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On the harmonic index of bicyclic graphs

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Abstract: The harmonic index of a graph G, denoted by $H(G)$, is defined as the sum of weights $2/[d(u)+d(v)]$ over all edges uv of G, where $d(u)$ denotes the degree of a vertex u. Hu and Zhou [Y. Hu and X. Zhou, WSEAS Trans. Math. **12** (2013) 716–726] proved that for any bicyclic graph G of order $n \geq 4$, $H(G) \leq \frac{n}{2} - \frac{1}{15}$ and characterized all extremal bicyclic graphs. In this paper, we prove that for any bicyclic graph G of order $n \geq 4$ and maximum degree Δ,

$$H(G) \leq \begin{cases} \frac{3n-1}{6} & \text{if } \Delta = 4 \\ 2(\frac{2\Delta-n-3}{\Delta+1} + \frac{n-\Delta+3}{\Delta+2} + \frac{1}{2} + \frac{n-\Delta-1}{3}) & \text{if } \Delta \geq 5 \text{ and } n \leq 2\Delta - 4 \\ 2(\frac{\Delta}{\Delta+2} + \frac{\Delta-4}{3} + \frac{n-2\Delta+4}{4}) & \text{if } \Delta \geq 5 \text{ and } n \geq 2\Delta - 3, \end{cases}$$

and characterize all extremal bicyclic graphs.

Keywords: Harmonic index, Bicyclic graphs

AMS Subject classification: 05C12, 92E10

1. Introduction

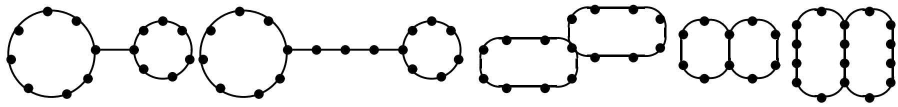
Let G be a simple connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$ and the size $|E|$ of G is denoted by $m = m(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A *leaf* of a graph G is a vertex of degree 1, a *support vertex*

is a vertex adjacent to a leaf, whereas a *strong support vertex* is a support vertex adjacent to at least two leaves. An *end support vertex* is a support vertex whose all neighbors with exception at most one are leaves. The *distance* between u and v in a graph G, denoted by $d(u,v)$, is the length of the shortest (u,v)-path in G. We use $G - uv$ to denote the graph obtained from G by deleting the edge $uv \in E(G)$. Similarly, $G+uv$ is the graph that arises from G by adding an edge $uv \notin E(G)$, where $u, v \in V(G)$. A path $P = u_0u_1 \dots u_k (k \geq 1)$ in G is called a pendent path if $d_{u_0} \geq 3$, $d_{u_k} = 1$ and the degree of any other vertex of the path is 2. We denote by C_n and K_n the cycle and the complete graph on n vertices, respectively. Let K_4^- be the graph obtained from K_4 by deleting one edge. For an edge $e = uv$, the *weight* of e in G is $w(e) = w_G(e) = \frac{1}{d_u+d_v}$.
A large variety of degree based topological indices has been defined in the mathematical and mathematico-chemical literature; for details we refer the reader to [7, 8]. Here, we focus on the harmonic index. For a simple graph G, the harmonic index of G, denoted $H(G)$, is defined in [5] as the sum of weights $2/[d(u) + d(v)]$ of all edges uv of G. That is,

