Communications in Combinatorics and Optimization
Vol. 3 No. 2, 2018 pp.93-119
DOI: 10.22049/CCO.2018.26060.1072

CCO
Commun. Comb. Optim.

Classification of rings with toroidal annihilating-ideal graph

K. Selvakumar* **and P. Subbulakshmi**

Department of Mathematics, Manonmaniam Sundaranar University
Tirunelveli 627 012, Tamil Nadu, India
`selva_158@yahoo.co.in`

Received: 20 October 2017; Accepted: 2 April 2018

*Communicated by **Seyed Mahmoud Sheikholeslami***

Abstract: Let R be a non-domain commutative ring with identity and $\mathbb{A}^*(R)$ be the set of non-zero ideals with non-zero annihilators. We call an ideal I_1 of R, an *annihilating-ideal* if there exists a non-zero ideal I_2 of R such that $I_1I_2 = (0)$. The *annihilating-ideal graph* of R is defined as the graph $\mathbb{AG}(R)$ with the vertex set $\mathbb{A}^*(R)$ and two distinct vertices I_1 and I_2 are adjacent if and only if $I_1I_2 = (0)$. In this paper, we characterize all commutative Artinian non-local rings R for which $\mathbb{AG}(R)$ has genus one.

Keywords: Annihilating-ideal, planar graph, genus, local ring, annihilating-ideal graph

AMS Subject classification: 13A15, 05C75, 13M05, 05C10

1. Terminology and introduction

The study of algebraic structures, using the properties of graphs, became an exciting research topic in the past twenty years, leading to many fascinating results and questions. In the literature, there are many papers assigning graphs to rings, groups and semigroups, see [3–7, 11, 17, 22]. For related graph, see the annihilator graph as in [9, 10]. For recent survey article on the zero-divisor graph see [6]. In ring theory, the structure of a ring R is closely tied to ideal's behavior more than elements, and so it is deserving to define a graph with vertex set as ideals instead of elements. Recently M. Behboodi and Z. Rakeei [12, 13] have introduced and investigated the annihilating-ideal graph of a commutative ring. For a non-domain commutative ring R, let $\mathbb{A}^*(R)$

---

* *Corresponding Author*

be the set of non-zero ideals with non-zero annihilators. We call an ideal I_1 of R, an *annihilating-ideal* if there exists a non-zero ideal I_2 of R such that $I_1I_2 = (0)$. The *annihilating-ideal graph* of R is defined as the graph $\mathbb{AG}(R)$ with the vertex set $\mathbb{A}^*(R)$ and two distinct vertices I_1 and I_2 are adjacent if and only if $I_1I_2 = (0)$. Several properties of $\mathbb{AG}(R)$ were studied by the authors in [1, 2, 12, 13, 15, 19]. In this paper, we characterize all commutative Artinian non-local rings R for which $\mathbb{AG}(R)$ has genus one.

By a graph $G = (V, E)$, we mean an undirected simple graph with vertex set V and edge set E. A graph in which each pair of distinct vertices is joined by the edge is called a complete graph. We use K_n to denote the complete graph with n vertices. An r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. The girth of G is the length of a shortest cycle in G and is denoted by $gr(G)$. If G has no cycles, we define the girth of G to be infinite. A graph G is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph G is *planar* if and only if it contains no subdivision of K_5 or $K_{3,3}$(see [14, p.153]).

A *minor* of G is a graph obtained from G by contracting edges in G or deleting edges and isolated vertices in G. A classical theorem due to K. Wagner [21] states that a graph G is planar if and only if G does not have K_5 or $K_{3,3}$ as a minor. It is well known that if G' is a minor of G, then $\gamma(G') \leq \gamma(G)$. For $xy \in E(G)$, we denote the contracted edge by the vertex $[x, y]$. Also if H is a subgraph of G and H' is a minor of H, then we call H' as a minor subgraph of G.

The main objective of topological graph theory is to embed a graph into a surfaces. By a surfaces, we mean a connected two-dimensional real manifold, i.e., a connected topological space such that each point has a neighborhood homeomorphic to an open disk. It is well known that any compact surfaces is either homeomorphic to a sphere, or to a connected sum of g tori, or to a connected sum of k projective planes (see [18, Theorem 5.1]). We denote S_g for the surfaces formed by a connected sum of g tori. The number g is called the genus of the surfaces S_g. When considering the orientability, the surfaces S_g and sphere are among the orientable class. In this paper, we mainly focus on the orientable cases.

A simple graph which can be embedded in S_g but not in S_{g-1} is called a graph of genus g. The notations $\gamma(G)$ is denoted for the genus. It is easy to see that $\gamma(H) \leq \gamma(G)$ for all subgraph H of G. For details on the notion of embedding of graphs in surfaces, one can refer to A. T. White [23].

The following results about the planarity are very useful in the subsequent sections.

Theorem 1. [19] *Let R be a commutative Artinian ring with identity. Then $\mathbb{AG}(R)$ is planar if and only if one of the following condition holds:*

(*i*) *$R \cong F_1 \times F_2$ or $R \cong F_1 \times F_2 \times F_3$ where F_i, $i = 1, 2, 3$ are Fields.*

(*ii*) *$R \cong R_1 \times R_2$ where $(R_i, \mathfrak{m}_i)$, $i = 1, 2$ is a local ring with $\mathfrak{m}_i \neq \{0\}$ and one of the following condition holds:*

  (*a*) *$n_1 = 2$, $n_2 = 3$ and $\mathfrak{m}_1$ is the only non-trivial ideal in R_1 and $\mathfrak{m}_2, \mathfrak{m}_2^2$ are the only non-trivial ideals in R_2.*

  (*b*) *$n_1 = 3$, $n_2 = 2$ and $\mathfrak{m}_1, \mathfrak{m}_1^2$ are the only non-trivial ideals in R_1 and $\mathfrak{m}_2$ is the only non-trivial ideal in R_2.*

  (*c*) *$n_1 = n_2 = 2$ and $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the only non-trivial ideal in R_1 and R_2 respectively.*

(*iii*) *$R = R_1 \times F_1 \times F_2$, $n_1 = 2$ and $\mathfrak{m}_1$ is the only non-trivial ideal in R_1.*

(*iv*) *$R = R_1 \times F_1$ and one of the following holds:*

  (*a*) *$n_1 = 2$ and $\mathfrak{m}_1$ is the only non-trivial ideal in R_1.*

  (*b*) *$n_1 = 3$ and $\mathfrak{m}_1, \mathfrak{m}_1^2$ are the only non-trivial ideals in R_1.*

  (*c*) *$n_1 = 4$ and $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3$ are the only non-trivial ideals in R_1.*

The following results about the genus are very useful in the subsequent sections.

Lemma 1. [23] *$\gamma(K_n) = \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil$, where $\lceil x \rceil$ is the least integer that is greater than or equal to x. In particular, $\gamma(K_n) = 1$ if $n = 5, 6, 7$.*

Lemma 2. [23] *$\gamma(K_{m,n}) = \left\lceil \frac{1}{4}(m-2)(n-2) \right\rceil$, where $\lceil x \rceil$ is the least integer that is greater than or equal to x. In particular, $\gamma(K_{4,4}) = \gamma(K_{3,n}) = 1$ if $n = 3, 4, 5, 6$.*

Lemma 3. [16] *Suppose that H and H' are two subgraphs of a graph G such that H and H' are isomorphic to $K_{3,3}$ or K_5. If $H \cap H' = \{v\}$, where v is a vertex of G, then $\gamma(G) > 1$.*

Lemma 4. [23] (Euler formula) *If G is a finite connected graph with n vertices, m edges, and genus γ, then $n - m + f = 2 - 2\gamma$, where f is the number of faces created when G is minimally embedded on a surfaces of genus γ.*

Lemma 5. [8] *If G is a graph with n vertices, m edges, girth $gr(G)$, and genus γ, then*

$$\frac{m(gr(G)-2)}{2gr(G)} - \frac{n}{2} + 1 \leq \gamma.$$

2. Genus of annihilating-ideal graph

The main goal of this section is to determine all commutative Artinian non-local rings R for which $\mathbb{AG}(R)$ has genus one.

Theorem 2. *Let $R = F_1 \times F_2 \times \cdots \times F_n$ be a commutative ring with identity where each F_i is a field and $n \geq 2$. Then $\gamma(\mathbb{AG}(R)) = 1$ if and only if $n = 4$.*

Proof. Assume that $\gamma(\mathbb{AG}(R)) = 1$. Suppose $n > 4$. Consider the non-trivial ideals $u_1 = F_1 \times (0) \times (0) \times (0) \times (0) \times \cdots \times (0)$, $u_2 = (0) \times F_2 \times (0) \times (0) \times (0) \times \cdots \times (0)$, $u_3 = F_1 \times F_2 \times (0) \times (0) \times (0) \times \cdots \times (0)$, $v_1 = (0) \times (0) \times F_3 \times (0) \times (0) \times \cdots \times (0)$, $v_2 = (0) \times (0) \times (0) \times F_4 \times (0) \times \cdots \times (0)$, $v_3 = (0) \times (0) \times (0) \times (0) \times F_5 \times \cdots \times (0)$, $v_4 = (0) \times (0) \times F_3 \times F_4 \times (0) \times \cdots \times (0)$, $v_5 = (0) \times (0) \times F_3 \times (0) \times F_5 \times \cdots \times (0)$, $v_6 = (0) \times (0) \times (0) \times F_4 \times F_5 \times \cdots \times (0)$, $v_7 = (0) \times (0) \times F_3 \times F_4 \times F_5 \times \cdots \times (0)$ in R. Then $u_i v_j = (0)$ for every i, j and so $K_{3,7}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence by Theorem 1, $n = 4$.

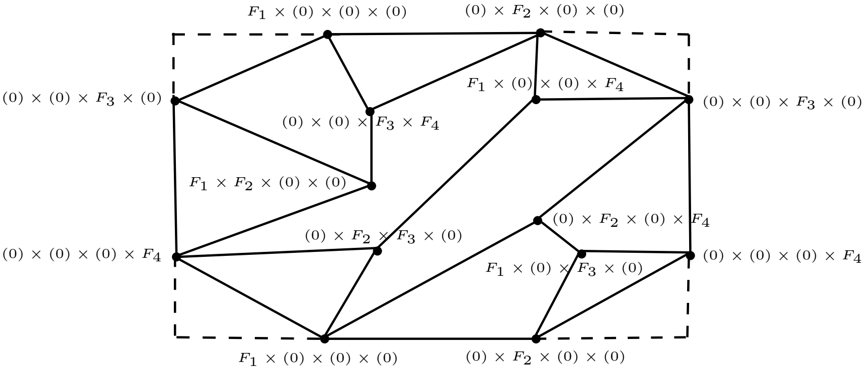


Fig 2.1: Torus embedding of $\mathbb{AG}(F_1 \times F_2 \times F_3 \times F_4)$

Converse follows from Fig 2.1. $\square$

The following two results are very useful in the subsequent sections.

Lemma 6. [20] *Let $(R, \mathfrak{m})$ be a local ring. If $\dim(\mathfrak{m}/\mathfrak{m}^2) = 1$ and for some positive integer t, $\mathfrak{m}^t = (0)$, then the set of all non-trivial ideals of R is the set $\{\mathfrak{m}^i : 1 \leq i < t\}$.*

Proposition 1. [20] *If $(R, \mathfrak{m})$ is a local ring and there is an ideal I of R such that $I \neq \mathfrak{m}^i$ for every i, then R has at least three distinct non-trivial ideals J, K and L such that J, K, $L \neq \mathfrak{m}^i$ for every i.*

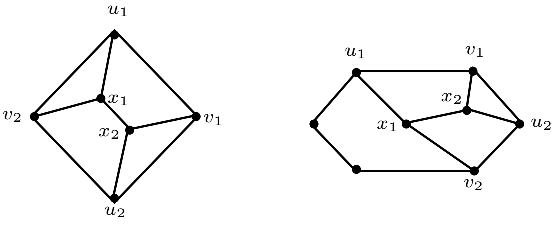
Theorem 3. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a commutative ring with identity where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$ and $n \geq 2$. Let n_i be the nilpotency of $\mathfrak{m}_i$. Then $\gamma(\mathbb{AG}(R)) = 1$ if and only if $n = 2$ and one of the following condition holds:*

- *(i) $n_1 = 2$, $n_2 = 4$, $\mathfrak{m}_1$ is the only non-trivial ideal in R_1 and $\mathfrak{m}_2, \mathfrak{m}_2^2, \mathfrak{m}_2^3$ are the only non-trivial ideals in R_2;*
- *(ii) $n_1 = 4$, $n_2 = 2$, $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3$ are the only non-trivial ideals in R_1 and $\mathfrak{m}_2$ is the only non-trivial ideal in R_2.*

Proof. Assume that $\gamma(\mathbb{AG}(R)) = 1$. Suppose that $n > 2$. Consider the non-trivial ideals $u_1 = \mathfrak{m}_1^{n_1-1} \times (0) \times (0) \times \cdots \times (0)$, $u_2 = (0) \times \mathfrak{m}_2^{n_2-1} \times (0) \times \cdots \times (0)$, $u_3 = \mathfrak{m}_1^{n_1-1} \times$

$\mathfrak{m}_2^{n_2-1}\times\cdots\times(0)$, $v_1=(0)\times(0)\times\mathfrak{m}_3\times(0)\cdots\times(0)$, $v_2=\mathfrak{m}_1\times(0)\times\mathfrak{m}_3\times(0)\times\cdots\times(0)$, $v_3=(0)\times\mathfrak{m}_2\times\mathfrak{m}_3\times(0)\times\cdots\times(0)$, $v_4=\mathfrak{m}_1\times\mathfrak{m}_2\times\mathfrak{m}_3\times(0)\times\cdots\times(0)$, $v_5=(0)\times(0)\times R_3\times(0)\times\cdots\times(0)$, $v_6=\mathfrak{m}_1\times(0)\times R_3\times(0)\times\cdots\times(0)$, $v_7=(0)\times\mathfrak{m}_2\times R_3\times(0)\times\cdots\times(0)$ in R. Then $u_iv_j=(0)$ for every i,j and so $K_{3,7}$ is a subgraph of $\mathbb{AG}(R)$. Hence by Lemma 2, $\gamma(\mathbb{AG}(R))>1$, a contradiction. Hence $n=2$.

