H_1, I_1)$  by operation  $O_2$ , i.e.  $(T, L) = (H_{11}, I^{11})$ .
- (iii) If  $k = 2$  then either  $(T, L)$  is  $(H_r, I^r)$  with  $r \in \{2, 3, 4, 5\}$ , or  $(T, L)$  is obtained from  $(H_{11}, I^{11})$  by operation  $O_1$  or by operation  $O_2$  (see the graphs  $(H_s, I^s)$  where  $s \in \{6, 7, 8, 9, 10\}$ ).

Let  $k \geq 3$  and suppose that each tree  $(H, L') \in \mathcal{T}_1$  with  $|L'_B(H) \cup L'_C(H)| < k$  is in  $\mathcal{T}$ . Let now  $(T, L) \in \mathcal{T}_1$  and  $k = |L_B(T) \cup L_C(T)|$ . To prove the required result, it suffices to show that  $T$  has a subtree, say  $U$ , such that  $(U, L|_U) \in \mathcal{T}_1$ , and  $(T, L)$  is obtained from  $(U, L|_U)$  by one of operations  $O_1, O_2, O_3$  and  $O_4$ . Consider any diametral path  $P : x_1, x_2, \dots, x_n$  in  $T$ . Clearly  $x_1$  is a leaf. Denote by  $x_i^1, x_i^2, \dots$  all neighbors of  $x_i$ , which do not belong to  $P$ ,  $2 \leq i \leq n - 1$ .

**Case 1:**  $sta(x_1) = A$  and  $sta(x_2) = B$ .

Then  $\deg(x_1) = 1$ ,  $\deg(x_2) = \deg(x_3) = 2$ ,  $sta(x_3) = B$  and  $sta(x_4) = A$ . Thus  $T$  is obtained from  $T - \{x_1, x_2, x_3\} \in \mathcal{T}_1$  and a copy of  $H_2$  by operation  $O_3$  (via  $x_4$ ).  $\square$

**Case 2:**  $sta(x_1) = A$  and  $sta(x_2) = C$ .

Hence  $\deg(x_2) \geq 3$ . By the choice of  $P$ ,  $\deg(x_2) = 3$ ,  $x_2^1$  is a leaf,  $sta(x_2^1) = A$ , and  $sta(x_3) = C$ . If  $\deg(x_3) \geq 4$  then  $T$  is obtained from  $T - \{x_2^1, x_1, x_2\} \in \mathcal{T}_1$  and a copy of  $F_1$  by operation  $O_1$ . So, let  $\deg(x_3) = 3$ . Assume first that  $sta(x_4) = A$ . Then either  $x_3^1$  is a leaf of status  $A$  or  $x_3^1$  is a support vertex,  $\deg(x_3^1) = 2$ , and both  $x_3^1$  and its leaf-neighbor have status  $D$ . Thus,  $T$  is obtained from  $T - (N[x_2] \cup N[x_3^1]) \in \mathcal{T}_1$  and a copy of  $H_3$  or  $H_4$ , respectively, by operation  $O_3$  (via  $x_4$ ). Finally let  $sta(x_4) = D$ . By the choice of  $P$ , either  $x_3^1$  is a leaf of status  $A$  and then  $T$  is obtained from

$T - (N[x_2] \cup \{x_3^1\}) \in \mathcal{T}_1$  and a copy of  $H_3$  by operation  $O_4$ , or  $x_3^1$  is a support vertex of degree 2 and both  $x_3^1$  and its leaf-neighbor have status  $D$ , and then  $T$  is obtained from  $T - \{x_2^1, x_1, x_2\} \in \mathcal{T}_1$  and a copy of  $F_1$  by operation  $O_1$ .  $\square$

In what follows, let  $sta(x_1) = D$ . Hence  $deg(x_2) = 2$ ,  $sta(x_2) = D$  and  $sta(x_3) = C$ . If  $deg(x_3) = 2$  then  $T$  is obtained from  $T - N[x_2] \in \mathcal{T}_1$  and a copy of  $F_4$  by operation  $O_2$ .

**Case 3:**  $deg(x_3) = 3$  and  $sta(x_4) \in \{A, D\}$ .

In this case  $sta(x_3^1) = C$ ,  $x_3^1$  is a support vertex,  $deg(x_3^1) = 3$ , and the leaf neighbors of  $x_3^1$  have status  $A$ . Now (a) if  $sta(x_4) = A$  then  $T$  is obtained from  $T - (N[x_2] \cup N[x_3^1]) \in \mathcal{T}_1$  and a copy of  $H_4$  by operation  $O_3$  (via  $x_4$ ), and (b) if  $sta(x_4) = D$  then  $T$  is obtained from  $T - (N[x_2] \cup N[x_3^1]) \in \mathcal{T}_1$  and a copy of  $H_4$  by operation  $O_4$  (via  $x_4$ ).  $\square$

**Case 4:**  $deg(x_3) = 3$ ,  $sta(x_4) = C$  and  $sta(x_3^1) = A$ .

Hence  $x_3^1$  is a leaf. If  $deg(x_4) = 3$  and  $sta(x_5) = sta(x_4^1) = D$ , or  $deg(x_4) \geq 4$ , then  $T$  is obtained from  $T - \{x_1, x_2, x_3, x_3^1\} \in \mathcal{T}_1$  and a copy of  $F_2$  by operation  $O_1$ . So, let  $deg(x_4) = 3$  and the status of at least one of  $x_5$  and  $x_4^1$  is  $A$ . Assume first that  $sta(x_4^1) = A$ . Hence  $x_4^1$  is a leaf (by the choice of  $P$ ). If  $sta(x_5) = A$  then  $T$  is obtained from a copy of  $H_4$  and a tree in  $\mathcal{T}_1$  by operation  $O_3$  (via  $x_5$ ). If  $sta(x_5) = D$  then  $T$  is obtained from a copy of  $H_4$  and a tree in  $\mathcal{T}_1$  by operation  $O_4$  (via  $x_5$ ). Second, let  $sta(x_4^1) = D$ . Hence  $sta(x_5) = A$ ,  $deg(x_4^1) = 2$  and the status of the leaf-neighbor of  $x_4^1$  is  $D$ . But then  $T$  is obtained from a copy of  $H_5$  and a tree in  $\mathcal{T}_1$  by operation  $O_3$  (via  $x_5$ ).  $\square$

**Case 5:**  $deg(x_3) = 3$ ,  $sta(x_4) = C$  and  $sta(x_3^1) = D$ .

Hence  $deg(x_3^1) = 2$ ,  $x_3^1$  is a support vertex, and the leaf-neighbor of  $x_3^1$  has status  $D$ . If  $deg(x_4) \geq 4$  or  $sta(x_5) = sta(x_4^1) = D$ , then  $T$  is obtained from  $T - N[\{x_2, x_3^1\}] \in \mathcal{T}_1$  and a copy of  $F_3$  by operation  $O_1$ . So, let  $deg(x_4) = 3$  and at least one of  $x_5$  and  $x_4^1$  has status  $A$ . Assume  $sta(x_4^1) = A$ . Hence  $x_4^1$  is a leaf. If  $sta(x_5) = A$  then  $T$  is obtained from  $T - N[\{x_2, x_3^1, x_4^1\}] \in \mathcal{T}_1$  and a copy of  $H_6$  by operation  $O_3$  (via  $x_5$ ). If  $sta(x_5) = D$  then  $T$  is obtained from  $T - N[\{x_2, x_3^1, x_4^1\}] \in \mathcal{T}_1$  and a copy of  $H_6$  by operation  $O_4$  (via  $x_5$ ). Now let  $sta(x_4^1) = D$ . Hence  $sta(x_5) = A$  and then  $T$  is obtained from a copy of  $H_7$  and a tree in  $\mathcal{T}_1$  by operation  $O_3$  (via  $x_5$ ).  $\square$

**Case 6:**  $deg(x_3) \geq 4$ .

Hence  $x_3$  has a neighbor, say  $y$ , such that  $y \neq x_4$  and  $sta(y) = C$ . By the choice of  $P$ ,  $y$  is a support vertex which is adjacent to exactly 2 leaves, say  $z_1$  and  $z_2$ , and  $sta(z_1) = sta(z_2) = A$ . But then  $T$  is obtained from  $T - \{y, z_1, z_2\} \in \mathcal{T}_1$  and a copy of  $F_1$  by operation  $O_1$ .