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)}.$$

In [10, 22–26], the minimum and maximum harmonic indices of simple connected graphs, trees, unicyclic, and bicyclic graphs were determined and the corresponding extremal graphs were characterized. For some related works see [13, 28–30]. Wu et al. [19] established a lower bound on H of a graph with minimum degree two. Favaron et al. [6] investigated the relation between graph eigenvalues of graphs and harmonic index. Deng et al. [3] considered the relation between $H(G)$ and the chromatic number $\chi(G)$, and proved that $\chi(G) \leq 2H(G)$. Liu [15] proposed a conjecture concerning the relation between the harmonic index and the diameter of a connected graph, and showed that the conjecture is true for trees. Liu's conjecture is proved by Amalorpava Jerline and L. Benedict Michaelraj for unicyclic graphs [1, 2]. Relationships between the harmonic index and several other topological indices were established in [9, 11, 21, 27]. For additional results on this index, see [12–14, 16–18, 20].
A *bicyclic graph* of order n is a connected graph with n vertices and $n+1$ edges. Let $\mathcal{B}_n$ be the set of connected bicyclic graphs of order $n(n \geq 5)$, and let $\widetilde{\mathcal{B}}_n$ be the set of connected bicyclic graphs on $n(n \geq 4)$ vertices without pendant vertices. Let $\widetilde{\mathcal{B}}_n^{(1)}$ be the set of bicyclic graphs obtained by joining two vertex-disjoint cycles C_a and C_b with $a + b = n(n \geq 6)$ by an edge. Let $\widetilde{\mathcal{B}}_n^{(2)}$ be the set of bicyclic graphs obtained by joining two vertex-disjoint cycles C_a and C_b with $a + b < n(n \geq 7)$ by a path of length $n - a - b + 1$. Let $\widetilde{\mathcal{B}}_n^{(3)}$ be the set of bicyclic graphs obtained by identifying a vertex of C_a and a vertex of C_b with $a + b = n + 1(n \geq 5)$, and let S_5^{++} denote the special case $n = 5$ and $a = b = 3$. Let $\widetilde{\mathcal{B}}_n^{(4)}$ be the set of bicyclic graphs obtained from $C_n(n \geq 4)$ by adding an edge. Let $\widetilde{\mathcal{B}}_n^{(5)}$ be the set of bicyclic graphs obtained by joining two non-adjacent vertices of $C_a(4 \leq a \leq n-1, n \geq 5)$ with a path of length $n - a + 1$. Clearly $\widetilde{\mathcal{B}}_n = \widetilde{\mathcal{B}}_n^{(1)} \cup \widetilde{\mathcal{B}}_n^{(2)} \cup \widetilde{\mathcal{B}}_n^{(3)} \cup \widetilde{\mathcal{B}}_n^{(4)} \cup \widetilde{\mathcal{B}}_n^{(5)}$. For $1 \leq i \leq 5$, let $\mathcal{B}_n^{(i)}$ be the set of bicyclic graphs G containing a member of $\widetilde{\mathcal{B}}_m^{(i)}$ as a subgraph, for some $m \leq n$.

Zhong and Xu [26], Hu and Zhou [10] and Zhu et al. [30], independently proved the



$\widetilde{\mathcal{B}}_{15}^{(1)}$ $\widetilde{\mathcal{B}}_{18}^{(2)}$ $\widetilde{\mathcal{B}}_{15}^{(3)}$ $\widetilde{\mathcal{B}}_{10}^{(4)}$ $\widetilde{\mathcal{B}}_{16}^{(5)}$

Figure 1. **Special families of bicyclic graphs.**

following upper bound on the harmonic index of bicyclic graphs and characterized all extremal bicyclic graphs.

Theorem A. *If G is a bicyclic graph of order $n \geq 4$, then*

$$H(G) \leq \frac{n}{2} - \frac{1}{15}$$

with equality if and only if $G \in \widetilde{\mathcal{B}}_n^{(1)} \cup \widetilde{\mathcal{B}}_n^{(4)}$.

Deng et al. [4] proved the following upper bound on the harmonic index of bicyclic graphs of order $n \geq 5$ and maximum degree 4 and characterized all extremal bicyclic graphs..

Theorem B. *If G is a bicyclic graph of order $n \geq 5$ with maximum degree $\Delta(G) = 4$, then*

$$H(G) \leq \frac{n}{2} - \frac{1}{6}.$$

In this paper, we establish an upper bound for the harmonic index of bicyclic graphs in terms of their order and maximum degree and characterize all extremal bicyclic graphs.
We make use of the following results in this paper.

Theorem C. *[25] Let H be a nontrivial connected graph with $u \in V(H)$. Let G be the graph obtained from H by attaching two paths $P := uu_1 \dots u_s$ and $Q := uv_1 \dots v_t (s \geq t \geq 1)$ at u, and let $G' = G - uv_1 + u_s v_1$. Then $H(G) < H(G')$.*

Theorem D. *[25] Let H be a nontrivial connected graph, and let u, v be two distinct vertices in H with $d_H(u), d_H(v) \geq 2$. Moreover, suppose that the two neighbors of v have degree sum at most 9 in H if $d_H(v) = 2$. Let G be the graph obtained from H by attaching two paths $P := uu_1 \dots u_s$ and $Q := vv_1 \dots v_t (s \geq t \geq 1)$ at u and v, respectively, and let $G' = G - vv_1 + u_s v_1$. Then $H(G) < H(G')$.*

Next result is an immediate consequence of Theorem A.

Corollary 1. If G is a bicyclic graph of order n with maximum degree 3, then

$$H(G) \leq \frac{n}{2} - \frac{1}{15}$$

with equality if and only if $n = 5$ and $G \cong \widetilde{\mathcal{B}}_n^{(4)}$ or $n \geq 6$ and $G \cong \widetilde{\mathcal{B}}_n^{(1)} \cup \widetilde{\mathcal{B}}_n^{(4)}$.