Suppose that $n_i\geq 3$ for every $i=1,2$. Consider the subgraph G of $\mathbb{AG}(R)$ induced by the non-trivial ideals $u_1=(0)\times\mathfrak{m}_2^{n_2-1}$, $u_2=\mathfrak{m}_1^{n_1-1}\times(0)$, $u_3=\mathfrak{m}_1^{n_1-1}\times\mathfrak{m}_2^{n_2-1}$, $v_1=\mathfrak{m}_1^{n_1-2}\times(0)$, $v_2=(0)\times\mathfrak{m}_2^{n_2-2}$, $v_3=\mathfrak{m}_1^{n_1-2}\times\mathfrak{m}_2^{n_2-2}$, $v_4=\mathfrak{m}_1^{n_1-1}\times\mathfrak{m}_2^{n_2-2}$, $v_5=\mathfrak{m}_1^{n_1-2}\times\mathfrak{m}_2^{n_2-1}$, $x_1=R_1\times(0)$, $x_2=(0)\times R_2$, $x_3=R_1\times\mathfrak{m}_2^{n_2-1}$, $x_4=\mathfrak{m}_1^{n_1-1}\times R_2$, $x_5=R_1\times\mathfrak{m}_2^{n_2-2}$, $x_6=\mathfrak{m}_1^{n_1-2}\times R_2$ of R. Let $G'=G-\{x_3,x_4,x_5,x_6\}-\{u_1u_2,u_2u_3,u_1u_3,v_1v_2,v_1v_4,v_2v_5,v_4v_5\}$ and $G''=G'-\{x_1,x_2\}$. Then $G''\cong K_{3,5}$ and so $\gamma(G'')=1$. Since $\gamma(G)=1$ and $\gamma(G'')\leq\gamma(G')\leq\gamma(G)$, we get $\gamma(G')=1$. Note that $|V(G')|=10$, $|E(G')|=20$. Then by Euler's formula, there are 10 faces when drawing G' on a torus. Fix a representation of G' and let $\{F'_1,\ldots,F'_{10}\}$ be the set of faces of G' corresponding to the representation. Let $\{F''_1,\ldots,F''_n\}$ be the set of faces of G'' obtained by deleting x_1,x_2 and all the edges incident with x_1,x_2 from the representation of G'. Notice that $G''\cong K_{3,5}$. From the fact that $n-m+f=2-2g$, $K_{3,5}$ has 7 faces, six with 4 boundary edges and one with 6 boundary edges. So $n=7$. Moreover, for every i, each boundary of F''_i cannot have consecutive repetition of a single edge. Therefore in $K_{3,5}$, the only way to have a closed walk of length 6 without consecutive repetition of single edge is to have 6−cycle. Then in $K_{3,5}$, all faces boundaries are 4−cycles but with one 6−cycle. We may assume that the boundary of F''_7 is 6. Now $\{F'_1,\ldots,F'_{10}\}$ can be recovered by inserting x_1,x_2 and all the edges incident with x_1,x_2 into the representation corresponds to $\{F''_1,\ldots,F''_7\}$.



(a) (b)

Fig 2.2

Note that $x_1x_2\in E(G')$. Hence x_1, x_2 should be inserted to the same face say F''_m of G'' to avoid crossing. Also note that $x_1u_1,x_1v_2,x_2u_2,x_2v_1\in E(G')$ and therefore u_1,v_2,u_2,v_1 are the boundary vertices of F''_m. Consider the following edges of G: $e_1=x_1u_1$, $e_2=x_1v_2$, $e_3=x_2u_2$, $e_4=x_2v_1$, $e_5=x_1x_2$, $e_6=u_1u_2$, $e_7=v_1v_2$. After inserting x_1, x_2 and e_i, $i=1$ to 5 into the face F''_m, $m\neq 7$, we obtain Fig 2.2(a) as above. Then the edge e_6 can be inserted into the face F''_7. But there is no other face with v_1 and v_2 as the boundary vertices and so there is no way to insert the edge e_7 without crossing in the embedding of G. After inserting x_1, x_2 and e_i, $i=1$ to 5 into the face F''_7, we obtain Fig 2.2(b) as above. Then the edge e_6 can be inserted into the face F''_m where $m\neq 7$. But there is no other face with v_1 and v_2 as the boundary vertices and so there is no way to insert the edge e_7 without crossing in the embedding of G. Hence we conclude that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $n_i = 2$ for some i.

Without loss of generality, assume that $n_1 = 2$. Suppose that $n_2 > 4$. Consider the non-trivial ideals $u_1 = (0) \times \mathfrak{m}_2^{n_2-1}$, $u_2 = \mathfrak{m}_1 \times \mathfrak{m}_2^{n_2-1}$, $u_3 = \mathfrak{m}_1 \times (0)$, $v_1 = (0) \times \mathfrak{m}_2$, $v_2 = (0) \times \mathfrak{m}_2^{n_2-2}$, $v_3 = (0) \times \mathfrak{m}_2^{n_2-3}$, $v_4 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $v_5 = \mathfrak{m}_1 \times \mathfrak{m}_2^{n_2-2}$, $v_6 = \mathfrak{m}_1 \times \mathfrak{m}_2^{n_2-3}$ in R. Then $u_iv_j = (0)$ for every i, j so $K_{3,6}$ is a subgraph of $\mathbb{AG}(R)$. Further, the subgraph K of $\mathbb{AG}(R)$ induced by the vertices $\{u_1, u_2, u_3\}$ is K_3, $V(K) \subset V(K_{3,6})$ and $E(K) \cap E(K_{3,6}) = \emptyset$. Since K_3 cannot be embedded in the torus along with an embedding with only rectangle as faces, one cannot have an embedding of K and $K_{3,6}$ together in a torus. This implies that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $n_2 \leq 4$. Suppose that $n_2 = 4$. Let J_1 be any non-trivial ideal in R_2 such that $J_1 \neq \mathfrak{m}_2^i$ for $i = 1, 2, 3$. Consider the non-trivial ideals $u_1 = (0) \times \mathfrak{m}_2^3$, $u_2 = \mathfrak{m}_1 \times \mathfrak{m}_2^3$, $u_3 = \mathfrak{m}_1 \times (0)$, $v_1 = (0) \times \mathfrak{m}_2^2$, $v_2 = \mathfrak{m}_1 \times \mathfrak{m}_2^2$, $v_3 = (0) \times \mathfrak{m}_2$, $v_4 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $v_5 = (0) \times J_1$, $v_6 = \mathfrak{m}_1 \times J_1$ in R. Then $u_iv_j = (0)$ for every i, j and so $K_{3,6}$ is a subgraph of $\mathbb{AG}(R)$. Further, the subgraph H of $\mathbb{AG}(R)$ induced by the vertices $\{u_1, u_2, u_3\}$ is K_3, $V(H) \subset V(K_{3,6})$ and $E(H) \cap E(K_{3,6}) = \emptyset$. Since K_3 cannot be embedded in the torus along with an embedding with only rectangle as faces, one cannot have an embedding of H and $K_{3,6}$ together in a torus. This implies that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $\mathfrak{m}_2, \mathfrak{m}_2^2, \mathfrak{m}_2^3$ are the only non-trivial ideal in R_2.

Let I_1 be any non-trivial ideal in R_1 such that $I_1 \neq \mathfrak{m}_1$. Consider the non-trivial ideals $u_1 = (0) \times \mathfrak{m}_2^3$, $u_2 = \mathfrak{m}_1 \times \mathfrak{m}_2^3$, $u_3 = I_1 \times \mathfrak{m}_2^3$, $v_1 = (0) \times \mathfrak{m}_2^2$, $v_2 = \mathfrak{m}_1 \times \mathfrak{m}_2^2$, $v_3 = I_1 \times \mathfrak{m}_2^2$, $v_4 = \mathfrak{m}_1 \times (0)$, $v_5 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $v_6 = I_1 \times \mathfrak{m}_2$, $v_7 = (0) \times \mathfrak{m}_2$ in R. Then $u_iv_j = (0)$ for every i, j and so $K_{3,7}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $\mathfrak{m}_1$ is the only non-trivial ideal in R_1.

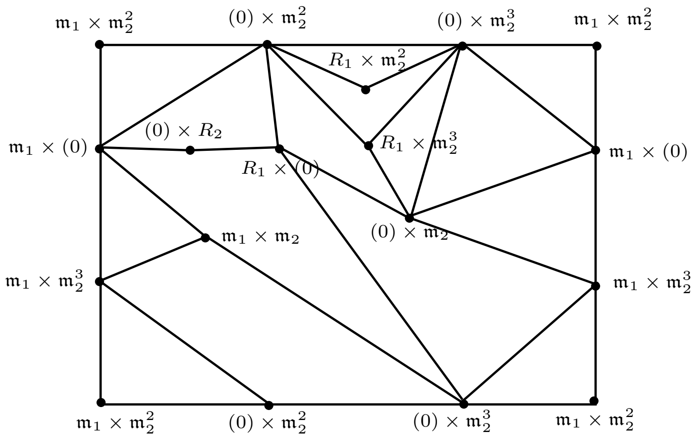


Fig 2.3: Torus embedding of $\mathbb{AG}(R_1 \times R_2)$ with $n_1 = 2$ and $n_2 = 4$

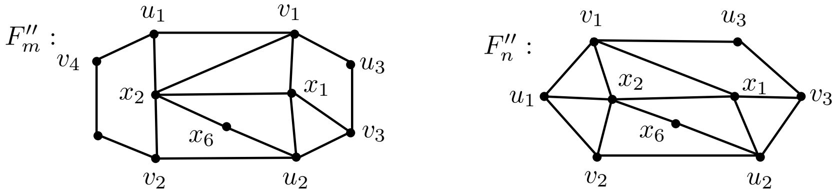
Suppose that $n_2 = 3$. Let J_1, J_2, J_3 be the distinct non-trivial ideals in R_2 such that

$J_i \neq \mathfrak{m}_1, \mathfrak{m}_1^2$, $(i = 1, 2)$. Consider the non-trivial ideals $u_1 = (0) \times \mathfrak{m}_2^2$, $u_2 = \mathfrak{m}_1 \times (0)$, $u_3 = \mathfrak{m}_1 \times \mathfrak{m}_2^2$, $v_1 = (0) \times \mathfrak{m}_2$, $v_2 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $v_3 = (0) \times J_1$, $v_4 = \mathfrak{m}_1 \times J_1$, $v_5 = (0) \times J_2$, $v_6 = \mathfrak{m}_1 \times J_2$, $v_7 = (0) \times J_3$ in R. Then $u_i v_j = (0)$ for every i, j and so $K_{3,7}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence by Proposition 1 and Lemma 6, $\mathfrak{m}_2, \mathfrak{m}_2^2$ are the only non-trivial ideals in R_2.
Let I_1 be any non-trivial ideal in R_1 such that $I_1 \neq \mathfrak{m}_1$. Consider the non-trivial ideals $u_1 = (0) \times \mathfrak{m}_2^2$, $u_2 = \mathfrak{m}_1 \times (0)$, $u_3 = I_1 \times (0)$, $v_1 = (0) \times \mathfrak{m}_2$, $v_2 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $v_3 = I_1 \times \mathfrak{m}_2$, $v_4 = \mathfrak{m}_1 \times \mathfrak{m}_2^2$, $v_5 = I_1 \times \mathfrak{m}_2^2$, $v_6 = [R_1 \times (0), (0) \times R_2]$ in R. Then $u_i v_j = (0)$ for every i, j and so $K_{3,6}$ is a subgraph of $\mathbb{AG}(R)$. Further, the subgraph H of $\mathbb{AG}(R)$ induced by the vertices $\{u_1, u_2, u_3\}$ is K_3, $V(H) \subset V(K_{3,6})$ and $E(H) \cap E(K_{3,6}) = \emptyset$. Since K_3 cannot be embedded in the torus along with an embedding with only rectangle as faces, one cannot have an embedding of H and $K_{3,6}$ together in a torus. This implies that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $\mathfrak{m}_1$ is the only non-trivial ideal in R_1. Therefore by Theorem 1, $\gamma(\mathbb{AG}(R)) = 0$, a contradiction.
Suppose that $n_2 = 2$. By Proposition 1, R_2 has at least 3 non-trivial ideals different from $\mathfrak{m}_2$. Let J_1, J_2, J_3 be the distinct non-trivial ideals in R_2 such that $J_i \neq \mathfrak{m}_2$ for all i. Consider the non-trivial ideals $u_1 = (0) \times \mathfrak{m}_2$, $u_2 = (0) \times J_1$, $u_3 = (0) \times J_2$, $v_1 = \mathfrak{m}_1 \times (0)$, $v_2 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $v_3 = \mathfrak{m}_1 \times J_1$, $v_4 = \mathfrak{m}_1 \times J_2$, $v_5 = R_1 \times (0)$, $v_6 = R_1 \times \mathfrak{m}_2$, $v_7 = R_1 \times J_1$ in R. Then $u_i v_j = (0)$ for every i, j and so $K_{3,7}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence by Lemma 6, $\mathfrak{m}_2$ is the only non-trivial ideal in R_2. Similarly one can prove that $\mathfrak{m}_1$ is the only non-trivial ideal in R_1. Hence by Theorem 1, $\gamma(\mathbb{AG}(R)) = 0$, a contradiction. Similar argument for other possibilities also.
Converse follows from Fig 2.3. □

Theorem 4. *Let $R = R_1 \times R_2 \times F_1$ be a commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$ and each F_1 is a field. Let n_i be the nilpotency of $\mathfrak{m}_i$. Then $\gamma(\mathbb{AG}(R)) > 1$.*

Proof. Suppose that $n_i > 2$ for some i. Let us assume that $n_2 > 2$. Consider the non-trivial ideals $u_1 = (0) \times (0) \times F_1$, $u_2 = \mathfrak{m}_1^{n_1-1} \times (0) \times F_1$, $u_3 = (0) \times \mathfrak{m}_2^{n_2-1} \times F_1$, $u_4 = \mathfrak{m}_1^{n_1-1} \times \mathfrak{m}_2^{n_2-1} \times F_1$, $v_1 = \mathfrak{m}_1 \times (0) \times (0)$, $v_2 = (0) \times \mathfrak{m}_2^{n_2-1} \times (0)$, $v_3 = \mathfrak{m}_1 \times \mathfrak{m}_2^{n_2-1} \times (0)$, $v_4 = (0) \times \mathfrak{m}_2 \times (0)$, $v_5 = \mathfrak{m}_1 \times \mathfrak{m}_2 \times (0)$ in R. Then $u_i v_j = (0)$ for every i, j and so $K_{4,5}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$. Similarly one can prove that $\gamma(\mathbb{AG}(R)) > 1$ in other possibilities also.
Suppose that $n_1 = 2$ and $n_2 = 2$. Assume that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the only non-trivial ideal in R_1 and R_2 respectively. Consider the non-trivial ideals $u_1 = \mathfrak{m}_1 \times (0) \times (0)$, $u_2 = (0) \times \mathfrak{m}_2 \times (0)$, $u_3 = \mathfrak{m}_1 \times \mathfrak{m}_2 \times (0)$, $v_1 = (0) \times (0) \times F_1$, $v_2 = \mathfrak{m}_1 \times (0) \times F_1$, $v_3 = (0) \times \mathfrak{m}_2 \times F_1$, $v_4 = \mathfrak{m}_1 \times \mathfrak{m}_2 \times F_1$, $x_1 = R_1 \times (0) \times (0)$, $x_2 = (0) \times R_2 \times (0)$, $x_3 = \mathfrak{m}_1 \times R_2 \times (0)$, $x_4 = R_1 \times \mathfrak{m}_2 \times (0)$, $x_5 = (0) \times R_2 \times F_1$, $x_6 = R_1 \times (0) \times F_1$, $x_7 = \mathfrak{m}_1 \times R_2 \times F_1$, $x_8 = R_1 \times \mathfrak{m}_2 \times F_1$, $x_9 = R_1 \times R_2 \times (0)$ of R. Let $G = \mathbb{AG}(R)$, $G' = G - \{x_3, x_4, x_7, x_8, x_9\} - \{u_1u_2, u_2u_3, u_1u_3\}$ and $G'' = G' - \{x_1, x_2, x_5, x_6\}$. Then $G'' \cong K_{3,4}$ and so $\gamma(G'') = 1$. Since $\gamma(G) = 1$ and $\gamma(G'') \leq \gamma(G') \leq \gamma(G)$, we get $\gamma(G') = 1$. Note that $|V(G')| = 11$, $|E(G')| = 23$. Then by Euler's formula, there are

12 faces when drawing G' on a torus. Fix a representation of G' and let $\{F'_1, \ldots, F'_{12}\}$ be the set of faces of G' corresponding to the representation. Let $\{F''_1, \ldots, F''_n\}$ be the set of faces of G'' obtained by deleting x_1, x_2, x_5, x_6 and all the edges incident with x_1, x_2, x_5, x_6 from the representation of G'. Notice that $G'' \cong K_{3,4}$. From the fact that $n - m + f = 2 - 2g$, $K_{3,4}$ has 5 faces, one octagonal face and 4 rectangular faces, or two hexagonal faces and 3 rectangular faces. So $n = 5$. Moreover, for every i, each boundary of F''_i cannot have consecutive repetition of a single edge. Therefore in $K_{3,4}$, the only way to have a closed walk of length 6 without consecutive repetition of single edge is to have $6-$cycle and the only way to have a closed walk of length 8 without consecutive repetition of single edge is to have $8-$cycle. Then in $K_{3,4}$, all faces boundaries are $4-$cycles but with two $6-$cycle or one $8-$cycle. Now $\{F'_1, \ldots, F'_{12}\}$ can be recovered by inserting x_1, x_2, x_5, x_6 and all the edges incident with x_1, x_2, x_5, x_6 into the representation corresponds to $\{F''_1, \ldots, F''_5\}$.