By Claim 2.1, there are no other possibilities.  $\square$

(II)  $(T, S) \in \mathcal{T} \Rightarrow (T, S) \in \mathcal{T}_1$ . Obvious.  $\square$

It remains the following.

(III) *Proof of  $(T, S) \in \mathcal{T} \Rightarrow T$  is  $\gamma_R$ -excellent and  $(P_1)$  holds.*

Let  $(T, S) \in \mathcal{T}$ . We know that  $(T, S) \in \mathcal{T}_1$ . We now proceed by induction on  $k = |S_B \cup S_C|$ . First let  $k \leq 2$ . By Claim 1.2,  $T \in \mathcal{H} = \{H_1, \dots, H_{11}\}$ . It is easy to see that all elements of  $\mathcal{H}$  are  $\gamma_R$ -excellent graphs and  $(\mathcal{P}_1)$  holds for each  $T \in \mathcal{H}$ . Let  $k \geq 3$  and suppose that if  $(H, S') \in \mathcal{T}$  and  $|S'_B(H) \cup S'_C(H)| < k$ , then  $H$  is  $\gamma_R$ -excellent and  $(\mathcal{P}_1)$  holds with  $(T, S)$  replaced by  $(H, S')$ . So, let  $(T, S) \in \mathcal{T}$  and  $k = |S_B(T) \cup S_C(T)|$ . Then there is a  $\mathcal{T}$ -sequence  $\tau : (T^1, S^1), \dots, (T^{j-1}, S^{j-1}), (T, S)$ . By induction hypothesis,  $T^{j-1}$  is  $\gamma_R$ -excellent and  $(\mathcal{P}_1)$  holds with  $(T, S)$  replaced by  $(T^{j-1}, S^{j-1})$ . We consider four possibilities depending on whether  $T$  is obtained from  $T^{j-1}$  by operation  $O_1, O_2, O_3$  or  $O_4$ .

**Case 7:**  $T$  is obtained from  $T^{j-1} \in \mathcal{T}$  and  $F_a$  by operation  $O_1$ ,  $a \in \{1, 2, 3\}$ . Hence  $T$  is obtained after adding the edge  $ux$  to the union of  $T^{j-1}$  and  $F_a$ , where  $sta_{T^{j-1}}(u) = sta_{F_a}(x) = C$  (see Fig. 2). First note that  $\gamma_R(F_a) = a + 1$ , and  $F_2$  and  $F_3$  are  $\gamma_R$ -excellent graphs. Since  $\gamma_R(F_a - x) = \gamma_R(F_a)$  and  $u \in V^{02}(T^{j-1})$ , Lemma 2 implies  $\gamma_R(T) = \gamma_R(T^{j-1}) + \gamma_R(F_a)$ . Hence for any  $\gamma_R$ -function  $g$  on  $T$ , the weight of  $g|_{F_a}$  is not more than  $\gamma_R(F_a)$ . But then  $g(x) \neq 1$  and either  $g|_{F_a}$  is a  $\gamma_R$ -function on  $F_a$  or  $g|_{F_a - x}$  is a  $\gamma_R$ -function on  $F_a - x$ . By inspection of all  $\gamma_R$ -functions on  $F_a$  and  $F_a - x$ , we obtain

$$(\alpha_1) \quad S_A(T) \cap V(F_a) = V^{01}(T) \cap V(F_a), \quad S_B(T) \cap V(F_a) = \emptyset, \quad \{x\} = S_C(T) \cap V(F_a) = V^{02}(T) \cap V(F_a), \quad \text{and} \quad S_D(T) \cap V(F_a) = V^{012}(T) \cap V(F_a).$$

By the definition of operation  $O_1$  it immediately follows

$$(\alpha_2) \quad S_X(T) \cap V(T^{j-1}) = S_X^{j-1}(T^{j-1}), \quad \text{for all } X \in \{A, B, C, D\}.$$

Let  $f_1$  be a  $\gamma_R$ -function on  $T^{j-1}$  and  $f_2$  a  $\gamma_R$ -function on  $F_a$ . Then the RD-function  $f$  on  $T$  defined as  $f|_{T^{j-1}} = f_1$  and  $f|_{F_a} = f_2$  is a  $\gamma_R$ -function on  $T$ . Since  $f_1$  was chosen arbitrarily, we have

$$(\alpha_3) \quad V^{01}(T^{j-1}) \subseteq V^{01}(T) \cup V^{012}(T), \quad V^{02}(T^{j-1}) \subseteq V^{02}(T) \cup V^{012}(T), \quad \text{and} \quad V^{012}(T^{j-1}) \subseteq V^{012}(T).$$

By  $(\alpha_1)$  and  $(\alpha_3)$  we conclude that  $T$  is  $\gamma_R$ -excellent.

Now we shall prove that

$$(\alpha_4) \quad V^{01}(T) \cap V(T^{j-1}) = V^{01}(T^{j-1}), \quad V^{02}(T) \cap V(T^{j-1}) = V^{02}(T^{j-1}), \quad \text{and} \quad V^{012}(T) \cap V(T^{j-1}) = V^{012}(T^{j-1}).$$

Assume there is a vertex  $z \in V^{02}(T^{j-1}) \cap V^{012}(T)$ . By Lemma 7,  $z$  is adjacent to at most 2 elements of  $V^-(T^{j-1})$ . Now by  $(\alpha_3)$  and since  $\Delta(\langle V^-(T) \rangle) \leq 1$  (by Lemma 6),  $z$  is adjacent to exactly one element of  $V^-(T^{j-1})$ . But then Lemma 7 implies that there is a path  $z_1, z, z_2, z_3$  in  $T^{j-1}$  such that  $deg_{T^{j-1}}(z) = deg_{T^{j-1}}(z_2) = 2$ ,  $z, z_2 \in V^{02}(T^{j-1})$  and  $z_1, z_3 \in V^{01}(T^{j-1})$ . Since  $(\mathcal{P}_1)$  is true for  $T^{j-1}$ ,  $sta_{T^{j-1}}(z_1) = sta_{T^{j-1}}(z_3) = A$ , and  $sta_{T^{j-1}}(z) = sta_{T^{j-1}}(z_2) = B$ . Clearly, at least one of  $z_1$  and  $z_3$  is a cut-vertex. Denote by  $Q$  the graph  $\langle \{z_1, z, z_2, z_3\} \rangle$  and let the vertices of  $Q$

are labeled as in  $T^{j-1}$ . Let  $U_s$  be the connected component of  $T - \{z, z_2\}$ , which contains  $z_s$ ,  $s = 1, 3$ .