Hence, we may assume from now on that G is a bicyclic graph of order n with maximum degree $\Delta(G) \geq 4$. Suppose $\mathcal{B}_n^{\Delta}$ denotes the set of all bicyclic graphs of order n with maximum degree $\Delta(G)$.

2. An upper bound on the harmonic index of bicyclic graph

In this section, we establish a sharp upper bound on the harmonic index of bicyclic graphs in terms of their order and maximum degree, and classify all extremal bicyclic graphs. Throughout this section, G denotes a bicyclic graph of order $n \geq 5$, $C_1 = (x_1x_2\dots x_r)$ and $C_2 = (y_1y_2\dots y_s)$ denote the cycles of G and $\omega \in V(G)$ denotes a vertex of maximum degree. Let $V_{cycle} = V(C_1) \cup V(C_2)$ and $V_c = V(C_1) \cup V(C_2) \cup \{\omega\}$. If $V(C_1) \cap V(C_2) = \emptyset$, then we assume $x_1y_1 \in E(G)$ when $d(V(C_1), V(C_2)) = 1$ and we assume $P := (x_1 =)w_0w_1\dots w_ky_1$ is the shortest (x_1, y_1)-path in G when $d(V(C_1), V(C_2)) \geq 2$. Similarly, if $V(C_1) \cap V(C_2) \neq \emptyset$, then we assume that $Q := x_1x_2\dots x_k$ be a longest path belonging to $V(C_1) \cap V(C_2)$ where $x_i = y_i$ for $1 \leq i \leq k$. Finally, if $\omega \notin V(C_1) \cup V(C_2) \cup V(P)$, then let $(\omega =)v_0v_1\dots v_t$ be a shortest path between ω and $V(C_1) \cap V(C_2) \cup V(P)$ (see Figure 2). In addition $h_\omega : E(G) \to \mathbb{R}$ is a function defined by $h_\omega(uv) = 1/[d(u) + d(v)]$. Hence $H(G) = 2\sum_{e \in E(G)} h_\omega(e)$.

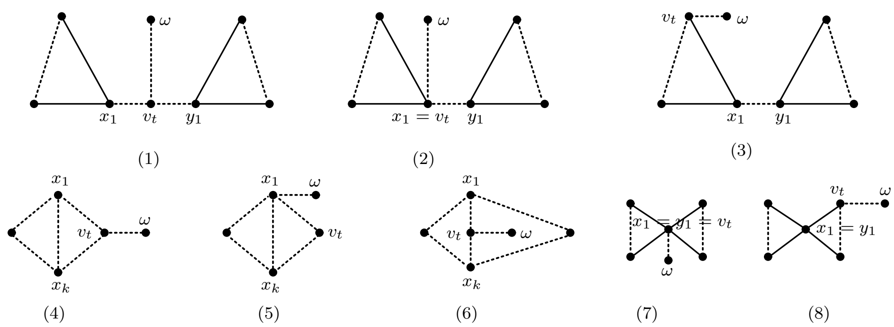


Figure 2. **Possible cases of bicyclic graphs with respect to vertices ω, x_1, y_1, x_k and v_t.**

Our first result is an immediate consequence of Theorem C.

Corollary 2. Let $G \in \mathcal{B}_n^{\Delta}$.

1. If G has an end-support vertex of degree at least three, different from ω, then there is a graph $G' \in \mathcal{B}_n^{\triangle}$ such that $H(G) < H(G')$.

2. If G has a vertex $v \neq \omega$ of degree at least three with two pendant paths $P := vu_1 \dots u_s$ and $Q := vv_1 \dots v_t (s,t \geq 1)$, then there is a bicyclic graph $G' \in \mathcal{B}_n^{\triangle}$ such that $H(G) < H(G')$.

Lemma 1. Let H be a nontrivial connected graph, and let u, v be two distinct vertices in H with $d_H(u) \geq 2, d_H(v) \geq 3$. Let G be the graph obtained from H by attaching two paths $P := uu_1 \dots u_s$ and $Q := vv_1 \dots v_t (s,t \geq 1)$ at u and v, respectively, and let $G' = G - vv_1 + u_s v_1$. Then $H(G) < H(G')$.