(a) (b)

Fig 2.4

Note that $x_1x_2 \in E(G')$. Hence x_1, x_2 should be inserted to the same faces say F''_m of G'' to avoid crossing. Also note that $x_1u_2, x_1v_1, x_1v_3, x_2u_1, x_2v_1, x_2v_2 \in E(G')$ and therefore u_1, u_2, v_1, v_2, v_3 are the boundary vertices of F''_m. Consider the following edges of G. Let $e_1 = x_1x_2$, $e_2 = x_1u_2$, $e_3 = x_1v_1$, $e_4 = x_1v_3$, $e_5 = x_2u_1$, $e_6 = x_2v_1$, $e_7 = x_2v_2$, $e_8 = x_2x_6$, $e_9 = x_6u_2$, $e_{10} = x_1x_5$, $e_{11} = x_5u_1$. From this, it is clear that x_1, x_2, x_5, x_6 should be inserted into the same face. Suppose if we insert x_1, x_2, x_6 and e_i, $i = 1$ to 9 in the octagonal face F''_m, then we obtain the Fig 2.4(a). However from Fig 2.4(a), it is clear that there is no way to insert the vertex x_5 into the faces F''_m without crossing in the embedding of G'. Suppose if we insert x_1, x_2, x_6 and e_i, $i = 1$ to 9 in the hexagonal face F''_n, then we obtain the Fig 2.4(b). However from Fig 2.4(b), it is clear that there is no way to insert x_5 into the face F''_n without crossing in the embedding of G'. Hence we conclude that $\gamma(\mathbb{AG}(R)) > 1$. □

Corollary 1. *Let $R = R_1 \times R_2 \times \cdots \times R_n \times F_1 \times F_2 \times \cdots \times F_m$ be a commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$ and $n \geq 2$ and each F_j is a field with $m \geq 1$. Let n_i be the nilpotency of $\mathfrak{m}_i$. Then $\gamma(\mathbb{AG}(R)) > 1$.*

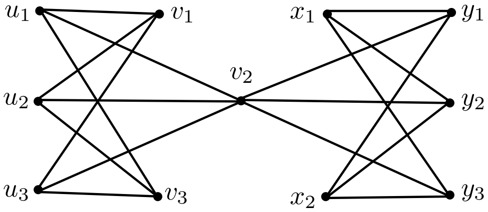


Fig 2.5

Theorem 5. *Let $R = R_1 \times F_1 \times F_2 \times \cdots \times F_m$ be a commutative ring with identity, where $(R_1, \mathfrak{m}_1)$ is a local ring with $\mathfrak{m}_1 \neq \{0\}$ and each F_j is a field with $m \geq 3$. Let n_1 be the nilpotency of $\mathfrak{m}_1$. Then $\gamma(\mathbb{AG}(R)) > 1$.*

Proof. Assume that $m > 2$. Consider the set $\Omega = \{u_1, u_2, u_3, v_1, v_2, v_3, x_1, x_2, y_1, y_2, y_3\}$ where $u_1 = (0) \times (0) \times F_2 \times (0) \times \cdots \times (0)$, $u_2 = (0) \times (0) \times (0) \times F_3 \times \cdots \times (0)$, $u_3 = (0) \times (0) \times F_2 \times F_3 \times \cdots \times (0)$, $v_1 = R_1 \times (0) \times (0) \times (0) \times \cdots \times (0)$, $v_2 = (0) \times F_1 \times (0) \times (0) \times \cdots \times (0)$, $v_3 = R_1 \times F_1 \times (0) \times (0) \times \cdots \times (0)$, $x_1 = \mathfrak{m}_1^{n_1-1} \times (0) \times (0) \times (0) \times \cdots \times (0)$, $x_2 = \mathfrak{m}_1^{n_1-1} \times F_1 \times (0) \times (0) \times \cdots \times (0)$, $y_1 = \mathfrak{m}_1 \times (0) \times F_2 \times (0) \times \cdots \times (0)$, $y_2 = \mathfrak{m}_1 \times (0) \times (0) \times F_3 \times \cdots \times (0)$, $y_3 = \mathfrak{m}_1 \times (0) \times F_2 \times F_3 \times \cdots \times (0)$ are the non-trivial ideals in R. Then the subgraph induced by Ω contains two blocks, both isomorphic to $K_{3,3}$ as in Fig 2.5 and by Lemma 2, $\gamma(K_{3,3}) = 1$. Hence by Lemma 3, $\gamma(\mathbb{AG}(R)) > 1$. □

Theorem 6. *Let $R = R_1 \times F_1 \times F_2$ be a commutative ring with identity, where $(R_1, \mathfrak{m}_1)$ is a local ring with $\mathfrak{m}_1 \neq \{0\}$ and F_1, F_2 are fields. Let n_1 be the nilpotency of $\mathfrak{m}_1$. Then $\gamma(\mathbb{AG}(R)) = 1$ if and only if $n_1 = 3$ and $\mathfrak{m}_1, \mathfrak{m}_1^2$ are the only non-trivial ideals in R_1.*

Proof. Assume that $\gamma(\mathbb{AG}(R)) = 1$. Suppose that $n_1 > 3$. Consider the set $\Omega_1 = \{u_1, u_2, u_3, v_1, v_2, v_3, x_1, x_2, y_1, y_2, y_3\}$ where $u_1 = (0) \times F_1 \times (0)$, $u_2 = (0) \times (0) \times F_2$, $u_3 = (0) \times F_1 \times F_2$, $v_1 = R_1 \times (0) \times (0)$, $v_2 = \mathfrak{m}_1^{n_1-1} \times (0) \times (0)$, $v_3 = \mathfrak{m}_1 \times (0) \times (0)$, $x_1 = \mathfrak{m}_1^{n_1-1} \times (0) \times F_2$, $x_2 = \mathfrak{m}_1^{n_1-2} \times (0) \times F_2$, $y_1 = \mathfrak{m}_1^{n_1-1} \times F_1 \times (0)$, $y_2 = \mathfrak{m}_1^{n_1-2} \times F_1 \times (0)$, $y_3 = \mathfrak{m}_1^{n_1-2} \times (0) \times (0)$ are non-trivial ideals in R. Then the subgraph induced by Ω_1 contains two blocks, both isomorphic to $K_{3,3}$ as in Fig 2.5 and by Lemma 2, $\gamma(K_{3,3}) = 1$. Hence by Lemma 3, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $n_1 \leq 3$.
Suppose $n_1 = 3$. Let I be any non-trivial ideal in R_1 such that $I \neq \mathfrak{m}_1, \mathfrak{m}_1^2$. Consider the non-trivial ideals $u_1 = \mathfrak{m}_1 \times (0) \times (0)$, $u_2 = \mathfrak{m}_1 \times (0) \times F_2$, $u_3 = I \times (0) \times F_2$, $v_1 = \mathfrak{m}_1^2 \times F_1 \times (0)$, $v_2 = \mathfrak{m}_1^2 \times (0) \times (0)$, $v_3 = (0) \times F_1 \times (0)$, $x_1 = \mathfrak{m}_1^2 \times (0) \times F_2$, $x_2 = (0) \times (0) \times F_2$, $y_1 = \mathfrak{m}_1 \times F_1 \times (0)$, $y_2 = I \times F_1 \times (0)$, $y_3 = I \times (0) \times (0)$ in R. Then the subgraph induced by $\Omega_2 = \{u_1, u_2, u_3, v_1, v_2, v_3, x_1, x_2, y_1, y_2, y_3\}$ contains two blocks, both isomorphic to $K_{3,3}$ as in Fig 2.5 and by Lemma 2, $\gamma(K_{3,3}) = 1$. Hence by Lemma 3, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $\mathfrak{m}_1, \mathfrak{m}_1^2$ are the only non-trivial ideals in R_1.

Suppose $n_1 = 2$. By Proposition 1, R_1 has at least three distinct non-trivial ideals different from $\mathfrak{m}_1$. Let I_1, I_2, I_3 be the distinct non-trivial ideals in R_1 such that $I_i \neq \mathfrak{m}_1$ for all i. Consider the non-trivial ideals $u_1 = \mathfrak{m}_1 \times (0) \times (0)$, $u_2 = \mathfrak{m}_1 \times F_1 \times (0)$, $u_3 = I_1 \times F_1 \times (0)$, $u_4 = (0) \times F_1 \times (0)$, $v_1 = I_1 \times (0) \times (0)$, $v_2 = \mathfrak{m}_1 \times (0) \times F_2$, $v_3 = I_1 \times (0) \times F_2$, $v_4 = (0) \times (0) \times F_2$, $v_5 = I_2 \times (0) \times (0)$, $v_6 = I_3 \times (0) \times (0)$. Then $u_i v_j = (0)$ for every i, j and so $K_{4,5}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence by Lemma 6, $\mathfrak{m}_1$ is the only non-trivial ideal in R_1. Then by Theorem 1, $\gamma(\mathbb{AG}(R)) = 0$, a contradiction.

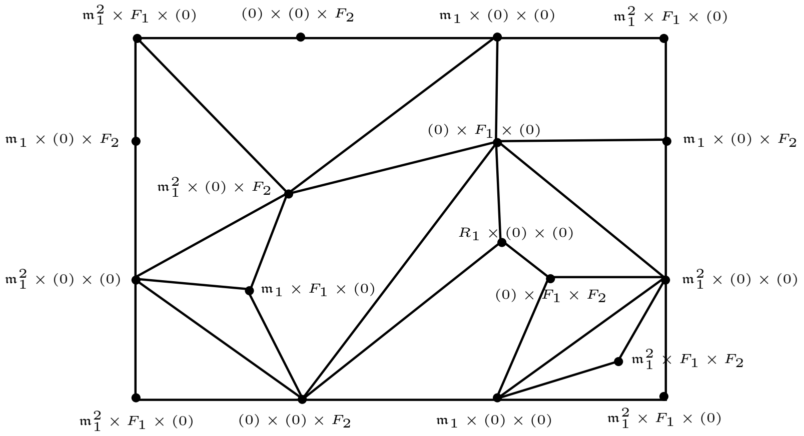


Fig 2.6: Torus embedding of $\mathbb{AG}(R_1 \times F_1 \times F_2)$ with $n_1 = 3$

Converse follows from Fig 2.6. □

Theorem 7. *Let $R = R_1 \times F_1$ be a commutative ring with identity, where each $(R_1, \mathfrak{m}_1)$ is a local ring with $\mathfrak{m}_1 \neq \{0\}$ and each F_1 is a field. Let n_1 be the nilpotency of $\mathfrak{m}_1$. Then $\gamma(\mathbb{AG}(R)) = 1$ if and only if one of the following condition holds:*

- (i) *$n_1 = 3$ and one of the following condition holds:*
    - (a) *R_1 has exactly 7 distinct non-trivial ideals, say $\mathfrak{m}_1, \mathfrak{m}_1^2, I_1, I_2, I_3, I_4, I_5$ with $I_i\mathfrak{m}_1 \neq (0)$ for every i and $I_jI_k = (0)$ for at most one $k \neq j$.*
    - (b) *R_1 has exactly 6 distinct non-trivial ideals, say $\mathfrak{m}_1, \mathfrak{m}_1^2, I_1, I_2, I_3, I_4$ with $I_i\mathfrak{m}_1 \neq (0)$ for every i and $I_jI_k = (0)$ for some $k \neq j$.*
    - (c) *R_1 has exactly 5 distinct non-trivial ideals, say $\mathfrak{m}_1, \mathfrak{m}_1^2, I_1, I_2, I_3$ with $I_i\mathfrak{m}_1 = (0)$ for some i and $I_jI_k \neq (0)$ for $k \neq j \neq i$.*
    - (d) *R_1 has exactly 5 distinct non-trivial ideals, say $\mathfrak{m}_1, \mathfrak{m}_1^2, I_1, I_2, I_3$ with $I_i\mathfrak{m}_1 \neq (0)$ for every i and $I_jI_k = (0)$ for every $k \neq j$.*
- (ii) *$n_1 = 4$ and one of the following condition holds:*

- (a) *R_1 has exactly 7 distinct non-trivial ideals, say $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, I_1, I_2, I_3, I_4$ with $I_i\mathfrak{m}_1 \neq (0)$ for every i and $I_1\mathfrak{m}_1^2 = (0)$, $I_j\mathfrak{m}_1^2 \neq (0)$ for every $j \neq 1$ and $I_1I_k \neq (0)$ for every $k \neq 1$, $I_sI_t = (0)$ for at most one $t \neq s$, $(s, t \neq 1)$.*
- (b) *R_1 has exactly 7 distinct non-trivial ideals, say $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, I_1, I_2, I_3, I_4$ with $I_i\mathfrak{m}_1 \neq (0)$, $I_i\mathfrak{m}_1^2 \neq (0)$ for every i and $I_jI_k = (0)$ for at most one $k \neq j$.*
- (c) *R_1 has exactly 6 distinct non-trivial ideals, say $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, I_1, I_2, I_3$ with $I_i\mathfrak{m}_1 \neq (0)$ for every i and $I_1\mathfrak{m}_1^2 = I_2\mathfrak{m}_1^2 = (0)$, $I_3\mathfrak{m}_1^2 \neq (0)$ and $I_1I_2 \neq (0)$, $I_jI_3 = (0)$ for some $j \neq 3$.*
- (d) *R_1 has exactly 6 distinct non-trivial ideals, say $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, I_1, I_2, I_3$ with $I_i\mathfrak{m}_1 \neq (0)$ for every i and $I_1\mathfrak{m}_1^2 = (0)$, $I_2\mathfrak{m}_1^2 \neq (0)$, $I_3\mathfrak{m}_1^2 \neq (0)$ and $I_jI_k = (0)$ for some $k \neq j$.*
- (e) *R_1 has exactly 6 distinct non-trivial ideals, say $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, I_1, I_2, I_3$ with $I_i\mathfrak{m}_1 \neq (0)$, $I_i\mathfrak{m}_1^2 \neq (0)$ for every i and $I_jI_k = (0)$ for some $k \neq j$.*