Assume first that  $T^1$  is a subtree of  $U \in \{U_1, U_3\}$ . Then there is  $i$  such that  $T^i$  is obtained from  $T^{i-1}$  and  $Q$  by operation  $O_3$ . Hence  $T^{i-1}$  is a subtree of  $U$ . Recall that if  $y \in V(T^r)$  and  $r \leq s \leq j-1$ , then  $sta_{T^r}(y) = sta_{T^s}(y)$ . Using this fact, we can choose  $\tau$  so, that  $T^{i-1} = U$ . Therefore  $U$  is in  $\mathcal{T}$ . Suppose that neither  $z_1$  nor  $z_3$  is a leaf of  $T^{j-1}$ . Define  $R^s = T^{i+s} - (V(T^{i-1}) \cup \{z, z_2\})$ ,  $s = 1, 2, \dots, j-1-i$ . Since clearly  $R^1$  is in  $\{H_2, H_3, \dots, H_7\}$ , the sequence  $R^1, R^2, \dots, R^{j-1-i}$  is a  $\mathcal{T}$ -sequence of  $U'$ , where  $\{U, U'\} = \{U_1, U_2\}$ . Thus, both  $U_1$  and  $U_3$  are in  $\mathcal{T}$ , and  $sta_{U_1}(z_1) = A$ . By the induction hypothesis,  $z_1 \in V^{01}(U_1)$ .

Suppose now that  $u \in V(U_3)$ . Consider the sequence of trees  $U_3, U_4, U_5$ , where  $U_4$  is obtained from  $U_3$  and  $Q$  by operation  $O_3$  (via  $z_3$ ), and  $U_5$  is obtained from  $U_4$  and  $F_a$  by operation  $O_1$ . Clearly  $U_5$  is in  $\mathcal{T}$ ,  $sta_{U_5}(z_1) = A$  and by the induction hypothesis,  $z_1 \in V^{01}(U_5)$ . Since  $T = (U_5 \cdot U_1)(z_1)$  and  $\{z_1\} = V^{01}(U_1) \cap V^{01}(U_5)$ , by Proposition 2 we have  $z_1 \in V^{01}(T)$ . But then Lemma 7 implies  $z_2 \in V^{02}(T)$ , a contradiction.

Now let  $u \in V(U_1)$ . Denote by  $U_2$  the graph obtained from  $U_1$  and  $F_a$  by operation  $O_3$ . Then  $U_2$  is in  $\mathcal{T}$ ,  $sta_{U_2}(z_1) = A$ , and by induction hypothesis,  $z_1 \in V^{01}(U_2)$ . Define also the graph  $U_6$  as obtained from  $U_3$  and  $Q$  by operation  $O_3$ , i.e.  $U_6 = (U_3 \cdot Q)(z_3)$ . Then  $U_6$  is in  $\mathcal{T}$ ,  $sta_{U_6}(z_1) = A$  and by induction hypothesis,  $z_1 \in V^{01}(U_6)$ . Now by Proposition 2,  $z_1 \in V^{01}(T)$ , which leads to  $z_2 \in V^{02}(T)$  (by Lemma 7), a contradiction.

Thus, in all cases we have a contradiction. Therefore  $V^{02}(T^{j-1}) \subseteq V^{02}(T)$  when both  $z_1$  and  $z_3$  are cut-vertices. If  $z_1$  or  $z_3$  is a leaf, then, by similar arguments, we can obtain the same result.

Let now  $T^1 \equiv Q$ . Then  $T^2$  is obtained from  $T^1$  and  $H_k$  by operation  $O_3$ . Consider the sequence of trees  $\tau_1 : T_1^1 = H_k, T^2, T^3, \dots, T^{j-1}$ . Clearly  $\tau_1$  is a  $\mathcal{T}$ -sequence of  $T^{j-1}$  and  $T_1^1 \neq Q$ . Therefore we are in the previous case. Thus,  $V^{02}(T^{j-1}) = V(T^{j-1}) \cap V^{02}(T)$ .

Assume now that there is a vertex  $w \in V^{01}(T^{j-1}) \cap V^{012}(T)$ . By Lemma 7(i)  $w$  has a neighbor in  $T$ , say  $w'$ , such that  $w' \in V^{012}(T)$ . Since  $w \neq u$ ,  $w' \in V(T^{j-1})$ . But all neighbors of  $w$  in  $T^{j-1}$  are in  $V^{02}(T^{j-1})$  (by Lemma 7 applied to  $T^{j-1}$  and  $w$ ). Since  $V^{02}(T^{j-1}) = V(T^{j-1}) \cap V^{02}(T)$ , we obtain a contradiction.

Thus  $(\alpha_4)$  is true.

Now we are prepared to prove that  $(\mathcal{P}_1)$  is valid. Using, in the chain of equalities below, consecutively  $(\alpha_2)$ , the induction hypothesis,  $(\alpha_1)$  and  $(\alpha_4)$ , we obtain

$$S_A(T) = S_A^{j-1}(T^{j-1}) \cup (S_A(T) \cap V(F_a)) = V^{01}(T^{j-1}) \cup (V^{01}(T) \cap V(F_a)) = V^{01}(T),$$

and similarly,  $S_D(T) = V^{012}(T)$ . Since  $u \notin S_B(T)$  and  $S_B(T) \cap V(F_a) = \emptyset$ , we have

$$\begin{aligned} S_B(T) &= S_B(T) \cap V(T^{j-1}) \stackrel{(\alpha_2)}{=} S_B^{j-1}(T^{j-1}) \\ &= \{t \in V^{02}(T^{j-1}) \mid \deg_{T^{j-1}}(t) = 2 \text{ and } |N_{T^{j-1}}(t) \cap V^{02}(T^{j-1})| = 1\} \\ &\stackrel{(\alpha_4)}{=} \{t \in V^{02}(T) \cap V(T^{j-1}) \mid \deg_T(t) = 2 \text{ and } |N_T(t) \cap V^{02}(T)| = 1\} \\ &= \{t \in V^{02}(T) \mid \deg_T(t) = 2 \text{ and } |N_T(t) \cap V^{02}(T)| = 1\}. \end{aligned}$$

The last equality follows from  $\deg_T(x) > 2$  and  $\{x\} = V^{02}(T) \cap V(F_a)$  (see  $(\alpha_1)$ ). Now the equality  $S_C(T) = V^{02}(T) - S_B(T)$  is obvious. Thus,  $(\mathcal{P}_1)$  holds and we are done.

**Case 8:**  $T$  is obtained from  $T^{j-1} \in \mathcal{T}$  by operation  $O_2$ .

Clearly,  $\gamma_R(F_4) = \gamma_R(F_4 - x) = 2$ . By Lemma 2,  $\gamma_R(T) = \gamma_R(T^{j-1}) + \gamma_R(H_4)$ . Let  $f_1$  be a  $\gamma_R$ -function on  $T^{j-1}$  and  $f_2$  a  $\gamma_R$ -function on  $F_4$ . Then the function  $f$  defined as  $f|_{T^{j-1}} = f_1$  and  $f|_{F_4} = f_2$  is a  $\gamma_R$ -function on  $T$ . Therefore  $V^{012}(T^{j-1}) \subseteq V^{012}(T)$ ,  $V^{01}(T^{j-1}) \subseteq V^{01}(T) \cup V^{012}(T)$ , and  $V^{02}(T^{j-1}) \subseteq V^{02}(T) \cup V^{012}(T)$ .