Proof. The result is immediate by Theorem D if $s \geq t$. Let $s < t$. If $s = 1$, then we have $\frac{1}{2}(H(G') - H(G)) \geq (\frac{1}{4} + \frac{1}{d(u)+2}) - (\frac{1}{d(u)+1} + \frac{1}{d(v)+2}) > 0$. Let $s \geq 2$ and $w \in N(v) - \{v_1\}$. Then

$$\begin{aligned}\frac{1}{2}(H(G') - H(G)) &\geq (\frac{1}{d(v)+d(w)-1} + \frac{1}{4} + \frac{1}{4}) - (\frac{1}{d(v)+d(w)} + \frac{1}{3} + \frac{1}{d(v)+2}) \\ &\geq (\frac{1}{d(v)+d(w)-1} - \frac{1}{d(v)+d(w)}) > 0,\end{aligned}$$

and the proof is complete. □

Lemma 2. Let $G \in \mathcal{B}_n^{\triangle}$ be a graph with the maximum value of the harmonic index. Then $d(v) \leq 2$ for any vertex $v \notin \{x_1, \omega, y_1, v_t, x_k\}$.

Proof. Suppose, to the contrary, that $d(v) \geq 3$ for some $v \notin \{x_1, \omega, y_1, v_t, x_k\}$. We claim that there is a pendant path beginning at v. Let $N(v) = \{z_1, z_2, \dots, z_r\}$. First let vz_i be a non-cut edge for some i, say $i = 1$. Then we may assume that the edges vz_1 and vz_2 are contained in the same cycle, say C_1. If vz_i is a non-cut edge for some $i \geq 3$, then clearly $v \in \{x_1, x_k\}$ which is a contradiction. Let vz_i be a cut edge for $i = 3, \dots, r$ and let G_i be the component of $G - vz_i$ containing z_i for $i = 1, \dots, r$. If G_i has a cycle for some $i \geq 3$, then clearly $v = x_1$, a contradiction. Let G_i be tree for each $i \geq 3$. If $\omega \notin V(G_i)$ for some $i \geq 3$, then we deduce from Corollary 2 and the choice of G that $G_i + vz_i$ is a pendant path, as desired. Hence we assume that $i = 3$ and $\omega \in V(G_3)$. Then clearly $v = x_1$ which is a contradiction. Now let vz_i is a cut edge for $i = 1, \dots, r$. Since G is bicyclic, we may assume that G_r is a tree. If $\omega \notin V(G_r)$, then it follows from Corollary 2 and the choice of G that $G_r + vz_r$ is a pendant path, as desired. Let $\omega \in V(G_r)$. If G_i is a tree for some $1 \leq i \leq r-1$, then as above $G_i + vz_i$ is a pendant path because of $\omega \notin V(G_i)$, as desired. Hence we assume that $r = 3$ and G_1, G_2 have cycle. Then clearly $v = v_t$ which is a contradiction. Thus, there is a pendant path $P := vu_1 \dots u_s$ $(s \geq 1)$ beginning at v. If $d(v) \geq 4$, then $G - \{u_1, \dots, u_s\}$ is a bicyclic graph and as above there is another pendant path beginning at v that leads to a contradiction by Corollary 2. Thus $d(v) = 3$. We consider two cases.