(iii) *$n_1 = 5$ and one of the following condition holds:*

- (a) *R_1 has exactly 7 distinct non-trivial ideals, say $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, \mathfrak{m}_1^4, I_1, I_2, I_3$ with $I_i\mathfrak{m}_1^j \neq (0)$ for every $i, j = 1, 2, 3$ and $I_kI_l = (0)$ for at most one $k \neq l$.*
- (b) *R_1 has exactly 4 distinct non-trivial ideals, say $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, \mathfrak{m}_1^4$.*

(iv) *$n_1 = 6$ and R_1 has exactly 5 distinct non-trivial ideals, say $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, \mathfrak{m}_1^4, \mathfrak{m}_1^5$.*

Proof. Assume that $\gamma(\mathbb{AG}(R)) = 1$. Suppose $n_1 > 6$. Consider the non-trivial ideals $u_1 = \mathfrak{m}_1^{n_1-1} \times (0)$, $u_2 = \mathfrak{m}_1^{n_1-2} \times (0)$, $u_3 = \mathfrak{m}_1^{n_1-3} \times (0)$, $v_1 = (0) \times F_1$, $v_2 = \mathfrak{m}_1^{n_1-1} \times F_1$, $v_3 = \mathfrak{m}_1^{n_1-2} \times F_1$, $v_4 = \mathfrak{m}_1^{n_1-3} \times F_1$, $v_5 = \mathfrak{m}_1^{n_1-4} \times F_1$, $v_6 = \mathfrak{m}_1^{n_1-4} \times (0)$ in R. Then $u_iv_j = (0)$ for every i, j and so $K_{3,6}$ is a subgraph of $\mathbb{AG}(R)$. Recall that the genus of $K_{3,6}$ is one and hence one can fix an embedding of $K_{3,6}$ on the surfaces of torus. By Euler's formula, there are 9 faces in the embedding of $K_{3,6}$, say $\{F_1, \ldots, F_9\}$. Let s_{F_i} be the length of the faces F_i. Note that $\sum\limits_{i=1}^{9} s_{F_i} = 36$ and $s_{F_i} \geq 4$ for every i. Thus $s_{F_i} = 4$ for every i. Further, the subgraph H of $\mathbb{AG}(R)$ induced by the vertices $\{u_1, u_2, u_3\}$ is K_3, $V(H) \subset V(K_{3,6})$ and $E(H) \cap E(K_{3,6}) = \emptyset$. Since K_3 cannot be embedded in the torus along with an embedding with only rectangle as faces, one cannot have an embedding of H and $K_{3,6}$ together in a torus. This implies that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $n_1 \leq 6$.

Case 1: $n_1 = 2$.

Suppose there is an ideal I of R_1 such that $I \neq \mathfrak{m}_1$. Then by Proposition 1, R_1 has at least three distinct non-trivial ideals I_1, I_2 and I_3 such that $\mathfrak{m}_1 \notin \{I_1, I_2, I_3\}$. Consider the non-trivial ideals $u_1 = \mathfrak{m}_1 \times (0)$, $u_2 = I_1 \times (0)$, $u_3 = I_2 \times (0)$, $u_4 = I_3 \times (0)$, $v_1 = (0) \times F_1$, $v_2 = \mathfrak{m}_1 \times F_1$, $v_3 = I_1 \times F_1$, $v_4 = I_2 \times F_1$, $v_5 = I_3 \times F_1$ in R. Then $u_iv_j = (0)$ for every i, j and so $K_{4,5}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Then by Proposition 1 and Lemma 6, $\mathfrak{m}_1$ is the only non-trivial ideal in R_1. Therefore by Theorem 1, $\gamma(\mathbb{AG}(R)) = 0$, a contradiction.

Case 2: $n_1 = 3$.

Suppose there is an ideal I of R_1 such that $I \neq \mathfrak{m}_1, \mathfrak{m}_1^2$. Then by Proposition 1, R_1 has at least three distinct non-trivial ideals different from $\mathfrak{m}_1, \mathfrak{m}_1^2$. Suppose that

R_1 has at least 6 non-trivial ideals I_1, I_2, I_3, I_4, I_5, I_6 such that $I_i \neq \mathfrak{m}_1$, $\mathfrak{m}_1^2$ for $1 \leq i \leq 6$. Consider the non-trivial ideals $u_1 = (0) \times F_1$, $u_2 = \mathfrak{m}_1^2 \times F_1$, $u_3 = \mathfrak{m}_1^2 \times (0)$, $v_1 = \mathfrak{m}_1 \times (0)$, $v_2 = I_1 \times (0)$, $v_3 = I_2 \times (0)$, $v_4 = I_3 \times (0)$, $v_5 = I_4 \times (0)$, $v_6 = I_5 \times (0)$, $v_7 = I_6 \times (0)$ in R. Then $u_i v_j = (0)$ for every i, j and so $K_{3,7}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence R_1 has at most 5 non-trivial ideals different from $\mathfrak{m}_1, \mathfrak{m}_1^2$. Then by Theorem 1 and Proposition 1, $5 \leq t_1 \leq 7$, where t_1 is the number of non-trivial ideals in R_1.

Subcase 2.1. Assume that R_1 has exactly 7 distinct non-trivial ideals, say $\mathfrak{m}_1$, $\mathfrak{m}_1^2$, I_1, I_2, I_3, I_4, I_5. Suppose $I_i \mathfrak{m}_1 = (0)$ for some i. Consider the non-trivial ideals $a_1 = (0) \times F_1$, $a_2 = \mathfrak{m}_1^2 \times F_1$, $a_3 = I_i \times F_1$, $b_1 = \mathfrak{m}_1^2 \times (0)$, $b_2 = I_1 \times (0)$, $b_3 = I_2 \times (0)$, $b_4 = I_3 \times (0)$, $b_5 = I_4 \times (0)$, $b_6 = I_5 \times (0)$, $b_7 = \mathfrak{m}_1 \times (0)$ in R. Then $a_i b_j = (0)$ for every i, j and so $K_{3,7}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_i \mathfrak{m}_1 \neq (0)$ for every i.

Suppose $I_j I_k = (0)$ for some $k \neq j$. Let us assume that $I_1 I_2 = (0)$ and $I_1 I_3 = (0)$. Consider the non-trivial ideals $u_1 = (0) \times F_1$, $u_2 = \mathfrak{m}_1^2 \times F_1$, $u_3 = \mathfrak{m}_1^2 \times (0)$, $v_1 = I_4 \times (0)$, $v_2 = I_5 \times (0)$, $v_3 = I_1 \times (0)$, $v_4 = I_2 \times (0)$, $v_5 = I_3 \times (0)$, $v_6 = \mathfrak{m}_1 \times (0)$, $x_1 = \mathfrak{m}_1 \times F_1$, $x_2 = I_4 \times F_1$, $x_3 = I_1 \times F_1$, $x_4 = I_2 \times F_1$, $x_5 = I_3 \times F_1$, $x_6 = I_5 \times F_1$, $x_7 = R_1 \times (0)$ in R. Let $G = \mathbb{AG}(R)$, $G' = G - \{x_1, x_2, x_6, x_7\} - \{u_1u_3, u_2u_3, v_3v_4, v_3v_5\}$ and $G'' = G' - \{x_3, x_4, x_5\}$. Then $G'' \cong K_{3,6}$ and so $\gamma(G'') = 1$. Since $\gamma(G) = 1$ and $\gamma(G'') \leq \gamma(G') \leq \gamma(G)$, we get $\gamma(G') = 1$. Note that $|V(G')| = 12$, $|E(G')| = 25$. Then by Euler's formula, there are 13 faces when drawing G' on a torus. Fix a representation of G' and let $\{F'_1, \ldots, F'_{13}\}$ be the set of faces of G' corresponding to the representation. Let $\{F''_1, \ldots, F''_n\}$ be the set of faces of G'' obtained by deleting x_3, x_4, x_5 and all the edges incident with x_3, x_4, x_5 from the representation of G'. Notice that $G'' \cong K_{3,6}$. By Euler formula, $K_{3,6}$ has 9 faces. So $n = 9$. Let s_{F_i} be the length of the faces F_i. Note that $\sum\limits_{i=1}^{9} s_{F_i} = 36$ and $s_{F_i} \geq 4$ for every i. Thus $s_{F_i} = 4$ for every i. Moreover, for every i, each boundary of F''_i cannot have consecutive repetition of a single edge. Therefore in $K_{3,6}$, the only way to have a closed walk of length 4 without consecutive repetition of single edge is to have 4−cycle. Then in $K_{3,6}$, all faces boundaries are 4−cycles. Now $\{F'_1, \ldots, F'_{13}\}$ can be recovered by inserting x_3, x_4, x_5 and all the edges incident with x_3, x_4, x_5 into the representation corresponds to $\{F''_1, \ldots, F''_9\}$.

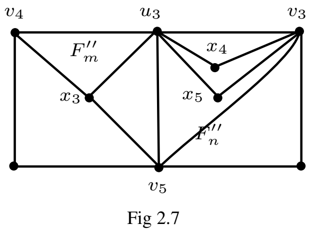


Fig 2.7

Consider the following edges of G. Let $e_1 = x_3u_3$, $e_2 = x_3v_4$, $e_3 = x_3v_5$, $e_4 = x_4u_3$, $e_5 = x_4v_3$, $e_6 = x_5u_3$, $e_7 = x_5v_3$, $e_8 = v_3v_5$, $e_9 = v_3v_4$. Now if we insert the vertices x_3, x_4, x_5 and the edges e_i where $1 \leq i \leq 8$ into the faces F''_m and F''_n in the embedding of G, then from Fig 2.7, it is clear that v_3, v_4 are in different faces and there is no other face containing v_3 and v_4 as boundary vertices. So there is no way to insert the edge e_9 without crossing in the embedding of G. Hence we conclude that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_1I_2 = (0)$ or $I_1I_3 = (0)$. Hence $I_jI_k = (0)$ for at most one $k \neq j$.

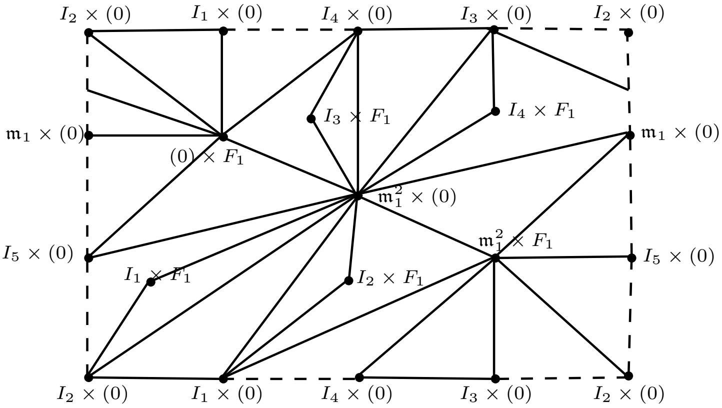


Fig 2.8: Torus embedding of $\mathbb{AG}(R_1 \times F_1)$ with $n_1 = 3$, $I_i\mathfrak{m}_1 \neq (0)\ \forall i$, $I_1I_2 = (0)$, $I_3I_4 = (0)$

Subcase 2.2. Assume that R_1 has exactly 6 distinct non-trivial ideals, say $\mathfrak{m}_1$, $\mathfrak{m}_1^2$, I_1, I_2, I_3, I_4. Suppose $I_i\mathfrak{m}_1 = (0)$ for some i. Consider the non-trivial ideals $a_1 = (0) \times F_1$, $a_2 = \mathfrak{m}_1^2 \times F_1$, $a_3 = I_i \times F_1$, $a_4 = \mathfrak{m}_1^2 \times (0)$, $b_1 = I_1 \times (0)$, $b_2 = I_2 \times (0)$, $b_3 = I_3 \times (0)$, $b_4 = I_4 \times (0)$, $b_5 = \mathfrak{m}_1 \times (0)$ in R. Then $a_ib_j = (0)$ for every i, j and so $K_{4,5}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_i\mathfrak{m}_1 \neq (0)$ for every i.

Suppose $I_jI_k = (0)$ for every $k \neq j$. Let us assume that $I_1I_2 = I_1I_3 = I_1I_4 = I_2I_3 = I_2I_4 = (0)$. Consider the non-trivial ideals $u_1 = (0) \times F_1$, $u_2 = \mathfrak{m}_1^2 \times F_1$, $u_3 = \mathfrak{m}_1^2 \times (0)$, $v_1 = I_1 \times (0)$, $v_2 = I_2 \times (0)$, $v_3 = I_3 \times (0)$, $v_4 = I_4 \times (0)$, $v_5 = \mathfrak{m}_1 \times (0)$, $x_1 = I_1 \times F_1$, $x_2 = I_2 \times F_1$, $x_3 = I_3 \times F_1$, $x_4 = I_4 \times F_1$, $x_5 = \mathfrak{m}_1 \times F_1$, $x_6 = R_1 \times (0)$ of R. Let $G = \mathbb{AG}(R)$, $G' = G - \{x_3, x_4, x_5, x_6\} - \{u_1u_3, u_2u_3, v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$ and $G'' = G' - \{x_1, x_2\}$. Then $G'' \cong K_{3,5}$ and so $\gamma(G'') = 1$. Since $\gamma(G) = 1$ and $\gamma(G'') \leq \gamma(G') \leq \gamma(G)$, we get $\gamma(G') = 1$. Note that $|V(G')| = 10$, $|E(G')| = 23$. Then by Euler's formula, there are 13 faces when drawing G' on a torus. Fix a representation of G' and let $\{F_1', \ldots, F_{13}'\}$ be the set of faces of G' corresponding to the representation. Let $\{F_1'', \ldots, F_n''\}$ be the set of faces of G'' obtained by deleting x_1, x_2 and all the edges incident with x_1, x_2 from the representation of G'. Notice that $G'' \cong K_{3,5}$. From the fact that $n - m + f = 2 - 2g$, $K_{3,5}$ has 7 faces, six with 4 boundary edges and one with 6 boundary edges. So $n = 7$. Moreover, for every i, each boundary of F_i'' cannot have consecutive repetition of a single edge. Therefore in $K_{3,5}$, the only way to have a closed walk of length 6 without consecutive repetition of single edge is to have 6−cycle. Then in $K_{3,5}$, all faces boundaries are 4−cycles but with one 6−cycle. We may assume that the boundary of F_7'' is 6. Now $\{F_1', \ldots, F_{13}'\}$

can be recovered by inserting x_1, x_2 and all the edges incident with x_1, x_2 into the representation corresponds to $\{F''_1, \ldots, F''_7\}$. Let $e_1 = x_1u_3$, $e_2 = x_1v_2$, $e_3 = x_1v_3$, $e_4 = x_1v_4$ be the edges incident with x_1 and $e_5 = x_2u_3$, $e_6 = x_2v_1$, $e_7 = x_2v_3$, $e_8 = x_2v_4$ be the edges incident with x_2. Since the vertices x_1 and x_2 have three neighbors in common, they should be inserted in different faces in the embedding of G'. Since x_1 is adjacent to u_3, v_2, v_3, v_4 and x_2 is adjacent to u_3, v_1, v_3, v_4, they should be inserted into the hexagonal faces. But $K_{3,5}$ contains only one hexagonal face. So there is no way to insert one of the vertices without crossing in the embedding of G'. Hence we conclude that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_jI_k \neq (0)$ for some $k \neq j$.