Assume that there is  $y \in V^{0s}(T^{j-1}) \cap V^{012}(T)$ ,  $s \in \{1, 2\}$ , and let  $f'$  be a  $\gamma_R$ -function on  $T$  with  $f'(y) = r \notin \{0, s\}$ . If  $f'|_{T^{j-1}}$  is an RD-function on  $T^{j-1}$ , then  $f'|_{T^{j-1}}(V(T^{j-1})) > \gamma_R(T^{j-1})$  and  $f'|_{F_4}(V(F_4)) \geq 2$ . This leads to  $f'(V(T)) > \gamma_R(T)$ , a contradiction. Hence  $f'|_{T^{j-1}}$  is no RD-function on  $T^{j-1}$  and  $f'|_{T^{j-1}-u}$  is a  $\gamma_R$ -function on  $T^{j-1} - u$ . Define now an RD-function  $f''$  on  $T^{j-1}$  as  $f''|_{T^{j-1}-u} = f'|_{T^{j-1}-u}$  and  $f''(u) = 1$ . Since  $u \in V^-(T^{j-1})$ ,  $f''$  is a  $\gamma_R$ -function on  $T^{j-1}$  with  $f''(y) = r \notin \{0, s\}$ , a contradiction with  $y \in V^{0s}(T^{j-1})$ . Thus

$$\begin{aligned} (\alpha_5) \quad V^{012}(T^{j-1}) &= V^{012}(T) \cap V(T^{j-1}), \quad V^{01}(T^{j-1}) = V^{01}(T) \cap V(T^{j-1}), \text{ and} \\ V^{02}(T^{j-1}) &= V^{02}(T) \cap V(T^{j-1}). \end{aligned}$$

Let  $x, x_1, x_2$  be a path in  $F_4$ ,  $h_1$  a  $\gamma_R$ -function on  $T^{j-1}$  with  $h_1(u) = 2$ , and  $h_2$  a  $\gamma_R$ -function on  $T^{j-1} - u$ . Define  $\gamma_R$ -functions  $g_1, \dots, g_4$  on  $T$  as follows:

- $g_1|_{T^{j-1}} = h_1$ ,  $g_1(x) = g_1(x_2) = 0$  and  $g_1(x_1) = 2$ ;
- $g_2|_{T^{j-1}} = h_1$ ,  $g_2(x) = 0$  and  $g_2(x_1) = g_2(x_2) = 1$ ;
- $g_3|_{T^{j-1}} = h_1$ ,  $g_3(x) = g_3(x_1) = 0$  and  $g_3(x_2) = 2$ ;
- $g_4|_{T^{j-1}-u} = h_2$ ,  $g_4(u) = g_4(x_1) = 0$ ,  $g_4(x) = 2$  and  $g_4(x_2) = 1$ .

This,  $(\alpha_5)$  and Lemma 6 allows us to conclude that  $T$  is  $\gamma_R$ -excellent,  $x_1, x_2 \in V^{012}(T)$  and  $x \in V^{02}(T)$ .

By induction hypothesis,  $(\mathcal{P}_1)$  holds with  $(T, S)$  replaced by  $(T^{j-1}, S^{j-1})$ . Then Since  $u \notin S_B(T)$  and  $S_B(T) \cap V(F_4) = \emptyset$ , we have

$$\begin{aligned} S_B(T) &= S_B^{j-1}(T^{j-1}) \\ &= \{t \in V^{02}(T^{j-1}) \mid \deg_{T^{j-1}}(t) = 2 \text{ and } |N_{T^{j-1}}(t) \cap V^{02}(T^{j-1})| = 1\} \\ &= \{t \in V^{02}(T) \mid \deg_T(t) = 2 \text{ and } |N_T(t) \cap V^{02}(T)| = 1\}. \end{aligned}$$

The last equality follows from  $\deg_T(x) > 2$  and  $\{x\} = V^{02}(T) \cap V(F_4)$ . Now the equality  $S_C(T) = V^{02}(T) - S_B(T)$  is obvious. Thus,  $(\mathcal{P}_1)$  is true.

**Case 9:**  $T$  is obtained from  $T^{j-1} \in \mathcal{T}$  by operation  $O_3$ .

Let  $T = (T^{j-1} \cdot H_k)(u, v : u)$ , where  $\text{sta}_{T^{j-1}}(u) = \text{sta}_{H_k}(v) = \text{sta}_T(u) = A$  and  $k \in \{2, \dots, 7\}$ . Hence  $S_X(T) = S_X^{j-1}(T^{j-1}) \cup I_X^k(H_k)$ , for any  $X \in \{A, B, C, D\}$ . We know that  $(\mathcal{P}_1)$  holds with  $(T, S)$  replaced by any of  $(T^{j-1}, S^{j-1})$  and  $(H_k, I^k)$ . Hence  $S_A(T) = S_A^{j-1}(T^{j-1}) \cup I_A^k(H_k) = V^{01}(T^{j-1}) \cup V^{01}(H_k)$ . Now, by Proposition 2, applied to  $T^{j-1}$  and  $H_k$ ,  $S_A(T) = V^{01}(T)$ . Similarly we obtain  $S_D(T) = V^{012}(T)$ . We also have

$$\begin{aligned} S_B(T) &= S_B^{j-1}(T^{j-1}) \cup I_B^k(H_k) \\ &= \{t \in V^{02}(T^{j-1}) \mid \deg_{T^{j-1}}(t) = 2 \text{ and } |N_{T^{j-1}}(t) \cap V^{02}(T^{j-1})| = 1\} \\ &\cup \{t \in V^{02}(H_k) \mid \deg_{H_k}(t) = 2 \text{ and } |N_{H_k}(t) \cap V^{02}(H_k)| = 1\} \\ &= \{t \in V^{02}(T^{j-1}) \cup V^{02}(H_k) \mid \deg_T(t) = 2 \text{ and } |N_T(t) \cap V^{02}(T)| = 1\}, \end{aligned}$$

as required, because  $V^{02}(T^{j-1}) \cup V^{02}(H_k) = V^{02}(T)$  (by Proposition 2). Now the equality  $S_C(T) = V^{02}(T) - S_B(T)$  is obvious.

**Case 10:**  $T$  is obtained from  $T^{j-1} \in \mathcal{T}$  and  $H_k \in \mathcal{T}$ ,  $k \in \{3, 4, 6\}$ , by operation  $O_4$ . By induction hypothesis and Lemma 4, we have  $\gamma_R(T) = \gamma_R(T^{j-1}) + \gamma_R(H_k) - 1$  and  $u \in V^{012}(T)$ . Let  $f_1$  be a  $\gamma_R$ -function on  $T^{j-1}$  and  $f_2$  a  $\gamma_R$ -function on  $H_k - v$ . Then the function  $f$  defined as  $f|_{T^{j-1}} = f_1$  and  $f|_{H_k - v} = f_2$  is a  $\gamma_R$ -function on  $T$ . Therefore  $V^{012}(T^{j-1}) \subseteq V^{012}(T)$ ,  $V^{01}(T^{j-1}) \subseteq V^{01}(T) \cup V^{012}(T)$ , and  $V^{02}(T^{j-1}) \subseteq V^{02}(T) \cup V^{012}(T)$ . Assume that there is  $y \in V^{0s}(T^{j-1}) \cap V^{012}(T)$ ,  $s \in \{1, 2\}$ , and let  $f'$  be a  $\gamma_R$ -function on  $T$  with  $f'(y) = r \notin \{0, s\}$ . But then  $f'|_{T^{j-1}}$  is no RD-function on  $T^{j-1}$ ,  $f'(u) = 0$ ,  $f'|_{T^{j-1}-u}$  is a  $\gamma_R$ -function on  $T^{j-1} - u$  and  $f'|_{H_k}$  is a  $\gamma_R$ -function on  $H_k$ . Define now an RD-function  $g_1$  on  $T^{j-1}$  as  $g_1|_{T^{j-1}-u} = f'|_{T^{j-1}-u}$  and  $g_1(u) = 1$ . Since  $g_1(V(T^{j-1})) = \gamma_R(T^{j-1} - u) + 1 = \gamma_R(T^{j-1})$ ,  $g_1$  is a  $\gamma_R$ -function on  $T^{j-1}$ . But  $g_1(y) = r \notin \{0, s\}$ , a contradiction. Thus

$$(\alpha_6) \quad V^{012}(T^{j-1}) = V^{012}(T) \cap V(T^{j-1}), \quad V^{01}(T^{j-1}) = V^{01}(T) \cap V(T^{j-1}), \quad \text{and} \\ V^{02}(T^{j-1}) = V^{02}(T) \cap V(T^{j-1}).$$

The next claim is obvious.