Case 1. $v \notin V_{cycle} \cup V(P)$.
Then there is a unique $(v, V_{cycle} \cup V(P))$-path such as $(v =)z_0 z_1 \dots z_m$ since G is bicyclic. Let T be component of $G - vz_1$ containing v. Clearly T is a tree. If $\omega \notin V(T)$, then there must be two pendant paths beginning at v which is a contradiction by Corollary 2. Assume that $\omega \in V(T)$ and $(v =)q_0 q_1 \dots q_p \omega$ is the unique (v, ω)-path in G. Then $\omega q_p \dots q_1 v z_1 \dots z_m$ is the unique $(\omega, V_{cycle} \cup V(P))$-path in G and so $z_m = v_t$. By Lemma 1 and the choice of G, there is no pendant path beginning at v_t and so $d(v_t) \leq 5$. If $s = 1$ or $d(z_1) \leq 3$, then let G' be the graph obtained from G by removing the edge vz_1 and adding the edge $u_s z_1$. Clearly G' is bicyclic. Let $S = \{vz_1, vq_1, vu_1, u_s u_{s-1}\}$ and $S' = \{vq_1, vu_1, u_s u_{s-1}, u_s z_1\}$. If $s = 1$, then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{1}{2+d(z_1)} - \frac{1}{3+d(z_1)}) + (\frac{1}{2+d(q_1)} - \frac{1}{3+d(q_1)}) > 0$ which is a contradiction. If $s \geq 2$ and $d(z_1) \leq 3$, then since $d(z_1) = 2$ or 3 we obtain $\frac{1}{2}(H(G') - H(G)) = (\frac{1}{2+d(z_1)} - \frac{1}{3+d(z_1)}) + (\frac{1}{2+d(y_1)} - \frac{1}{3+d(y_1)}) + (\frac{1}{2} - \frac{8}{15}) > 0$ which is a contradiction. Assume that $s \geq 2$ and $d(z_1) \geq 4$. Similarly, we may assume that $d(q_1) \geq 4$. If $m \geq 2$ (resp. $p \geq 2$), then there must be two pendant paths beginning at z_1 (resp. q_1) which is a contradiction by Corollary 2. Thus $z_1 = v_t$ and $q_1 = \omega$. First let ω be a support vertex and p be a leaf adjacent to ω. Assume G' is the graph obtained from $G - vu_1$ by adding the edge $u_1 p$. It is easy to verify that
$\frac{1}{2}(H(G') - H(G)) = (\frac{1}{2+\Delta} + \frac{1}{2+d(v_t)} + \frac{1}{\Delta+2} + \frac{1}{4}) - (\frac{1}{3+\Delta} + \frac{1}{3+d(v_t)} + \frac{1}{\Delta+1} + \frac{1}{5}) > 0$ which is a contradiction.
Assume ω is not a support vertex and $(\omega =)b_0 b_1 \dots b_{\ell'}$ be a pendant path beginning at ω where $\ell' \geq 2$. First let $4 \leq \Delta \leq 5$. Assume G' is the graph obtained from $G - vv_t$ by adding the edge $u_s v_t$. Since $4 \leq d(v_t) \leq \Delta \leq 5$, we deduce that $\frac{1}{2}(H(G') - H(G)) = (\frac{2}{4} + \frac{1}{\Delta+2} + \frac{1}{d(v_t)+2}) - (\frac{8}{15} + \frac{1}{\Delta+3} + \frac{1}{3+d(v_t)}) > 0$ which is a contradiction. Now let $\Delta \geq 6$. Assume G' is the graph obtained from $G - \{vv_t, v\omega\}$ by adding the edges $v_t\omega$ and $vb_{\ell'}$. It is easy to verify that $\frac{1}{2}(H(G') - H(G)) = (\frac{3}{4} + \frac{1}{\Delta+d(v_t)}) - (\frac{8}{15} + \frac{1}{3+d(v_t)} + \frac{1}{\Delta+3}) > 0$, a contradiction.

Case 2. $v \in (V_c \cup V(P)) \setminus \{x_1, \omega, y_1, v_t, x_k\}$.
Then clearly v has exactly two neighbors in $V_c \cup V(P)$, say α, β. Assume without loss of generality that $\alpha \neq \{\omega, v_t\}$. Since G is a graph with the maximum value of the harmonic index, we conclude from Lemma 1 that there is no pendant path beginning at α when $d(\alpha) \geq 4$. This implies that $d(\alpha) \leq 4$. As Case 1, we may assume that $s \geq 1$ and $\deg(\alpha) = 4$. Since there is no pendant path beginning at α, we deduce that $V(C_1) \cap V(C_2) = \{\alpha\}$. Using the argument described in Case 1, we may assume that $\beta \geq 4$. This implies that there is a pendant path beginning at β and we deduce from Lemma 1 and the choice of G that $\beta = \omega$. Now, an argument similar to that described in Case 1, leads to a contradiction and the proof is complete. □

Corollary 3. If $G \in \mathcal{B}_n^{\Delta}$ be a graph with the maximum value of the harmonic index where $\Delta \geq 4$, then any pendant path is beginning at ω.

Proof. Suppose, to the contrary, that there is a pendant path $vu_1 \dots u_\ell$ beginning at $v \neq \omega$. Lemma 2 yields $v \in \{x_1, y_1, v_t, x_k\} \setminus \{\omega\}$ and this implies that $\deg(v) \geq 4$.

If $\Delta \geq 5$, then clearly there is a pendant path beginning at ω and this leads to a contradiction by Lemma 1 and the choice of G. If $\Delta = 4$, then we consider v as ω. By repeating above argument, we deduce that all pendant paths are beginning at ω. $\square$

Lemma 3. If $G \in \mathcal{B}_n^{\Delta}$ is a graph with the maximum value of the harmonic index, then $V(C_1) \cap V(C_2) \neq \emptyset$.

Proof. Suppose, to the contrary, that $V(C_1) \cap V(C_2) = \emptyset$. Let $P := (x_1 = w_0)w_1 \dots w_k y_1$ be the shortest $(V(C_1), V(C_2))$-path in G. We consider four cases.

Case 1. $\omega \in \{x_1, y_1\}$.