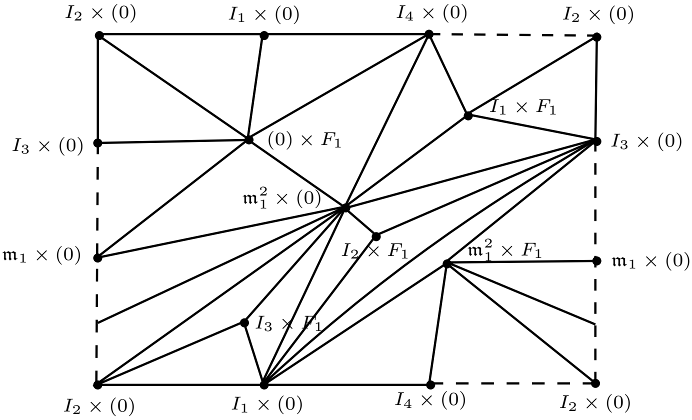


Fig 2.9: Torus embedding of $\mathbb{AG}(R_1 \times F_1)$ with $n_1 = 3$, $I_i\mathfrak{m}_1 \neq (0)$ and $I_1I_2 = I_1I_3 = I_1I_4 = I_2I_3 = (0)$

Subcase 2.3. Assume that R_1 has exactly 5 distinct non-trivial ideals, say $\mathfrak{m}_1$, $\mathfrak{m}_1^2$, I_1, I_2, I_3. Suppose $I_i\mathfrak{m}_1 = (0)$ for some i. Let us assume that $I_1\mathfrak{m}_1 = I_2\mathfrak{m}_1 = (0)$. Consider the non-trivial ideals $c_1 = (0) \times F_1$, $c_2 = \mathfrak{m}_1^2 \times F_1$, $c_3 = I_1 \times F_1$, $c_4 = I_2 \times F_1$, $d_1 = \mathfrak{m}_1^2 \times (0)$, $d_2 = I_1 \times (0)$, $d_3 = I_2 \times (0)$, $d_4 = I_3 \times (0)$, $d_5 = \mathfrak{m}_1 \times (0)$ in R. Then $c_id_j = (0)$ for every i, j and so $K_{4,5}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_i\mathfrak{m}_1 = (0)$ for at most one i.
Suppose that $I_1\mathfrak{m}_1 = (0)$ and $I_i\mathfrak{m}_1 \neq (0)$ for every $i \neq 1$. Suppose $I_2I_3 = (0)$. Consider the non-trivial ideals $u_1 = (0) \times F_1$, $u_2 = \mathfrak{m}_1^2 \times F_1$, $u_3 = I_1 \times F_1$, $v_1 = \mathfrak{m}_1^2 \times (0)$, $v_2 = I_1 \times (0)$, $v_3 = I_2 \times (0)$, $v_4 = I_3 \times (0)$, $v_5 = \mathfrak{m}_1 \times (0)$, $x_1 = I_2 \times F_1$, $x_2 = I_3 \times F_1$, $x_3 = \mathfrak{m}_1 \times F_1$, $x_4 = R_1 \times (0)$ in R. Let $G = \mathbb{AG}(R)$, $G' = G - \{x_3, x_4\} - \{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_3, v_2v_4, v_2v_5, v_3v_4\}$ and $G'' = G' - \{x_1, x_2\}$. Then $G'' \cong K_{3,5}$ and so $\gamma(G'') = 1$. Since $\gamma(G) = 1$ and $\gamma(G'') \leq \gamma(G') \leq \gamma(G)$, we get $\gamma(G') = 1$. Note that $|V(G')| = 10$, $|E(G')| = 21$. Then by Euler's formula, there are 11 faces when drawing G' on a torus. Fix a representation of G' and let $\{F'_1, \ldots, F'_{11}\}$ be the set of faces of G' corresponding to the representation. Let $\{F''_1, \ldots, F''_n\}$ be the set of faces of G'' obtained by deleting x_1, x_2 and all the edges incident with x_1, x_2 from the representation of G'. Notice that $G'' \cong K_{3,5}$. From the fact that $n - m + f = 2 - 2g$,

$K_{3,5}$ has 7 faces, six with 4 boundary edges and one with 6 boundary edges. So $n = 7$. Moreover, for every i, each boundary of F''_i cannot have consecutive repetition of a single edge. Therefore in $K_{3,5}$, the only way to have a closed walk of length 6 without consecutive repetition of single edge is to have $6-$cycle. Then in $K_{3,5}$, all faces boundaries are $4-$cycles but with one $6-$cycle. We may assume that the boundary of F''_7 is 6. Now $\{F'_1, \dots, F'_{11}\}$ can be recovered by inserting x_1, x_2 and all the edges incident with x_1, x_2 into the representation corresponds to $\{F''_1, \dots, F''_7\}$. Consider the following edges of G'. Let $e_1 = x_1v_1$, $e_2 = x_1v_2$, $e_3 = x_1v_4$, $e_4 = x_2v_1$, $e_5 = x_2v_2$, $e_6 = x_2v_3$. Since x_1 is adjacent to v_1, v_2, v_4 and x_2 is adjacent to v_1, v_2, v_3, they should be inserted into the faces with 6 boundary edges. But in $K_{3,5}$, there is only one face with 6 boundary edges. So there is no way to insert one of the vertices x_1, x_2 in the embedding of G' without crossing. Hence we conclude that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_2I_3 \neq (0)$.

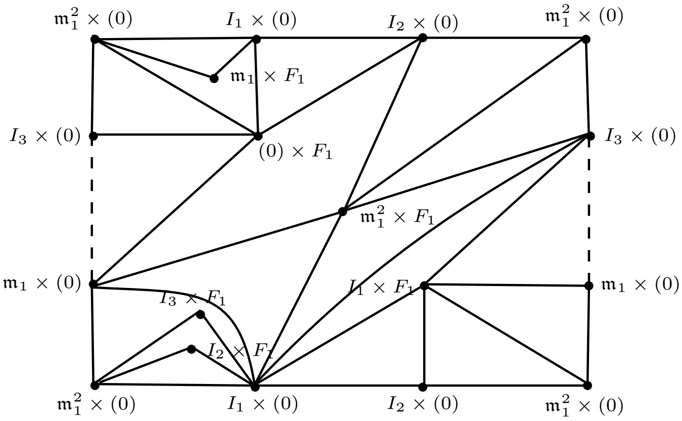


Fig 2.10: Torus embedding of $\mathbb{AG}(R_1 \times F_1)$ with $n_1 = 3$, $I_1\mathfrak{m}_1 = (0)$, $I_i\mathfrak{m}_1 \neq (0)\ \forall i \neq 1$ and $I_2I_3 \neq (0)$

Clearly proof of $(ii)(d)$ follows from proof of $(ii)(b)$.
Case 3: Suppose $n_1 = 4$.
Suppose there is an ideal I of R_1 such that $I \neq \mathfrak{m}_1^i$ for all $i = 1, 2$. Then by Proposition 1, R_1 has at least three distinct non-trivial ideals different from $\mathfrak{m}_1^i$ for all $i = 1, 2, 3$. Suppose that R_1 has at least 5 distinct non-trivial ideals I_1, I_2, I_3, I_4, I_5 such that $I_i \neq \mathfrak{m}_1^j$ for $i = 1$ to 5 and $j = 1$ to 3. Consider the non-trivial ideals $u_1 = (0) \times F_1$, $u_2 = \mathfrak{m}_1^3 \times F_1$, $u_3 = \mathfrak{m}_1^3 \times (0)$, $v_1 = \mathfrak{m}_1^2 \times (0)$, $v_2 = \mathfrak{m}_1 \times (0)$, $v_3 = I_1 \times (0)$, $v_4 = I_2 \times (0)$, $v_5 = I_3 \times (0)$, $v_6 = I_4 \times (0)$, $v_7 = I_5 \times (0)$ in R. Then $u_iv_j = (0)$ for every i, j and so $K_{3,7}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence R_1 has at most 4 non-trivial ideals different from $\mathfrak{m}_1^i$ for all $i = 1, 2, 3$. Then by Theorem 1 and Proposition 1, $6 \leq t_1 \leq 7$, where t_1 is the number of non-trivial ideals in R_1.
Subcase 3.1. Suppose R_1 has exactly 7 distinct non-trivial ideals, say $\mathfrak{m}_1$, $\mathfrak{m}_1^2$, $\mathfrak{m}_1^3$, I_1, I_2, I_3, I_4. Suppose $I_i\mathfrak{m}_1 = (0)$ for some i. Consider the non-trivial ideals

$a_1 = (0)\times F_1$, $a_2 = \mathfrak{m}_1^3\times F_1$, $a_3 = \mathfrak{m}_1^3\times(0)$, $a_4 = I_i\times F_1$, $b_1 = \mathfrak{m}_1^2\times(0)$, $b_2 = \mathfrak{m}_1\times(0)$, $b_3 = I_1\times(0)$, $b_4 = I_2\times(0)$, $b_5 = I_3\times(0)$ in R. Then $a_ib_j = (0)$ for every i, j and so $K_{4,5}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_i\mathfrak{m}_1 \neq (0)$ for every i.

Suppose that $I_i\mathfrak{m}_1^2 = (0)$ for $i = 1, 2$ and $I_j\mathfrak{m}_1^2 \neq (0)$ for $j = 3, 4$. Consider the non-trivial ideals $u_1 = (0)\times F_1$, $u_2 = \mathfrak{m}_1^3\times F_1$, $u_3 = \mathfrak{m}_1^3\times(0)$, $v_1 = \mathfrak{m}_1^2\times(0)$, $v_2 = I_1\times(0)$, $v_3 = I_2\times(0)$, $v_4 = I_3\times(0)$, $v_5 = I_4\times(0)$, $v_6 = \mathfrak{m}_1\times(0)$, $x_1 = \mathfrak{m}_1^2\times F_1$, $x_2 = I_1\times F_1$, $x_3 = I_2\times F_1$, $x_4 = I_3\times F_1$, $x_5 = I_4\times F_1$, $x_6 = \mathfrak{m}_1\times F_1$, $x_7 = R_1\times(0)$ of R. Let $G = \mathbb{AG}(R)$, $G' = G - \{x_2, x_3, x_4, x_5, x_6, x_7\} - \{u_1u_3, u_2u_3, v_1v_2, v_1v_3\}$ and $G'' = G' - \{x_1\}$. Then $G'' \cong K_{3,6}$ and so $\gamma(G'') = 1$. Since $\gamma(G) = 1$ and $\gamma(G'') \leq \gamma(G') \leq \gamma(G)$, we get $\gamma(G') = 1$. Note that $|V(G')| = 10$, $|E(G')| = 22$. Then by Euler's formula, there are 12 faces when drawing G' on a torus. Fix a representation of G' and let $\{F'_1, \ldots, F'_{12}\}$ be the set of faces of G' corresponding to the representation. Let $\{F''_1, \ldots, F''_n\}$ be the set of faces of G'' obtained by deleting x_1 and all the edges incident with x_1 from the representation of G'. Notice that $G'' \cong K_{3,6}$. By Euler formula, $K_{3,6}$ has 9 faces. So $n = 9$. Let s_{F_i} be the length of the faces F_i. Note that $\sum\limits_{i=1}^{9} s_{F_i} = 36$ and $s_{F_i} \geq 4$ for every i. Thus $s_{F_i} = 4$ for every i. Moreover, for every i, each boundary of F''_i cannot have consecutive repetition of a single edge. Therefore in $K_{3,6}$, the only way to have a closed walk of length 4 without consecutive repetition of single edge is to have 4−cycle. Then in $K_{3,6}$, all faces boundaries are 4−cycles. Now $\{F'_1, \ldots, F'_{12}\}$ can be recovered by inserting x_1 and all the edges incident with x_1 into the representation corresponds to $\{F''_1, \ldots, F''_9\}$. Also note that $x_1u_3, x_1v_1, x_1v_2, x_1v_3 \in E(G')$ and so u_3, v_1, v_2, v_3 should be the boundary vertices of F''_m. Since $G'' \cong K_{3,6}$ and $s_{F_i} = 4$ for every i, there is no faces containing the vertices u_3, v_1, v_2, v_3. So there is no way to insert x_1 without crossing in the embedding of G'. Hence we conclude that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_i\mathfrak{m}_1^2 = (0)$ for at most one i.

Suppose $I_1\mathfrak{m}_1^2 = (0)$ and $I_i\mathfrak{m}_1^2 \neq (0)$ for every $i \neq 1$. Suppose that $I_1I_2 = (0)$. Consider the non-trivial ideals $u_1 = (0)\times F_1$, $u_2 = \mathfrak{m}_1^3\times F_1$, $u_3 = \mathfrak{m}_1^3\times(0)$, $v_1 = \mathfrak{m}_1^2\times(0)$, $v_2 = I_1\times(0)$, $v_3 = I_2\times(0)$, $v_4 = I_3\times(0)$, $v_5 = I_4\times(0)$, $v_6 = \mathfrak{m}_1\times(0)$, $x_1 = \mathfrak{m}_1^2\times F_1$, $x_2 = I_1\times F_1$, $x_3 = I_2\times F_1$, $x_4 = I_3\times F_1$, $x_5 = I_4\times F_1$, $x_6 = \mathfrak{m}_1\times F_1$, $x_7 = R_1\times(0)$ of R. Let $G = \mathbb{AG}(R)$, $G' = G - \{x_3, x_4, x_5, x_6, x_7\} - \{u_1u_3, u_2u_3, v_1v_2, v_2v_3\}$ and $G'' = G' - \{x_1, x_2\}$. Then $G'' \cong K_{3,6}$ and so $\gamma(G'') = 1$. Since $\gamma(G) = 1$ and $\gamma(G'') \leq \gamma(G') \leq \gamma(G)$, we get $\gamma(G') = 1$. Note that $|V(G')| = 11$, $|E(G')| = 24$. Then by Euler's formula, there are 13 faces when drawing G' on a torus. Fix a representation of G' and let $\{F'_1, \ldots, F'_{12}\}$ be the set of faces of G' corresponding to the representation. Let $\{F''_1, \ldots, F''_n\}$ be the set of faces of G'' obtained by deleting x_1, x_2 and all the edges incident with x_1, x_2 from the representation of G'. Notice that $G'' \cong K_{3,6}$. By Euler formula, $K_{3,6}$ has 9 faces. So $n = 9$. Let s_{F_i} be the length of the faces F_i. Note that $\sum\limits_{i=1}^{9} s_{F_i} = 36$ and $s_{F_i} \geq 4$ for every i. Thus $s_{F_i} = 4$ for every i. Moreover, for every i, each boundary of F''_i cannot have consecutive repetition of a single edge. Therefore in $K_{3,6}$, the only way to have a closed walk of length

4 without consecutive repetition of single edge is to have 4−cycle. Then in $K_{3,6}$, all faces boundaries are 4−cycles. Now $\{F'_1, \dots, F'_{12}\}$ can be recovered by inserting x_1, x_2 and all the edges incident with x_1, x_2 into the representation corresponds to $\{F''_1, \dots, F''_9\}$.