**Claim 1.3** Let  $x$  be the neighbor of  $v$  in  $H_k$ ,  $k \in \{3, 4, 6\}$ . Then  $\gamma_R(H_3) = 4$ ,  $\gamma_R(H_4) = 5$ ,  $\gamma_R(H_6) = 6$ ,  $\gamma_R(H_k - v) = \gamma_R(H_k - \{v, x\}) = \gamma_R(H_k)$ , and  $l(x) = 0$  for any  $\gamma_R$ -function  $l$  on  $H_k - v$ .

Let  $h$  be a  $\gamma_R$ -function on  $T$ . We know that  $u \in V^{012}(T)$ ,  $u \in V^{012}(T^{j-1})$ ,  $v \in V^{01}(H_k)$ , and  $\gamma_R(T) = \gamma_R(T^{j-1}) + \gamma_R(H_k) - 1$ . Then by Claim 1.3 we clearly have:

(a1) If  $h(u) = 2$  then at least one of the following holds:

(a1.1)  $h|_{H_k - v}$  is a  $\gamma_R$ -function on  $H_k - v$ , and

(a1.2)  $h|_{H_k - \{v, x\}}$  is a  $\gamma_R$ -function on  $H_k - \{v, x\}$ .



(a2) If  $h(u) = 1$  then  $h|_{H_k-v}$  is a  $\gamma_R$ -function on  $H_k - v$ .

(a3) If  $h(u) = 0$  then either  $h|_{H_k}$  is a  $\gamma_R$ -function on  $H_k$ , or  $h|_{H_k-v}$  is a  $\gamma_R$ -function on  $H_k - v$ .

Let  $l_1, l_2, l_3, l_4$  and  $l_5$  be  $\gamma_R$ -functions on  $H_k, H_k - v, H_k - \{v, x\}, T^{j-1} - u$  and  $T^{j-1}$ , respectively, and let  $l_5(u) = 2$ . Define the functions  $h_1, h_2$ , and  $h_3$  on  $T$  as follows: (i)  $h_1|_{T^{j-1}} = l_5, h_1(x) = 0$  and  $h_1|_{H_k-\{v,x\}} = l_3$ , (ii)  $h_2|_{T^{j-1}} = l_5$  and  $h_2|_{H_k-v} = l_2$ , and (iii)  $h_3|_{T^{j-1}-u} = l_4$  and  $h_3|_{H_k} = l_1$ . Clearly  $h_1, h_2$ , and  $h_3$  are  $\gamma_R$ -functions on  $T$ . After inspection of all  $\gamma_R$ -functions of  $H_k, H_k - v$  and  $H_k - \{v, x\}$ , we conclude that  $V^{01}(H_k) - \{v\} \subseteq V^{01}(T), V^{02}(H_k) \subseteq V^{02}(T)$ , and  $V^{012}(H_k) \subseteq V^{012}(T)$ . This and  $(\alpha_6)$  imply

$$(\alpha_7) \quad V^{012}(T) = V^{012}(T^{j-1}) \cup V^{012}(H_k), \quad V^{02}(T) = V^{02}(T^{j-1}) \cup V^{02}(H_k), \quad \text{and} \\ V^{01}(T) = V^{01}(T^{j-1}) \cup (V^{01}(H_k) - \{v\}).$$

Since  $(\mathcal{P}_1)$  holds with  $T$  replaced by  $H_k$  or by  $T^{j-1}$  (by induction hypothesis), using  $(\alpha_7)$  we obtain that  $(\mathcal{P}_1)$  is satisfied.  $\square$

## 5. Corollaries

The next three results immediately follow by Theorem 1.

**Corollary 1.** *If  $(T, S_1), (T, S_2) \in \mathcal{T}$  then  $S_1 \equiv S_2$ .*

If  $(T, S) \in \mathcal{T}$  then we call  $S$  the  $\mathcal{T}$ -labeling of  $T$ .

**Corollary 2.** *Let  $T$  be a  $\gamma_R$ -excellent tree of order  $n \geq 5$ , and  $S$  the  $\mathcal{T}$ -labeling of  $T$ . Then  $\frac{n}{5} \leq |V^{02}(T)| \leq \frac{2}{3}(n-1)$  and  $\frac{4}{5}n \geq |V^-(T)| \geq \frac{1}{3}(n+2)$ . Moreover,*

(i)  $\frac{n}{5} = |V^{02}(T)|$  if and only if  $(T, S)$  has a  $\mathcal{T}$ -sequence  $\tau : (T^1, S^1), \dots, (T^j, S^j)$ , such that  $(T^1, S^1) = (F_3, J^3)$  and if  $j \geq 2$ ,  $(T^{i+1}, S^{i+1})$  can be obtained recursively from  $(T^i, S^i)$  and  $(F_3, J^3)$  by operation  $O_1$ .

(ii)  $|V^{02}(T)| \leq \frac{2}{3}(n-1)$  if and only if  $(T, S)$  has a  $\mathcal{T}$ -sequence  $\tau : (T^1, S^1), \dots, (T^j, S^j)$ , such that  $(T^1, S^1) = (H_2, I^2)$  and if  $j \geq 2$ ,  $(T^{i+1}, S^{i+1})$  can be obtained recursively from  $(T^i, S^i)$  and  $(H_2, I^2)$  by operation  $O_3$ .

**Corollary 3.** *Let  $G$  be an  $n$ -order  $\gamma_R$ -excellent connected graph of minimum size. Then either  $G = K_3$  or  $n \neq 3$  and  $G$  is a tree.*

## 6. Special cases

Let  $G$  be a graph and  $\{a_1, \dots, a_k\} \subseteq \{0, 1, 2, 01, 02, 12, 012\}$ . We say that  $G$  is a  $\mathcal{R}_{a_1, \dots, a_k}$ -graph if  $V(G) = \bigcup_{i=1}^k V^{a_i}(G)$  and all  $V^{a_1}(G), \dots, V^{a_k}(G)$  are nonempty. Now let  $T$  be a  $\gamma_R$ -excellent tree of order at least 2. By Theorem 1, we immediately conclude that  $T \in \mathcal{R}_{012} \cup \mathcal{R}_{01,02} \cup \mathcal{R}_{02,012} \cup \mathcal{R}_{01,02,012}$ . Moreover,

- (i)  $T \in \mathcal{R}_{012}$  if and only if  $T = K_2$ , and
- (ii)  $T \in \mathcal{R}_{01,02,012}$  if and only if none of  $S_A(T)$ ,  $S_C(T)$  and  $S_D(T)$  is empty, where  $S$  is the  $\mathcal{T}$ -labeling of  $T$ .