Suppose without loss of generality that $\omega = x_1$. By Lemma 2, we have $d(v) \leq 2$ for each $v \in V(G) - \{y_1, \omega\}$. Since $d(\omega) \geq 4$, there is a pendant path $(\omega =)u_0u_1 \dots u_\ell$ $(\ell \geq 1)$. It follows from Corollary 3 that there is no pendant path beginning at y_1 and hence $d(y_1) = 3$. If $k = 0$, then let G' be the graph obtained from G by removing the edge yy_s and adding the edge $u_\ell y_s$ (see Figure 3). Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{3}{4} + \frac{1}{\Delta+2}) - (\frac{2}{5} + \frac{1}{3} + \frac{1}{\Delta+3}) > 0$ when $\ell \geq 2$, and $\frac{1}{2}(H(G') - H(G)) = (\frac{2}{4} + \frac{1}{2+\Delta} + \frac{1}{2+\Delta}) - (\frac{2}{5} + \frac{1}{3+\Delta} + \frac{1}{3+\Delta}) > 0$ if $\ell = 1$, as desired.

If $k \geq 1$, then let G' be the graph obtained from G by removing the edge yy_s and adding the edge $w_k y_s$. Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{1}{4} + \frac{1}{5} + \frac{1}{5}) - (\frac{1}{5} + \frac{1}{5} + \frac{1}{5}) > 0$, as desired.

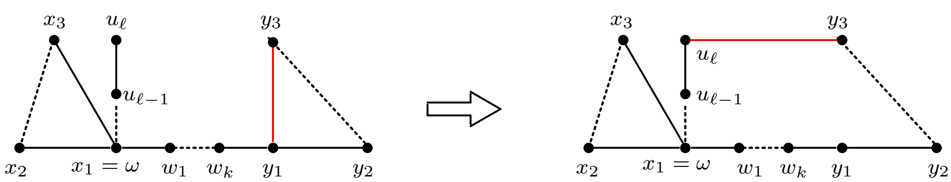


Figure 3. **The transformation used in proof of Lemma 3, case $\omega \in \{x_1, y_1\}$.**

Case 2. $\omega \in V(P) - \{x_1, y_1\}$.

By Lemma 2, $d(v) \leq 2$ for each $v \in V(G) - \{x_1, y_1, \omega\}$. Since $\Delta(G) \geq 4$, there are two pendant paths $(\omega =)u_0u_1 \dots u_\ell$ and $(\omega =)u'_0u'_1 \dots u'_{\ell'}$ $(\ell, \ell' \geq 1)$ beginning at ω. By Corollary 3, there is no pendant path beginning at x_1 or y_1 and this implies that $d(x_1) = d(y_1) = 3$. Let G' be the graph obtained from G by removing the edges x_1x_r, y_1y_s and adding new edges $u_\ell x_r$ and $u'_{\ell'} y_s$ (see Figure 4). We show that $H(G) < H(G')$. We distinguish the following subcases.

Subcase 2.1. $\ell = \ell' = k = 1$.

Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{4}{\Delta+2} + \frac{4}{4}) - (\frac{2}{\Delta+3} + \frac{4}{5} + \frac{2}{\Delta+1}) > 0$, as desired.

Subcase 2.2. $\ell = \ell' = 1$ and $k \geq 2$.

We may assume that $\omega \neq w_1$. If $\omega \neq w_k$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4} + \frac{2}{\Delta+2}) - (\frac{6}{5} + \frac{2}{\Delta+1}) > 0$, and if $\omega = w_k$, then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{3}{\Delta+2}) - (\frac{5}{5} + \frac{2}{\Delta+1} + \frac{1}{\Delta+3}) > 0$, as desired.

Subcase 2.3. $\ell, \ell' \geq 2$ and $k = 1$.
Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4}\frac{2}{\Delta+2}) - (\frac{4}{5} + \frac{2}{3} + \frac{2}{\Delta+3}) > 0$.

Subcase 2.4. $\ell, \ell' \geq 2$ and $k \geq 2$.
We may assume that $\omega \neq w_1$. If $\omega \neq w_k$, then clearly $\frac{1}{2}(H(G') - H(G)) = (\frac{8}{4}) - (\frac{6}{5} + \frac{2}{3}) > 0$, and if $\omega = w_k$, then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{7}{4} + \frac{1}{\Delta+2}) - (\frac{5}{5} + \frac{2}{3} + \frac{1}{\Delta+3}) > 0$, as desired

Subcase 2.5. $\ell = 1, \ell' \geq 2$ and $k = 1$ (the case $\ell \geq 2, \ell' = 1$ and $k = 1$ is similar).
Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{3}{\Delta+2}) - (\frac{4}{5} + \frac{1}{3} + \frac{1}{\Delta+1} + \frac{2}{\Delta+3}) > 0$.