Consider the following edges of G: $e_1 = x_1u_3$, $e_2 = x_1v_1$, $e_3 = x_1v_2$, $e_4 = x_2u_3$, $e_5 = x_2v_1$, $e_6 = x_2v_3$, $e_7 = v_2v_3$. After inserting x_1, x_2 and e_i, $i = 1$ to 6 into the faces F''_m and F''_n in the embedding of G, we obtain Fig 2.11. Then from Fig 2.11, it is clear that v_2 and v_3 are in different faces. So there is no way to insert the edge e_7 without crossing in the embedding of G. Hence we conclude that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_1I_2 \neq (0)$. Similarly one can prove that $I_1I_3 \neq (0)$ and $I_1I_4 \neq (0)$.

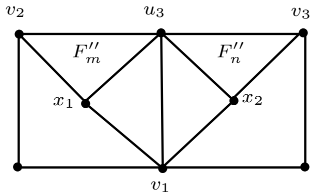


Fig 2.11

Suppose $I_2I_3 = (0)$ and $I_2I_4 = (0)$. Consider the following edges of G: $e_1 = x_3u_3$, $e_2 = x_3v_4$, $e_3 = x_3v_5$, $e_4 = x_4u_3$, $e_5 = x_4v_3$, $e_6 = x_5u_3$, $e_7 = x_5v_3$, $e_8 = v_3v_5$, $e_9 = v_3v_4$. If we insert the vertices x_3, x_4, x_5 and the edges e_i where $1 \leq i \leq 8$ into the faces in the embedding of G, then from Fig 2.7, it is clear that v_3 and v_4 are in different faces. So there is no way to insert the edge e_9 without crossing in the embedding of G. Hence we conclude that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_2I_3 = (0)$ or $I_2I_4 = (0)$. By the similar argument, $I_3I_4 \neq (0)$.

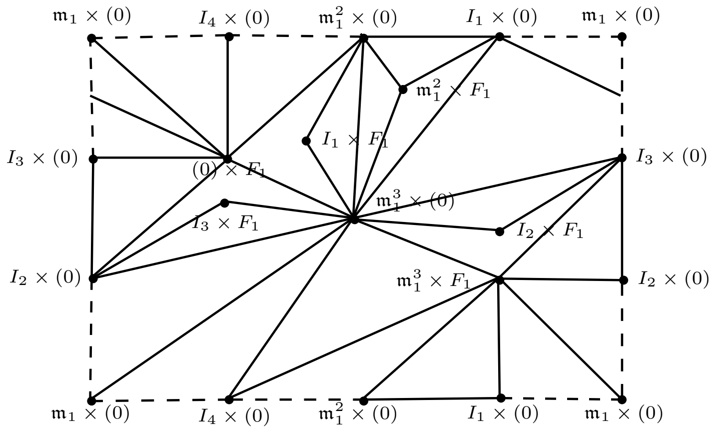


Fig 2.12: Torus embedding of $\mathbb{AG}(R_1 \times F_1)$ with $n_1 = 4$, $I_1\mathfrak{m}_1^2 = (0)$, $I_i\mathfrak{m}_1^2 \neq (0)$ $\forall i \neq 1$ and $I_2I_3 = (0)$

Suppose $I_i\mathfrak{m}_1^2 \neq (0)$ for every i. Suppose $I_iI_j = (0)$ for some $j \neq i$. Without loss of generality, assume that $I_1I_2 = (0)$ and $I_1I_3 = (0)$. Consider the non-trivial ideals $u_1 = (0)\times F_1$, $u_2 = \mathfrak{m}_1^3\times F_1$, $u_3 = \mathfrak{m}_1^3\times(0)$, $v_1 = \mathfrak{m}_1^2\times(0)$, $v_2 = \mathfrak{m}_1\times(0)$, $v_3 = I_1\times(0)$, $v_4 = I_2\times(0)$, $v_5 = I_3\times(0)$, $v_6 = I_4\times(0)$, $x_1 = \mathfrak{m}_1^2\times F_1$, $x_2 = \mathfrak{m}_1\times F_1$, $x_3 = I_1\times F_1$, $x_4 = I_2\times F_1$, $x_5 = I_3\times F_1$, $x_6 = I_4\times F_1$, $x_7 = R_1\times(0)$ in R. Let $G = \mathbb{AG}(R)$, $G' = G - \{x_1, x_2, x_6, x_7, x_8\} - \{u_1u_3, u_2u_3, v_3v_4, v_3v_5\}$ and $G'' = G' - \{x_3, x_4, x_5\}$. Then $G'' \cong K_{3,6}$ and so $\gamma(G'') = 1$. Since $\gamma(G) = 1$ and $\gamma(G'') \leq \gamma(G') \leq \gamma(G)$, we get $\gamma(G') = 1$. Note that $|V(G')| = 12$, $|E(G')| = 25$. Then by Euler's formula, there are 13 faces when drawing G' on a torus. Fix a representation of G' and let $\{F_1', \ldots, F_{13}'\}$ be the set of faces of G' corresponding to the representation. Let $\{F_1'', \ldots, F_n''\}$ be the set of faces of G'' obtained by deleting x_3, x_4, x_5 and all the edges incident with x_3, x_4, x_5 from the representation of G'. Notice that $G'' \cong K_{3,6}$. By Euler formula, $K_{3,6}$ has 9 faces. So $n = 9$. Let s_{F_i} be the length of the faces F_i. Note that $\sum\limits_{i=1}^{9} s_{F_i} = 36$ and $s_{F_i} \geq 4$ for every i. Thus $s_{F_i} = 4$ for every i. Moreover, for every i, each boundary of F_i'' cannot have consecutive repetition of a single edge. Therefore in $K_{3,6}$, the only way to have a closed walk of length 4 without consecutive repetition of single edge is to have $4-$cycle. Then in $K_{3,6}$, all faces boundaries are $4-$cycles. Now $\{F_1', \ldots, F_{13}'\}$ can be recovered by inserting x_3, x_4, x_5 and all the edges incident with x_3, x_4, x_5 into the representation corresponds to $\{F_1'', \ldots, F_9''\}$. Consider the following edges of G: $e_1 = x_3u_3$, $e_2 = x_3v_4$, $e_3 = x_3v_5$, $e_4 = x_4u_3$, $e_5 = x_4v_3$, $e_6 = x_5u_3$, $e_7 = x_5v_3$, $e_8 = v_3v_5$, $e_9 = v_3v_4$. If we insert the vertices x_3, x_4, x_5 the edges e_i where $1 \leq i \leq 8$ into the faces F_m'' and F_n'' in the embedding of G, then from Fig 2.7, it is clear that v_3 and v_4 are in different faces. So there is no way to insert the edge e_9 without crossing in the embedding of G. Hence we conclude that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_1I_2 = (0)$ or $I_1I_3 = (0)$. Hence we conclude that $I_iI_j = (0)$ for at most one $j \neq i$.

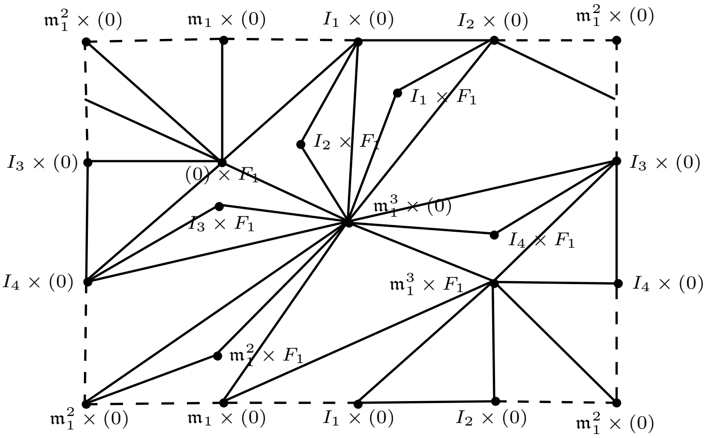


Fig 2.13: Torus embedding of $\mathbb{AG}(R_1 \times F_1)$ with $n_1 = 4$, $I_i\mathfrak{m}_1 \neq (0) \forall i$ $I_i\mathfrak{m}_1^2 \neq (0)\ \forall i$, $I_1I_2 = (0)$ and $I_3I_4 = (0)$

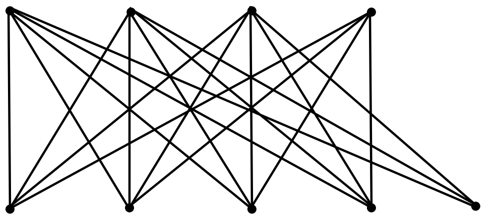


Fig 2.14

Subcase 3.2. Suppose R_1 has exactly 6 distinct non-trivial ideals, say $\mathfrak{m}_1$, $\mathfrak{m}_1^2$, $\mathfrak{m}_1^3$, I_1, I_2, I_3. Suppose $I_i\mathfrak{m}_1 = (0)$ for some i. Consider the non-trivial ideals $a_1 = (0) \times F_1$, $a_2 = \mathfrak{m}_1^3 \times F_1$, $a_3 = \mathfrak{m}_1^3 \times (0)$, $a_4 = I_i \times F_1$, $b_1 = \mathfrak{m}_1^2 \times (0)$, $b_2 = \mathfrak{m}_1 \times (0)$, $b_3 = I_1 \times (0)$, $b_4 = I_2 \times (0)$, $b_5 = I_3 \times (0)$ in R. Then $a_ib_j = (0)$ for every i, j and so $K_{4,5}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_i\mathfrak{m}_1 \neq (0)$ for every i.

Suppose $I_i\mathfrak{m}_1^2 = (0)$ for every i. Consider the set $S = \{c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4, d_5\}$ where $c_1 = (0) \times F_1$, $c_2 = \mathfrak{m}_1^3 \times F_1$, $c_3 = \mathfrak{m}_1^3 \times (0)$, $c_4 = \mathfrak{m}_1^2 \times F_1$, $d_1 = \mathfrak{m}_1^2 \times (0)$, $d_2 = I_1 \times (0)$, $d_3 = I_2 \times (0)$, $d_4 = I_3 \times (0)$, $d_5 = \mathfrak{m}_1 \times (0)$ are the non-trivial ideals in R. Then the subgraph induced by S in $\mathbb{AG}(R)$ contains a subgraph isomorphic to the graph given in Fig 2.14. By Lemma 5, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_i\mathfrak{m}_1^2 = (0)$ for some i.

Suppose $I_i\mathfrak{m}_1^2 = (0)$ for $i = 1, 2$. Suppose that $I_1I_2 = (0)$. Consider the non-trivial ideals $u_1 = (0) \times F_1$, $u_2 = \mathfrak{m}_1^3 \times F_1$, $u_3 = \mathfrak{m}_1^3 \times (0)$, $v_1 = \mathfrak{m}_1^2 \times (0)$, $v_2 = I_1 \times (0)$, $v_3 = I_2 \times (0)$, $v_4 = I_3 \times (0)$, $v_5 = \mathfrak{m}_1 \times (0)$, $x_1 = \mathfrak{m}_1^2 \times F_1$, $x_2 = I_1 \times F_1$, $x_3 = I_2 \times F_1$, $x_4 = I_3 \times F_1$, $x_5 = \mathfrak{m}_1 \times F_1$, $x_6 = R_1 \times (0)$ of R. Let $G = \mathbb{AG}(R)$, $G' = G - \{x_3, x_4, x_5, x_6\} - \{u_1u_3, u_2u_3, v_1v_2, v_1v_3, v_2v_3\}$ and $G'' = G' - \{x_1, x_2\}$.

Then $G'' \cong K_{3,5}$ and so $\gamma(G'') = 1$. Since $\gamma(G) = 1$ and $\gamma(G'') \leq \gamma(G') \leq \gamma(G)$, we get $\gamma(G') = 1$. Note that $|V(G')| = 10$, $|E(G')| = 22$. Then by Euler's formula, there are 12 faces when drawing G' on a torus. Fix a representation of G' and let $\{F'_1, \ldots, F'_{12}\}$ be the set of faces of G' corresponding to the representation. Let $\{F''_1, \ldots, F''_n\}$ be the set of faces of G'' obtained by deleting x_1, x_2 and all the edges incident with x_1, x_2 from the representation of G'. Notice that $G'' \cong K_{3,5}$. From the fact that $n - m + f = 2 - 2g$, $K_{3,5}$ has 7 faces, six with 4 boundary edges and one with 6 boundary edges. So $n = 7$. Moreover, for every i, each boundary of F''_i cannot have consecutive repetition of a single edge. Therefore in $K_{3,5}$, the only way to have a closed walk of length 6 without consecutive repetition of single edge is to have $6-$cycle. Then in $K_{3,5}$, all faces boundaries are $4-$cycles but with one $6-$cycle. We may assume that the boundary of F''_7 is 6. Now $\{F'_1, \ldots, F'_{12}\}$ can be recovered by inserting x_1, x_2 and all the edges incident with x_1, x_2 into the representation corresponds to $\{F''_1, \ldots, F''_7\}$. Let $e_1 = x_1u_3$, $e_2 = x_1v_1$, $e_3 = x_1v_2$, $e_4 = x_1v_3$ be the edges incident with x_1 and $e_5 = x_2u_3$, $e_6 = x_2v_1$, $e_7 = x_2v_3$ be the edges incident with x_2. Since the vertices x_1 and x_2 have three neighbors in common, they should be inserted in different faces in the embedding of G'. Since x_1 is adjacent to u_3, v_1, v_2, v_3, it should be inserted into the faces F''_7 and x_2 is adjacent to u_3, v_1, v_3, it should be inserted into the faces F''_m where $m \neq 7$. Since u_1, u_2, u_3 are in F''_7, any faces of length 4 should contain two of the $u_i's$ and so from Fig 2.15, it is clear that there is no other faces F''_m containing the vertices u_3, v_1, v_3. So there is no way to insert x_2 into a faces F''_m without crossing in the embedding of G'. Hence we conclude that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_1I_2 \neq (0)$.