In this section, we turn our attention to the classes  $\mathcal{R}_{01,02}$  and  $\mathcal{R}_{02,012}$ .

### 6.1. $\mathcal{R}_{01,02}$ -graphs.

Here we give necessary and sufficient conditions for a tree to be in  $\mathcal{R}_{01,02}$ . We define a subfamily  $\mathcal{T}_{01,02}$  of  $\mathcal{T}$  as follows. A labeled tree  $(T, S) \in \mathcal{T}_{01,02}$  if and only if  $(T, S)$  can be obtained from a sequence of labeled trees  $\tau : (T^1, S^1), \dots, (T^j, S^j)$ , ( $j \geq 1$ ), such that  $(T^1, S^1)$  is in  $\{(H_2, I^2), (H_3, I^3)\}$  (see Figure 1) and  $(T, S) = (T^j, S^j)$ , and, if  $j \geq 2$ ,  $(T^{i+1}, S^{i+1})$  can be obtained recursively from  $(T^i, S^i)$  by one of the operations  $O_5$  and  $O_6$  listed below; in this case  $\tau$  is said to be a  $\mathcal{T}_{01,02}$ -sequence of  $T$ .

**Operation  $O_5$ .** The labeled tree  $(T^{i+1}, S^{i+1})$  is obtained from  $(T^i, S^i)$  and  $(F_1, J^1)$  (see Figure 2) by adding the edge  $ux$ , where  $u \in V(T_i)$ ,  $x \in V(F_1)$  and  $sta_{T^i}(u) = sta_{F_1}(x) = C$ .

**Operation  $O_6$ .** The labeled tree  $(T^{i+1}, S^{i+1})$  is obtained from  $(T^i, S^i)$  and  $(H_k, I^k)$ ,  $k \in \{2, 3\}$  (see Figure 1), in such a way that  $T^{i+1} = (T^i \cdot H_k)(u, v : u)$ , where  $sta_{T^i}(u) = sta_{H_k}(v) = A$ , and  $sta_{T^{i+1}}(u) = A$ .

Remark that once a vertex is assigned a status, this status remains unchanged as the labeled tree  $(T, S)$  is recursively constructed. By the above definitions we see that  $S_D(T)$  is empty when  $(T, S) \in \mathcal{T}_{01,02}$ . So, in this case, it is naturally to consider a labeling  $S$  as  $S : V(T) \rightarrow \{A, B, C\}$ . From Theorem 1 we immediately obtain the following result.

**Corollary 4.** *Let  $T$  be a tree of order at least 2. Then  $T \in \mathcal{R}_{01,02}$  if and only if there is a labeling  $S : V(T) \rightarrow \{A, B, C\}$  such that  $(T, S)$  is in  $\mathcal{T}_{01,02}$ . Moreover, if  $(T, S) \in \mathcal{T}_{01,02}$  then*

$$(\mathcal{P}_3) \quad S_B(T) = \{x \in V^{02}(T) \mid \deg(x) = 2 \text{ and } |N(x) \cap V^{02}(T)| = 1\}, \quad S_A(T) = V^{01}(T), \text{ and} \\ S_C(T) = V^{02}(T) - S_B(T).$$

As an immediate consequence of Corollary 1 we obtain:

**Corollary 5.** *If  $(T, S_1), (T, S_2) \in \mathcal{T}_{01,02}$  then  $S_1 \equiv S_2$ .*

A graph  $G$  is called a 2-corona if each vertex of  $G$  is either a support vertex or a leaf, and each support vertex of  $G$  is adjacent to exactly 2 leaves. In a labeled 2-corona all leaves have status  $A$  and all support vertices have status  $C$ .

**Proposition 3.** *Every connected  $n$ -order graph  $H$ ,  $n \geq 2$ , is an induced subgraph of a  $\mathcal{R}_{01,02}$ -graph with the domination number equals to  $2|V(H)|$ .*

*Proof.* Let a graph  $G$  be a 2-corona such that the induced subgraph by the set of all support vertices of  $G$  is isomorphic to  $H$ . Let  $x$  be a support vertex of  $G$  and  $y, z$  the leaf neighbors of  $x$  in  $G$ . Then clearly for any  $\gamma_R$ -function  $f$  on  $G$ ,  $f(x) + f(y) + f(z) \geq 2$ ,  $f(y) \neq 2 \neq f(z)$  and  $f(x) \neq 1$ . Define RD-functions  $h$  and  $g$  on  $G$  as follows: (a)  $h(u) = 2$  when  $u$  is a support vertex of  $G$  and  $h(u) = 0$ , otherwise, and (b)  $g(v) = h(v)$  when  $v \notin \{x, y, z\}$ , and  $g(x) = 0$ ,  $g(y) = g(z) = 1$ . Therefore  $\gamma_R(G) = 2|V(H)|$  and  $G$  is in  $\mathcal{R}_{01,02}$ .  $\square$

**Corollary 6.** *There does not exist a forbidden subgraph characterization of the class of  $\mathcal{R}_{01,02}$ -graphs. There does not exist a forbidden subgraph characterization of the class of  $\gamma_R$ -excellent graphs.*

Let  $\mathcal{T}'_{01,02}$  be the family of all labeled trees  $(T, L)$  that can be obtained from a sequence of labeled trees  $\lambda : (T^1, L^1), \dots, (T^j, L^j)$ , ( $j \geq 1$ ), such that  $(T, L) = (T^j, L^j)$ ,  $(T^1, L^1)$  is either  $(H_2, I^2)$  (see Figure 1) or a labeled 2-corona tree, and, if  $j \geq 2$ ,  $(T^{i+1}, L^{i+1})$  can be obtained recursively from  $(T^i, L^i)$  by one of the operations  $O_7$  and  $O_8$  listed below; in this case  $\lambda$  is said to be a  $\mathcal{T}'_{01,02}$ -sequence of  $T$ .

**Operation  $O_7$ .** The labeled tree  $(T^{i+1}, L^{i+1})$  is obtained from  $(T^i, L^i)$  and  $(H_2, I^2)$ , in such a way that  $T^{i+1} = (T^i \cdot H_2)(u, v : u)$ , where  $sta_{T^i}(u) = sta_{H_2}(v) = A$ , and  $sta_{T^{i+1}}(u) = A$ .

**Operation  $O_8$ .** The labeled tree  $(T^{i+1}, L^{i+1})$  is obtained from  $(T^i, L^i)$  and a labeled 2-corona tree, say  $U_i$ , in such a way that  $T^{i+1} = (T^i \cdot U_i)(u, v : u)$ , where  $sta_{T^i}(u) = sta_{U_i}(v) = A$ , and  $sta_{T^{i+1}}(u) = A$ .

Again, once a vertex is assigned a status, this status remains unchanged as the 2-labeled tree  $T$  is recursively constructed.

**Theorem 2.** *For any tree  $T$  the following are equivalent.*

(A<sub>1</sub>)  $T$  is in  $\mathcal{R}_{01,02}$ .

(A<sub>2</sub>) There is a labeling  $S : V(T) \rightarrow \{A, B, C\}$  such that  $(T, S)$  is in  $\mathcal{T}_{01,02}$ .

(A<sub>3</sub>) There is a labeling  $L : V(T) \rightarrow \{A, B, C\}$  such that  $(T, L)$  is in  $\mathcal{T}'_{01,02}$ .