Subcase 2.6. $\ell = 1, \ell' \geq 2$ and $k \geq 2$ (the case $\ell \geq 2, \ell' = 1$ and $k \geq 2$ is similar).
We may assume that $\omega \neq w_1$. If $\omega \neq w_k$, then clearly $\frac{1}{2}(H(G') - H(G)) = (\frac{7}{4} + \frac{1}{\Delta+2}) - (\frac{6}{5} + \frac{1}{3} + \frac{1}{\Delta+1}) > 0$, and if $\omega = w_k$ then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4} + \frac{2}{\Delta+2}) - (\frac{5}{5} + \frac{1}{3} + \frac{1}{\Delta+1} + \frac{1}{\Delta+3}) > 0$, as desired.

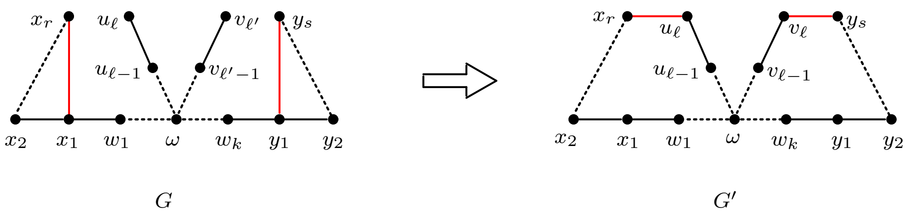


Figure 4. **The transformation used in proof of Lemma 3, case $\omega \in V(P) - \{x_1, y_1\}$.**

Case 3. $\omega \in V_{cycle} - \{x_1, y_1\}$.
Suppose without loss of generality that $\omega \in V(C_1)$ and that $\omega \neq x_r$. As Case 2, we can see that there are two pendant paths $(\omega =)u_0u_1 \ldots u_\ell$ and $(\omega =)u'_0u'_1 \ldots u'_{\ell'}$ $(\ell, \ell' \geq 1)$ beginning at ω and that $d(x_1) = d(y_1) = 3$. Let G' be the graph obtained from G by removing the edges x_1x_r, y_1y_s and adding new edges $u_\ell x_r$ and $u'_{\ell'}y_s$ (see Figure 5). We show that $H(G) < H(G')$. We consider the following subcases.

1. $\omega \neq x_2$, $\ell = \ell' = 1$ and $k = 0$.
   It is easy to verify that $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{2}{\Delta+2}) - (\frac{4}{5} + \frac{1}{6} + \frac{2}{\Delta+1}) > 0$.

2. $\omega = x_2$, $\ell = \ell' = 1$ and $k = 0$.
   Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{4}{4} + \frac{3}{\Delta+2}) - (\frac{3}{5} + \frac{1}{6} + \frac{1}{\Delta+3} + \frac{2}{\Delta+1} > 0$ as desired.

3. $\omega \neq x_2$, $\ell = \ell' = 1$ and $k \geq 1$.
   It is easy to see that $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4} + \frac{2}{\Delta+2}) - (\frac{6}{5} + \frac{2}{\Delta+1}) > 0$ as desired.

4. $\omega = x_2$, $\ell = \ell' = 1$ and $k \geq 1$.
   Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{3}{\Delta+2}) - (\frac{5}{5} + \frac{1}{\Delta+3} + \frac{2}{\Delta+1}) > 0$.

5. $\omega \neq x_2$, $\ell, \ell' \geq 2$ and $k = 0$.
   Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{7}{4}) - (\frac{4}{5} + \frac{1}{6} + \frac{2}{3}) > 0$.

6. $\omega = x_2$, $\ell, \ell' \geq 2$ and $k = 0$.
   Clearly, we have $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4} + \frac{1}{\Delta+2}) - (\frac{3}{5} + \frac{1}{6} + \frac{2}{3} + \frac{1}{\Delta+3}) > 0$.

7. $\omega \neq x_2$, $\ell, \ell' \geq 2$ and $k \geq 1$.
   It is easy to see that $\frac{1}{2}(H(G') - H(G)) = (\frac{8}{4}) - (\frac{6}{5} + \frac{2}{3}) > 0$ as desired.

8. $\omega = x_2$, $\ell, \ell' \geq 2$ and $k \geq 1$.
   Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{7}{4} + \frac{1}{\Delta+2}) - (\frac{5}{5} + \frac{2}{3} + \frac{1}{\Delta+3}) > 0$, as desired.