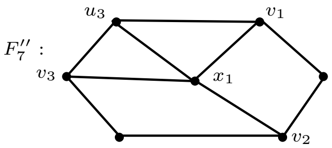


Fig 2.15

Suppose $I_3I_j = (0)$ for every $j \neq 3$. Let $e_1 = x_1u_3$, $e_2 = x_1v_1$, $e_3 = x_1v_2$, $e_4 = x_1v_3$ be the edges incident with x_1 and $e_5 = x_4u_3$, $e_6 = x_4v_2$, $e_7 = x_4v_3$ be the edges incident with x_4. Since the vertices x_1 and x_4 have three neighbors in common, they should be inserted in different faces in the embedding of G'. Since x_1 is adjacent to u_3, v_1, v_2, v_3, it should be inserted into the faces F''_7 and x_4 is adjacent to u_3, v_2, v_3, it should be inserted into the faces F''_m where $m \neq 7$. Since u_1, u_2, u_3 are in F''_7, any faces of length 4 should contain two of the $u_i's$ and so from Fig 2.15, it is clear that there is no other faces F''_m containing the vertices u_3, v_2, v_3. So there is no way to insert x_4 into a faces F''_m without crossing in the embedding of G'. Hence we conclude that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_3I_j = (0)$ for some $j \neq 3$.

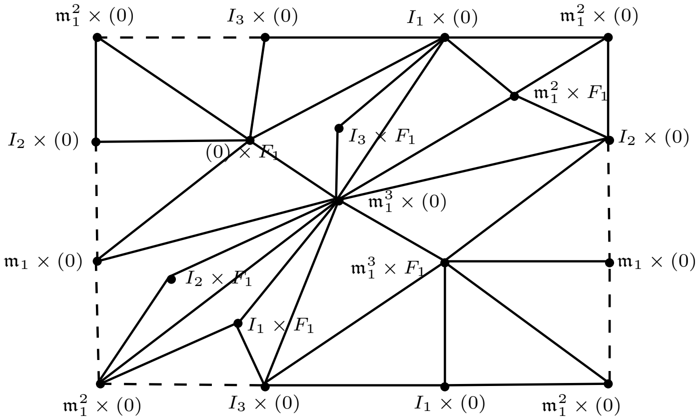


Fig 2.16: Torus embedding of $\mathbb{AG}(R_1 \times F_1)$ with $n_1 = 4$,

$I_1\mathfrak{m}_1^2 = I_2\mathfrak{m}_1^2 = (0), I_3\mathfrak{m}_1^2 \neq (0)$, $I_1I_3 = (0)$

Suppose $I_1\mathfrak{m}_1^2 = (0)$ and $I_i\mathfrak{m}_1^2 \neq (0)$ for every $i \neq 1$. Suppose that $I_jI_k = (0)$ for every $k \neq j$. Consider the non-trivial ideals $u_1 = (0) \times F_1$, $u_2 = \mathfrak{m}_1^3 \times F_1$, $u_3 = \mathfrak{m}_1^3 \times (0)$, $v_1 = \mathfrak{m}_1^2 \times (0)$, $v_2 = I_1 \times (0)$, $v_3 = I_2 \times (0)$, $v_4 = I_3 \times (0)$, $v_5 = \mathfrak{m}_1 \times (0)$, $x_1 = \mathfrak{m}_1^2 \times F_1$, $x_2 = I_1 \times F_1$, $x_3 = I_2 \times F_1$, $x_4 = I_3 \times F_1$, $x_5 = \mathfrak{m}_1 \times F_1$, $x_6 = R_1 \times (0)$ of R. Let $G = \mathbb{AG}(R)$, $G' = G - \{x_5, x_6\} - \{u_1u_3, u_2u_3, v_1v_2, v_2v_3, v_2v_4, v_3v_4\}$ and $G'' = G' - \{x_1, x_2, x_3, x_4\}$. Then $G'' \cong K_{3,5}$ and so $\gamma(G'') = 1$. Since $\gamma(G) = 1$ and $\gamma(G'') \leq \gamma(G') \leq \gamma(G)$, we get $\gamma(G') = 1$. Note that $|V(G')| = 12$, $|E(G')| = 28$. Then by Euler's formula, there are 16 faces when drawing G' on a torus. Fix a representation of G' and let $\{F'_1, \ldots, F'_{16}\}$ be the set of faces of G' corresponding to the representation. Let $\{F''_1, \ldots, F''_n\}$ be the set of faces of G'' obtained by deleting x_1, x_2, x_3, x_4 and all the edges incident with x_1, x_2, x_3, x_4 from the representation of G'. Notice that $G'' \cong K_{3,5}$. From the fact that $n - m + f = 2 - 2g$, $K_{3,5}$ has 7 faces, six with 4 boundary edges and one with 6 boundary edges. So $n = 7$. Moreover, for every i, each boundary of F''_i cannot have consecutive repetition of a single edge. Therefore in $K_{3,5}$, the only way to have a closed walk of length 6 without consecutive repetition of single edge is to have 6−cycle. Then in $K_{3,5}$, all faces boundaries are 4−cycles but with one 6−cycle. We may assume that the boundary of F''_7 is 6. Now $\{F'_1, \ldots, F'_{16}\}$ can be recovered by inserting x_1, x_2, x_3, x_4 and all the edges incident with x_1, x_2, x_3, x_4 into the representation corresponds to $\{F''_1, \ldots, F''_7\}$.

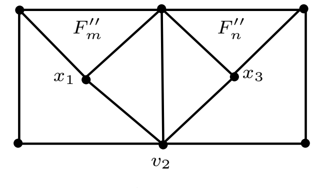


Fig 2.17

Let $e_1 = x_1u_3$, $e_2 = x_1v_1$, $e_3 = x_1v_2$ be the edges incident with x_1 and $e_4 = x_2u_3$, $e_5 = x_2v_1$, $e_6 = x_2v_3$, $e_7 = x_2v_4$ be the edges incident with x_2 and $e_8 = x_3u_3$, $e_9 = x_3v_2$, $e_{10} = x_3v_4$ be the edges incident with x_3 and $e_{11} = x_4u_3$, $e_{12} = x_4v_2$, $e_{13} = x_4v_3$ be the edges incident with x_4. So the vertices x_1, x_3, x_4 and x_2 should be inserted in different faces in the embedding of G' and u_3, v_2 are the common neighbors of x_1, x_3 and x_4. Since x_2 is adjacent to u_3, v_1, v_3, v_4, it should be inserted into the faces F''_7 and x_1 is adjacent to u_3, v_1, v_2 and x_3 is adjacent to u_3, v_2, v_4 and x_4 is adjacent to u_3, v_2, v_3, they should be inserted into the faces F''_m where $m \neq 7$. After inserting x_1, x_3, e_1, e_2, e_3, e_8, e_9 and e_{10} into the faces F''_m and F''_n in the embedding of G'', we obtain Fig 2.17. From 2.17, it is clear that there is no other face containing the boundary vertices u_3, v_2, v_3. So there is no way to insert the vertex x_4 and the edges e_{11}, e_{12} and e_{13} without crossing in the embedding of G'. Hence we conclude that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_1I_2 \neq (0)$ or $I_2I_3 \neq (0)$ or $I_1I_3 \neq (0)$.

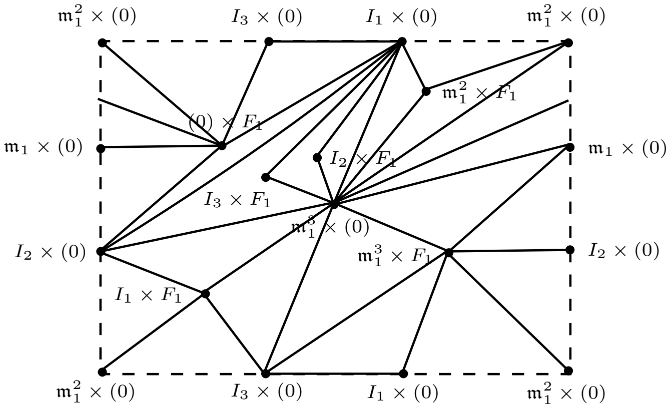


Fig 2.18: Torus embedding of $\mathbb{AG}(R_1 \times F_1)$ with $n_1 = 4$, $I_i\mathfrak{m}_1 \neq (0)\ \forall i$, $I_1\mathfrak{m}_1^2 = (0)$, $I_i\mathfrak{m}_1^2 \neq (0)\ \forall i \neq 1$ and $I_1I_2 = I_1I_3 = (0)$

Clearly proof of (ii)(e) follows from proof of (ii)(d).

Case 4. $n_1 = 5$.

Suppose there is an ideal I of R_1 such that $I \neq \mathfrak{m}_1^i$ for all $1 \leq i \leq 4$. Then by Proposition 1, R_1 has at least three distinct non-trivial ideals different from $\mathfrak{m}_1^i$ for all $1 \leq i \leq 4$. Suppose that R_1 has at least 4 non-trivial ideals I_1, I_2, I_3, I_4 such that $I_i \neq \mathfrak{m}_1^j$ for $i = 1$ to 4 and $j = 1$ to 4. Consider the non-trivial ideals $u_1 = (0) \times F_1$, $u_2 = \mathfrak{m}_1^4 \times F_1$, $u_3 = \mathfrak{m}_1^4 \times (0)$, $v_1 = \mathfrak{m}_1^3 \times (0)$, $v_2 = \mathfrak{m}_1^2 \times (0)$, $v_3 = \mathfrak{m}_1 \times (0)$, $v_4 = I_1 \times (0)$, $v_5 = I_2 \times (0)$, $v_6 = I_3 \times (0)$, $v_7 = I_4 \times (0)$ in R. Then $u_iv_j = (0)$ for every i, j and so $K_{3,7}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence R_1 has exactly 3 non-trivial ideals different from $\mathfrak{m}_1^i$ for all $1 \leq i \leq 4$.

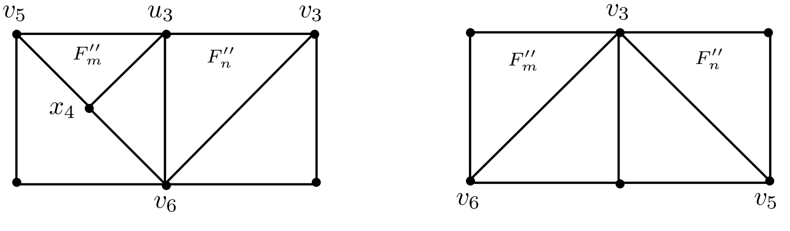
Subcase 4.1. Suppose that R_1 has exactly 3 distinct non-trivial ideals I_1, I_2, I_3 such that $I_i \neq \mathfrak{m}_1^j$ for $i = 1$ to 3 and $j = 1$ to 4. Suppose that $I_i\mathfrak{m}_1 = (0)$ for some i. Consider the non-trivial ideals $a_1 = (0) \times F_1$, $a_2 = \mathfrak{m}_1^4 \times F_1$, $a_3 = \mathfrak{m}_1^4 \times (0)$, $a_4 = I_i \times F_1$, $b_1 = \mathfrak{m}_1^3 \times (0)$, $b_2 = \mathfrak{m}_1^2 \times (0)$, $b_3 = \mathfrak{m}_1 \times (0)$, $b_4 = I_1 \times (0)$, $b_5 = I_2 \times (0)$, $b_6 = I_3 \times (0)$ in R. Then $a_ib_j = (0)$ for every i, j and so $K_{4,6}$ is a subgraph of $\mathbb{AG}(R)$.

Therefore by Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_i\mathfrak{m}_1 \neq (0)$ for every i.

Suppose that $I_i\mathfrak{m}_1^2 = (0)$ for some i. Without loss of generality, assume that $I_1\mathfrak{m}_1^2 = (0)$. Consider the non-trivial ideals $u_1 = (0) \times F_1$, $u_2 = \mathfrak{m}_1^4 \times F_1$, $u_3 = \mathfrak{m}_1^4 \times (0)$, $v_1 = \mathfrak{m}_1^3 \times (0)$, $v_2 = \mathfrak{m}_1^2 \times (0)$, $v_3 = \mathfrak{m}_1 \times (0)$, $v_4 = I_1 \times (0)$, $v_5 = I_2 \times (0)$, $v_6 = I_3 \times (0)$, $x_1 = \mathfrak{m}_1^3 \times F_1$, $x_2 = \mathfrak{m}_1^2 \times F_1$, $x_3 = \mathfrak{m}_1 \times F_1$, $x_4 = I_1 \times F_1$, $x_5 = I_2 \times F_1$, $x_6 = I_3 \times F_1$, $x_7 = R_1 \times (0)$ of R. Let $G = \mathbb{AG}(R)$, $G' = G - \{x_2, x_3, x_4, x_5, x_6, x_7\} - \{u_1u_3, u_2u_3, v_1v_2, v_1v_4, v_2v_4\}$ and $G'' = G' - \{x_1\}$. Then $G'' \cong K_{3,6}$ and so $\gamma(G'') = 1$. Since $\gamma(G) = 1$ and $\gamma(G'') \leq \gamma(G') \leq \gamma(G)$, we get $\gamma(G') = 1$. Note that $|V(G')| = 10$, $|E(G')| = 22$. Then by Euler's formula, there are 12 faces when drawing G' on a torus. Fix a representation of G' and let $\{F_1', \ldots, F_{12}'\}$ be the set of faces of G' corresponding to the representation. Let $\{F_1'', \ldots, F_n''\}$ be the set of faces of G'' obtained by deleting x_1 and all the edges incident with x_1 from the representation of G'. Notice that $G'' \cong K_{3,6}$. From the fact that $n - m + f = 2 - 2g$, $K_{3,6}$ has 9 faces. So $n = 9$. Let s_{F_i} be the length of the faces F_i. Note that $\sum\limits_{i=1}^{9} s_{F_i} = 36$ and $s_{F_i} \geq 4$ for every i. Thus $s_{F_i} = 4$ for every i. Moreover, for every i, each boundary of F_i'' cannot have consecutive repetition of a single edge. Therefore in $K_{3,6}$, the only way to have a closed walk of length 4 without consecutive repetition of single edge is to have 4−cycle. Then in $K_{3,6}$, all faces boundaries are 4−cycles. Now $\{F_1', \ldots, F_{12}'\}$ can be recovered by inserting x_1 and all the edges incident with x_1 into the representation corresponds to $\{F_1'', \ldots, F_9''\}$. Also note that $x_1u_3, x_1v_1, x_2v_2, x_2v_4 \in E(G')$ and so u_3, v_1, v_2, v_4 should be the boundary vertices of F_m''. Since $G'' \cong K_{3,6}$ and $s_{F_i} = 4$ for every i, there is no faces containing the vertices u_3, v_1, v_2, v_4. So there is no way to insert x_1 without crossing in the embedding of G'. Hence we conclude that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_i\mathfrak{m}_1^2 \neq (0)$ for every i