**Proof.** (A<sub>1</sub>)  $\Leftrightarrow$  (A<sub>2</sub>): By Corollary 4.

(A<sub>3</sub>)  $\Rightarrow$  (A<sub>2</sub>):

Let a tree  $(T, L) \in \mathcal{T}'_{01,02}$ . It is clear that all  $\mathcal{T}'_{01,02}$ -sequences of  $(T, L)$  have the same number of elements. Denote this number by  $r(T)$ . We shall prove that  $(T, L) \in \mathcal{T}'_{01,02} \Rightarrow (T, L) \in \mathcal{T}_{01,02}$ . We proceed by induction on  $r(T)$ . If  $r(T) = 1$  then either

$(T, L)$  is a labeled 2-corona tree, or  $(T, L) = (H_2, I^2)$ . In both cases  $(T, L) \in \mathcal{T}_{01,02}$ . We need the following obvious claim.

**Claim 2.1** If  $(T', L')$  is a labeled 2-corona tree,  $w \in V(T')$  and  $sta(w) = A$ , then either  $(T', L')$  is  $(H_3, I^3)$  or there is a  $\mathcal{T}$ -sequence  $\tau : (T^1, S^1), \dots, (T^l, S^l)$ , ( $l \geq 2$ ), such that  $(T^1, S^1) = (H_3, I^3)$ ,  $w \in V(T^1)$ ,  $(T^l, S^l) = (T', L')$ , and  $(T^{i+1}, S^{i+1})$  can be obtained recursively from  $(T^i, S^i)$  and  $(F_1, J^1)$  by operation  $O_5$ .

Suppose now that each tree  $(H, L_H) \in \mathcal{T}'_{01,02}$  with  $r(H) < k$  is in  $\mathcal{T}_{01,02}$ , where  $k \geq 2$ . Let  $\lambda : (T^1, L^1), \dots, (T^k, L^k)$ , be a  $\mathcal{T}'_{01,02}$ -sequence of a labeled tree  $(T, L) \in \mathcal{T}'_{01,02}$ . By the induction hypothesis,  $(T^{k-1}, L^{k-1})$  is in  $\mathcal{T}_{01,02}$ . Let  $\tau_1 : (U^1, S^1), \dots, (U^m, S^m)$  be a  $\mathcal{T}$ -sequence of  $(T^{k-1}, L^{k-1})$ . Hence  $U^m = T^{k-1}$  and  $S^m = L^{k-1}$ . If  $(T^k, L^k)$  is obtained from  $(T^{k-1}, L^{k-1})$  and  $(H_2, I^2)$  by operation  $O_7$ , then  $(U^1, S^1), \dots, (U^m, S^m), (T^k, L^k) = (T, L)$  is a  $\mathcal{T}$ -sequence of  $(T, L)$ . So, let  $(T^k, L^k)$  is obtained from  $(T^{k-1}, L^{k-1})$  and a labeled 2-corona tree, say  $(Q, L_q)$  by operation  $O_8$ . Hence  $T^{k-1}$  and  $Q$  have exactly one vertex in common, say  $w$ , and  $sta_{T^{k-1}}(w) = sta_Q(w) = sta_{T^k}(w) = A$ . By Claim 2.1,  $(Q, L_q) \in \mathcal{T}_{01,02}$  and it has a  $\mathcal{T}_{01,02}$ -sequence, say  $(Q^1, L_q^1), \dots, (Q^s, L_q^s)$  such that  $Q^s = Q$ ,  $L_q = L_q^s$ , and  $w \in V(Q^1)$ . Denote  $W^{m+i} = \langle V(U^m) \cup V(Q^i) \rangle$  and let a labeling  $S^{m+i}$  be such that  $S^{m+i}|_{U^m} = S^m$  and  $S^{m+i}|_{Q^i} = L_q^i$ . Then the sequence of labeled trees  $(U^1, S^1), \dots, (U^m, S^m), (W^{m+1}, S^{m+1}), \dots, (W^{m+s}, S^{m+s}) = (T, L)$  is a  $\mathcal{T}_{01,02}$ -sequence of  $(T, L)$ .

$(A_2) \Rightarrow (A_3)$ :

Let a labeled tree  $(T, S) \in \mathcal{T}_{01,02}$ . Then  $(T, S)$  has a  $\mathcal{T}$ -sequence  $\tau : (T^1, S^1), \dots, (T^j, S^j) = (T, S)$ , where  $(T^1, S^1) \in \{(H_2, I^2), (H_3, I^3)\} \subset \mathcal{T}'_{01,02}$ . We proceed by induction on  $p(T) = \sum_{z \in \mathcal{C}(T)} deg_T(z)$ , where  $\mathcal{C}(T)$  is the set of all cut-vertices of  $T$  that belong to  $S_A(T)$ . Assume first  $p(T) = 0$ . If  $j = 1$  then we are done. If  $j \geq 2$  then  $(T^1, S^1) = (H_3, I^3)$  and  $(T^{i+1}, S^{i+1})$  is obtained from  $(F_1, J^1)$  and  $(T^i, S^i)$  by operation  $O_5$ . Thus,  $(T, S)$  is a labeled 2-corona tree, which allow us to conclude that  $(T, S)$  is in  $\mathcal{T}'_{01,02}$ .

Suppose now that  $p(T) = k \geq 1$  and for each labeled tree  $(H, S_H) \in \mathcal{T}_{01,02}$  with  $p(H) < k$  is fulfilled  $(H, S_H) \in \mathcal{T}'_{01,02}$ . Then there is a cut-vertex, say  $z$ , such that (a)  $z \in S_A(T)$ , (b)  $(T, S)$  is a coalescence of 2 graphs, say  $(T', S|_{T'})$  and  $(T'', S|_{T''})$ , via  $z$ , and (c) no vertex in  $S_A(T) \cap V(T'')$  is a cut-vertex of  $T''$ . Hence  $(T', S|_{T'}) \in \mathcal{T}'_{01,02}$  (by induction hypothesis) and  $(T'', S|_{T''})$  is either a labeled 2-corona tree or  $H_2$ . Thus  $(T, S)$  is in  $\mathcal{T}'_{01,02}$ .  $\square$

## 6.2. $\mathcal{R}_{02,012}$ -trees.

Our aim in this section is to present a characterization of  $\mathcal{R}_{02,012}$ -trees. For this purpose, we need the following definitions. Let  $\mathcal{T}_{02,012} \subset \mathcal{T}$  be such that  $(T, S) \in \mathcal{T}_{02,012}$  if and only if  $(T, S)$  can be obtained from a sequence of labeled trees  $\tau : (T^1, S^1), \dots, (T^j, S^j)$ , ( $j \geq 1$ ), such that  $(T^1, S^1) = (F_3, J^3)$  (see Figure 2) and  $(T, S) = (T^j, S^j)$ , and, if  $j \geq 2$ ,  $(T^{i+1}, S^{i+1})$  can be obtained recursively from  $(T^i, S^i)$  by one of the operations  $O_9$  and  $O_{10}$  listed below.

**Operation  $O_9$ .** The labeled tree  $(T^{i+1}, S^{i+1})$  is obtained from  $(T^i, S^i)$  and  $(F_3, J^3)$  by adding the edge  $ux$ , where  $u \in V(T^i)$ ,  $x \in V(F_3)$  and  $sta_{T^i}(u) = sta_{F_3}(x) = C$ .

**Operation  $O_{10}$ .** The labeled tree  $(T^{i+1}, S^{i+1})$  is obtained from  $(T^i, S^i)$  and  $(F_4, J^4)$  (see Figure 2) by adding the edge  $ux$ , where  $u \in V(T^i)$ ,  $x \in V(F_4)$ ,  $sta_{T^i}(u) = D$ , and  $sta_{F_4}(x) = C$ .