Suppose that $I_i\mathfrak{m}_1^3 = (0)$ for some i. Without loss of generality, assume that $I_1\mathfrak{m}_1^3 = (0)$. Consider the non-trivial ideals $a_1 = (0) \times F_1$, $a_2 = \mathfrak{m}_1^4 \times F_1$, $a_3 = \mathfrak{m}_1^4 \times (0)$, $b_1 = \mathfrak{m}_1^3 \times (0)$, $b_2 = \mathfrak{m}_1^2 \times (0)$, $b_3 = I_1 \times (0)$, $b_4 = \mathfrak{m}_1 \times (0)$, $b_5 = I_2 \times (0)$, $b_6 = I_3 \times (0)$, $c_1 = \mathfrak{m}_1^3 \times F_1$, $c_2 = \mathfrak{m}_1^2 \times F_1$, $c_3 = \mathfrak{m}_1 \times F_1$, $c_4 = I_1 \times F_1$, $c_5 = I_2 \times F_1$, $c_6 = I_3 \times F_1$, $c_7 = R_1 \times (0)$ of R. Let $G = \mathbb{AG}(R)$, $G' = G - \{c_2, c_3, c_4, c_5, c_6, c_7\} - \{a_1a_3, a_2a_3, b_1b_2, b_1b_3\}$ and $G'' = G' - \{c_1\}$. Then $G'' \cong K_{3,6}$ and so $\gamma(G'') = 1$. Since $\gamma(G) = 1$ and $\gamma(G'') \leq \gamma(G') \leq \gamma(G)$, we get $\gamma(G') = 1$. Note that $|V(G')| = 10$, $|E(G')| = 22$. Then by Euler's formula, there are 12 faces when drawing G' on a torus. Fix a representation of G' and let $\{F_1', \ldots, F_{12}'\}$ be the set of faces of G' corresponding to the representation. Let $\{F_1'', \ldots, F_n''\}$ be the set of faces of G'' obtained by deleting c_1 and all the edges incident with c_1 from the representation of G'. Notice that $G'' \cong K_{3,6}$. From the fact that $n - m + f = 2 - 2g$, $K_{3,6}$ has 9 faces. So $n = 9$. Let s_{F_i} be the length of the faces F_i. Note that $\sum\limits_{i=1}^{9} s_{F_i} = 36$ and $s_{F_i} \geq 4$ for every i. Thus $s_{F_i} = 4$ for every i. Moreover, for every i, each boundary of F_i'' cannot have consecutive repetition of a single edge. Therefore in $K_{3,6}$, the only way to have a closed walk of length 4 without consecutive repetition of single edge is to have 4−cycle. Then in $K_{3,6}$, all faces boundaries are 4−cycles. Now $\{F_1', \ldots, F_{12}'\}$ can be recovered by inserting c_1 and all the edges incident with c_1 into the representation corresponds to $\{F_1'', \ldots, F_9''\}$. Also note that $c_1a_3, c_1b_1, c_2b_2, c_2b_3 \in E(G')$ and so a_3, b_1, b_2, b_3 should be the boundary vertices of F_m''. Since $G'' \cong K_{3,6}$ and $s_{F_i} = 4$ for every i, there is no faces containing the vertices a_3, b_1, b_2, b_3. So there is no way to insert c_1 without crossing in the embedding of G'. Hence we conclude that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_i\mathfrak{m}_1^3 \neq (0)$ for every i. Therefore I_i is adjacent only to $\mathfrak{m}_1^4$ for every i.

Suppose $I_jI_k = (0)$ for some $j \neq k$. Without loss of generality, assume that $I_1I_2 = (0)$ and $I_1I_3 = (0)$. Consider the non-trivial ideals $u_1 = (0) \times F_1$, $u_2 = \mathfrak{m}_1^4 \times F_1$, $u_3 = \mathfrak{m}_1^4 \times (0)$, $v_1 = \mathfrak{m}_1^3 \times (0)$, $v_2 = \mathfrak{m}_1^2 \times (0)$, $v_3 = I_1 \times (0)$, $v_4 = \mathfrak{m}_1 \times (0)$, $v_5 = I_2 \times (0)$, $v_6 = I_3 \times (0)$, $x_1 = \mathfrak{m}_1^3 \times F_1$, $x_2 = \mathfrak{m}_1^2 \times F_1$, $x_3 = \mathfrak{m}_1 \times F_1$, $x_4 = I_1 \times F_1$, $x_5 = I_2 \times F_1$, $x_6 = I_3 \times F_1$, $x_7 = R_1 \times (0)$ of R. Then $u_iv_j = (0)$ for every i, j and so $K_{3,6}$ is a subgraph of $\mathbb{AG}(R)$. Let $G = \mathbb{AG}(R)$, $G' = G - \{x_1, x_2, x_3, x_5, x_6, x_7\} - \{u_1u_3, u_2u_3, v_1v_2, v_3v_5, v_3v_6\}$ and $G'' = G' - \{x_4\}$. Then $G'' \cong K_{3,6}$ and so $\gamma(G'') = 1$. Since $\gamma(G) = 1$ and $\gamma(G'') \leq \gamma(G') \leq \gamma(G)$, we get $\gamma(G') = 1$. Note that $|V(G')| = 10$, $|E(G')| = 21$. Then by Euler's formula, there are 11 faces when drawing G' on a torus. Fix a representation of G' and let $\{F_1', \ldots, F_{11}'\}$ be the set of faces of G' corresponding to the representation. Let $\{F_1'', \ldots, F_n''\}$ be the set of faces of G'' obtained by deleting x_4 and all the edges incident with x_4 from the representation of G'. Notice that $G'' \cong K_{3,6}$. From the fact that $n - m + f = 2 - 2g$, $K_{3,6}$ has 9 faces. So $n = 9$. Let s_{F_i} be the length of the faces F_i. Note that $\sum\limits_{i=1}^{9} s_{F_i} = 36$ and $s_{F_i} \geq 4$ for every i. Thus $s_{F_i} = 4$ for every i. Moreover, for every i, each boundary of F_i'' cannot have consecutive repetition of a single edge. Therefore in $K_{3,6}$, the only way to have a closed walk of length 4 without consecutive repetition of single edge is to have 4$-$cycle. Then in $K_{3,6}$, all faces boundaries are 4$-$cycles. Now $\{F_1', \ldots, F_{11}'\}$ can be recovered by inserting x_4 and all the edges incident with x_4 into the representation corresponds to $\{F_1'', \ldots, F_9''\}$.



(a) (b)

Fig 2.19

Also note that $x_4u_3, x_4v_5, x_4v_6 \in E(G')$ and so x_4 should be inserted into the faces F_m'' with boundary vertices u_3, v_5, v_6. Consider the edges in G: $e_1 = v_3v_5$, $e_2 = v_3v_6$. If we insert the edges e_1, e_2 in the embedding of G, then from Fig 2.19(b) it is clear that there is no way to insert the vertices x_4 without crossing in the embedding of G. If we insert the vertex x_4 and the edge e_2 in the embedding of G', then from Fig 2.19(a) it is clear that the vertex v_3 and v_5 are in different faces. So there is no way to insert the edges e_1 without crossing in the embedding of G. Hence we conclude that $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_1 I_2 = (0)$ or $I_1 I_3 = (0)$. That is $I_j I_k = (0)$ for at most $j \neq k$.

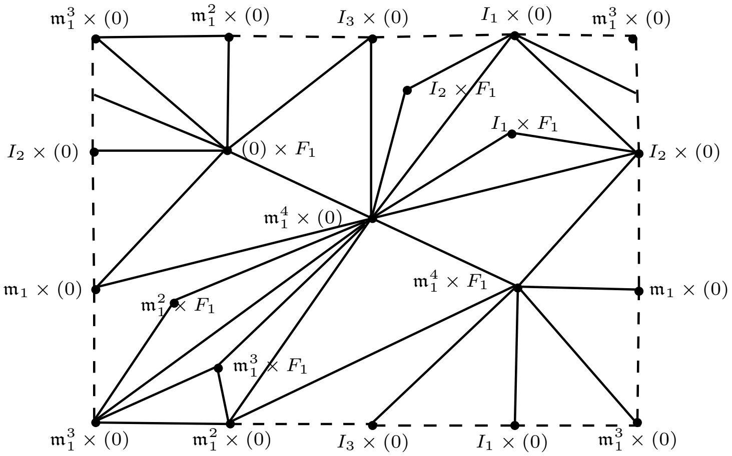


Fig 2.20: Torus embedding of $\mathbb{AG}(R_1 \times F_1)$ with $n_1 = 5$ and $I_i \mathfrak{m}_1^j \neq (0) \forall i, j = 1, 2, 3$ and $I_1 I_2 = (0)$

Proof of iii(b) follows from proof of iii(a).

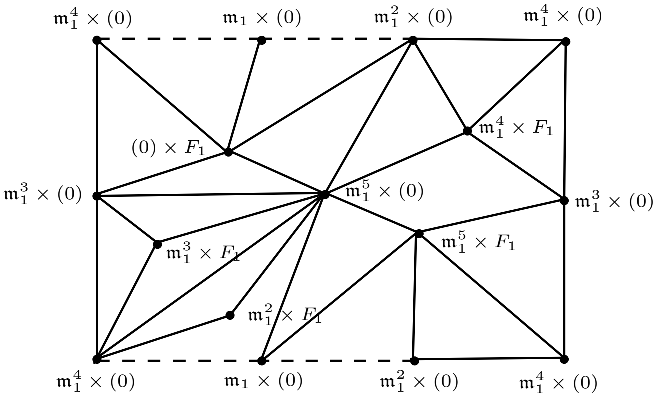


Fig 2.21: Torus embedding of $\mathbb{AG}(R_1 \times F_1)$ with $n_1 = 6$

Case 5. $n_1 = 6$.
Suppose there is an ideal I of R_1 such that $I \neq \mathfrak{m}_1^i$ for $1 \leq i \leq 5$. Then by Proposition 1, R_1 has at least three distinct non-trivial ideals I_1, I_2 and I_3 such that I_1, I_2, $I_3 \neq \mathfrak{m}_1$. Consider the set $S = \{a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$ where $a_1 = (0) \times F_1$, $a_2 = \mathfrak{m}_1^5 \times F_1$, $a_3 = \mathfrak{m}_1^5 \times (0)$, $b_1 = \mathfrak{m}_1^4 \times (0)$, $b_2 = \mathfrak{m}_1^3 \times (0)$, $b_3 = \mathfrak{m}_1^2 \times (0)$, $b_4 = \mathfrak{m}_1 \times (0)$, $b_5 = I_1 \times (0)$, $b_6 = I_2 \times (0)$, $b_7 = I_3 \times (0)$ are the non-trivial ideals in R. Then $a_i b_j = (0)$

for every i, j and so $K_{3,7}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 2, $\gamma(\mathbb{AG}(R)) > 1$, a contradiction. Hence $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, \mathfrak{m}_1^4, \mathfrak{m}_1^5$ are the only non-trivial ideals in R_1. Converse follows from embedding given in Figs 2.8, 2.9, 2.10, 2.12, 2.13, 2.16, 2.18, 2.20, and Fig. 2.21. ☐

ACKNOWLEDGMENTS

The authors are deeply grateful to the referee for careful reading of the manuscript and helpful suggestions. The work reported here is supported by the UGC Major Research Project (F. No. 42-8/2013(SR)) awarded to K. Selvakumar by the University Grants Commission, Government of India.

References

[1] G. Aalipour, S. Akbari, R. Nikandish, M.J. Nikmehr, and F. Shaveisi, *On the coloring of the annihilating-ideal graph of a commutative ring*, Discrete Math. **312** (2012), no. 17, 2620–2626.

[2] ———, *Minimal prime ideals and cycles in annihilating-ideal graphs*, Rocky Mountain J. Math. **43** (2013), no. 5, 1415–1425.

[3] M. Afkhami and K. Khashyarmanesh, *The cozero-divisor graph of a rommutative ring.*, Southeast Asian Bull. Math. **35** (2011), no. 5, 753–762.

[4] S. Akbari, A. Alilou, J. Amjadi, and S.M. Sheikholeslami, *The co-annihilating-ideal graphs of commutative rings*, Canad. Math. Bull. **60** (2017), no. 1, 3–11.

[5] D.F. Anderson, M. Axtell, and J. Stickles, *Zero-divisor graphs in commutative rings: A survey*, Commutative Algebra: Noetherian and Non-Noetherian Perspectives, (M. Fontana, S. E. Kabbaj, B. Olberding, I. Swanson) Springer-Verlag, New York, Springer-Verlag, New York, 2011, pp. 23–45.

[6] D.F. Anderson and A. Badawi, *The zero-divisor graph of a commutative semigroup: a survey*, Groups, Modules, and Model Theory-Surveys and Recent Developments, Springer, 2017, pp. 23–39.

[7] D.F. Anderson and P.S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), no. 2, 434–447.

[8] D. Archdeacon, *Topological graph theory: a survey*, Congr. Number. **115** (1996), 5 – 54.

[9] A. Badawi, *On the annihilator graph of a commutative ring*, Comm. Algebra **42** (2014), no. 1, 108–121.

[10] ———, *Recent results on the annihilator graph of a commutative ring: A survey*, Nearrings, Nearfields and Related Topics, edited by K. Prasad et al., World Scientific, 2017, pp. 170–184.

[11] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), no. 1, 208–226.

[12] M. Behboodi and Z. Rakeei, *The annihilating-ideal graph of commutative rings I*, J. Algebra Appl. **10** (2011), no. 4, 727–739.

[13] ______, *The annihilating-ideal graph of commutative rings II*, J. Algebra Appl. **10** (2011), no. 4, 741 – 753.

[14] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, vol. 290, American Elsevier, New York, 1976.

[15] T.T. Chelvam and K. Selvakumar, *Central sets in the annihilating-ideal graph of commutative rings*, J. Combin. Math. Combin. Comput. **88** (2014), 277–288.

[16] H.-J. Chiang-Hsieh, *Classification of rings with projective zero-divisor graphs*, J. Algebra **319** (2008), no. 7, 2789–2802.

[17] H.R. Maimani, M. Salimi, A. Sattari, and S. Yassemi, *Comaximal graph of commutative rings*, J. Algebra **319** (2008), no. 4, 1801–1808.

[18] W.S. Massey, *Algebraic topology: an introduction*, Harcourt, Brace & World, Inc., New York, 1967.

[19] K. Selvakumar and P. Subbulakshmi, *On the crosscap of the annihilating-ideal graph of a commutative ring*, Palestine J. Math. **7** (2018), no. 1, 151–160.

[20] K. Selvakumar, P. Subbulakshmi, and J. Amjadi, *On the genus of the graph associated to a commutative ring*, Discrete Math. Algorithms Appl. **9** (2017), no. 5, ID: 1750058 (11 pages).

[21] K. Wagner, *Über eine erweiterung des satzes von kuratowski*, Deutsche Math. **2** (1937), 280–285.

[22] H.-J. Wang, *Graphs associated to co-maximal ideals of commutative rings*, J. Algebra **320** (2008), no. 7, 2917–2933.

[23] A. T. White, *Graphs, Groups and Surfaces*, North-Holland, Amsterdam, 1973.