Note that once a vertex is assigned a status, this status remains unchanged as the labeled tree  $(T, S)$  is recursively constructed. By the above definitions we see that if  $(T, S) \in \mathcal{R}_{01,02}$ , then  $S_A(T) = S_B(T) = \emptyset$ . Therefore it is naturally to consider a labeling  $S$  as  $S : V(T) \rightarrow \{C, D\}$ .

From Theorem 1 we immediately obtain the following result.

**Corollary 7.** *A tree  $T$  is in  $\mathcal{R}_{02,012}$  if and only if there is a labeling  $S : V(T) \rightarrow \{C, D\}$  such that  $(T, S)$  is in  $\mathcal{T}_{02,012}$ . Moreover, if  $(T, S) \in \mathcal{T}_{02,012}$  then  $S_C(T) = V^{02}(T)$  and  $S_D(T) = V^{012}(T)$ .*

As an immediate consequence of Corollary 1 we obtain:

**Corollary 8.** *If  $(T, S_1), (T, S_2) \in \mathcal{T}_{02,012}$  then  $S_1 \equiv S_2$ .*

**Theorem 3.** [3] *If  $G$  is a connected graph of order  $n \geq 3$ , then  $\gamma_R(G) \leq 4n/5$ . The equality holds if and only if  $G$  is  $C_5$  or is obtained from  $\frac{n}{5}P_5$  by adding a connected subgraph on the set of centers of the components of  $\frac{n}{5}P_5$ .*

As a consequence of Theorem 3 and Corollary 7 we have:

**Corollary 9.** *Let  $G$  be a connected  $n$ -vertex graph with  $n \geq 6$  and  $\gamma_R(G) = 4n/5$ . Then  $G$  is in  $\mathcal{R}_{02,012}$  and  $V^{012}(G)$  consists of all leaves and all support vertices. Moreover, if  $G$  is a tree, then  $G$  has a  $\mathcal{T}$ -sequence  $\tau : (G^1, S^1), \dots, (G^j, S^j)$ , ( $j \geq 1$ ), such that  $(G^1, S^1) = (F_3, J^3)$  (see Figure 2) and if  $j \geq 2$ , then  $(G^{i+1}, S^{i+1})$  can be obtained recursively from  $(G^i, S^i)$  by operation  $O_9$ .*

A graph  $G$  is said to be in class  $UVR$  if  $\gamma(G - v) = \gamma(G)$  for each  $v \in V(G)$ . Constructive characterizations of trees belonging to  $UVR$  are given in [14] by Samodivkin, and independently in [11] by Haynes and Henning. We need the following result in [14] (reformulated in our present terminology).

**Theorem 4.** [14] *A tree  $T$  of order at least 5 is in  $UVR$  if and only if there is a labeling  $S : V(T) \rightarrow \{C, D\}$  such that  $(T, S)$  is in  $\mathcal{T}_{02,012}$ . Moreover, if  $(T, S) \in \mathcal{T}_{02,012}$  then  $S_C(T)$  and  $S_D(T)$  are the sets of all  $\gamma$ -bad and all  $\gamma$ -good vertices of  $T$ , respectively.*

We end with our main result in this subsection.

**Theorem 5.** For any tree  $T$  the following are equivalent:

- (A<sub>4</sub>)  $T$  is in  $\mathcal{R}_{02,012}$ ,      (A<sub>5</sub>)  $T$  is in  $\mathcal{T}_{02,012}$ ,      (A<sub>6</sub>)  $T$  is in  $UVR$ .

**Proof.** Corollary 7 and Theorem 4 together imply the required result.  $\square$

## 7. Open problems and questions

We conclude the paper by listing some interesting problems and directions for further research. Let first note that if  $n \geq 3$  and  $\mathbf{G}_{n,k}$  is not empty, then  $k \leq 4n/5$  (Theorem 3).

An element of  $\mathbb{R}\mathbb{E}_{n,k}$  is said to be *isolated*, whenever it is both maximal and minimal. In other words, a graph  $H \in \mathbf{G}_{n,k}$  is isolated in  $\mathbb{R}\mathbb{E}_{n,k}$  if and only if  $H \in \mathcal{R}_{CEA}$  and for each  $e \in E(H)$  at least one of the following holds: (a)  $H - e$  is not connected, (b)  $\gamma_R(H) \neq \gamma_R(H - e)$ , (c)  $H - e$  is not  $\gamma_R$ -excellent.

**Example 1.** (i) All  $\gamma_R$ -excellent graphs with the Roman domination number equals to 2 are  $\overline{K_2}$  and  $K_n$ ,  $n \geq 2$ . If a graph  $G \in \mathcal{R}_{CEA}$  and  $\gamma_R(G) = 2$ , then  $G$  is complete.  $K_n$  is isolated in  $\mathbb{R}\mathbb{E}_{n,2}$ ,  $n \geq 2$ .

(ii) [8]  $K_2$ ,  $H_7$  and  $H_8$  (see Fig. 1) are the only trees in  $\mathcal{R}_{CEA}$ .

(iii) If  $\mathbb{R}\mathbb{E}_{n,k}$  has a tree  $T$  as an isolated element, then either  $(n, k) = (2, 2)$  and  $T = K_2$ , or  $(n, k) = (9, 7)$  and  $T = H_7$ , or  $(n, k) = (10, 8)$  and  $T = H_8$ .

- Find results on the isolated elements of  $\mathbb{R}\mathbb{E}_{n,k}$ .
- What is the maximum number of edges  $m(\mathbf{G}_{n,k})$  of a graph in  $\mathbf{G}_{n,k}$ ? Note that (a)  $m(\mathbf{G}_{n,2}) = n(n-1)/2$ , (b)  $m(\mathbf{G}_{n,3}) = n(n-1)/2 - \lceil n/2 \rceil$ .
- Find results on those minimal elements of  $\mathbb{R}\mathbb{E}_{n,k}$  that are not trees.

**Example 2.** (a) A cycle  $C_n$  is a minimal element of  $\mathbb{R}\mathbb{E}_{n,k}$  if and only if  $n \equiv 0 \pmod{3}$  and  $k = 2n/3$ . (b) A graph  $G$  obtained from the complete bipartite graph  $K_{p,q}$ ,  $p \geq q \geq 3$ , by deleting an edge is a minimal element of  $\mathbb{R}\mathbb{E}_{p+q,4}$ .

The height of a poset is the maximal number of elements of a chain.

- Find the height of  $\mathbb{R}\mathbb{E}_{n,k}$ .

**Example 3.** (a) It is easy to check that any longest chain in  $\mathbb{R}\mathbb{E}_{6,4}$  has as the first element  $H_3$  (see Fig 1) and as the last element one of the two 3-regular 6-vertex graphs. Therefore the height of  $\mathbb{R}\mathbb{E}_{6,4}$  is 5.

- (b) Let us consider the poset  $\mathbb{RE}_{5r,4r}$ ,  $r \geq 2$ . All its minimal elements are  $\gamma_R$ -excellent trees (by Theorem 3 and Corollary 9), which are characterized in Corollary 9. Moreover, the graph obtained from  $rP_5$  by adding a complete graph on the set of centers of the components of  $rP_5$  is the largest element of  $\mathbb{RE}_{5r,4r}$ . Therefore the height of  $\mathbb{RE}_{5r,4r}$  is  $(r-1)(r-2)/2 + 1$ .
- Find results on  $\gamma_{YR}$ -excellent graphs at least when  $Y$  is one of  $\{-1, 0, 1\}$ ,  $\{-1, 1\}$  and  $\{-1, 1, 2\}$ .

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