9. $\omega \neq x_2$, $\ell = 1, \ell' \geq 2$ and $k = 0$ (the case $\omega \neq x_2$, $\ell \geq 2, \ell' = 1$ and $k = 0$ is similar).
   It is easy to verify that $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4} + \frac{1}{\Delta+2}) - (\frac{4}{5} + \frac{1}{6} + \frac{1}{3} + \frac{1}{\Delta+1}) > 0$, as desired.

10. $\omega = x_2$, $\ell = 1, \ell' \geq 2$ and $k = 0$ (the case $\omega = x_2$, $\ell \geq 2, \ell' = 1$ and $k = 0$ is similar).
    Then $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{2}{\Delta+2}) - (\frac{3}{5} + \frac{1}{6} + \frac{1}{\Delta+1} + \frac{1}{\Delta+3} + \frac{1}{3}) > 0$.

11. $\omega \neq x_2$, $\ell = 1, \ell' \geq 2$ and $k \geq 1$ (the case $\omega \neq x_2$, $\ell \geq 2, \ell' = 1$ and $k = 0$ is similar).
    Clearly, we have $\frac{1}{2}(H(G') - H(G)) = (\frac{7}{4} + \frac{1}{\Delta+2}) - (\frac{6}{5} + \frac{1}{3} + \frac{1}{\Delta+1}) > 0$, as desired.

12. $\omega = x_2$, $\ell = 1, \ell' \geq 2$ and $k \geq 1$ (the case $\omega = x_2$, $\ell \geq 2, \ell' = 1$ and $k = 0$ is similar).
    Obviously, we have $\frac{1}{2}(H(G') - H(G)) = (\frac{3}{2} + \frac{1}{\Delta+2} + \frac{1}{\Delta+2}) - (\frac{4}{3} + \frac{1}{\Delta+1} + \frac{1}{\Delta+3}) > 0$ as desired.

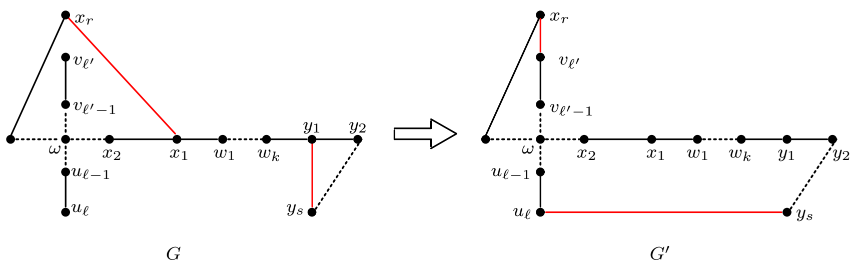


Figure 5. **The transformation used in proof of Lemma 3, case $\omega \in V_{cycle} - \{x_1, y_1\}$.**

Case 4. $\omega \notin V_{cycle} \cup V(P)$.
Let $(\omega =)u_0u_1 \ldots u_\ell$ ($\ell \geq 1$) be a pendant path beginning at ω such that ℓ is as large as possible. Assume $\omega v_1 \ldots v_t$ is the shortest path between ω and $V(C_1) \cup V(C_2) \cup V(P)$. We may assume without loss of generality that $v_t \in V(C_1) \cup V(P) - \{y_1\}$. As Case 2, we can see that $d(y_1) = 3$. Let G' be the graph obtained from G by removing the edge y_1y_s and adding new edges $u_\ell y_s$ (see Figure 6). We show that $H(G) < H(G')$. Consider the following subcases.

- $\ell = 1$ and $k = 0$.
  Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{2}{4} + \frac{1}{5} + \frac{1}{\Delta+2}) - (\frac{2}{5} + \frac{1}{6} + \frac{1}{\Delta+1}) > 0$.
- $\ell = 1$ and $k \geq 1$.
  Then clearly $\frac{1}{2}(H(G') - H(G)) = (\frac{3}{4} + \frac{1}{\Delta+2}) - (\frac{3}{5} + \frac{1}{\Delta+1}) > 0$.
- $\ell \geq 2$ and $k = 0$.
  Then $\frac{1}{2}(H(G') - H(G)) = (\frac{3}{4} + \frac{1}{5}) - (\frac{2}{5} + \frac{1}{6} + \frac{1}{3}) > 0$.
- $\ell \geq 2$ and $k \geq 1$.
  Then $\frac{1}{2}(H(G') - H(G)) = (\frac{4}{4}) - (\frac{3}{5} + \frac{1}{3}) > 0$,

and the proof is complete. □

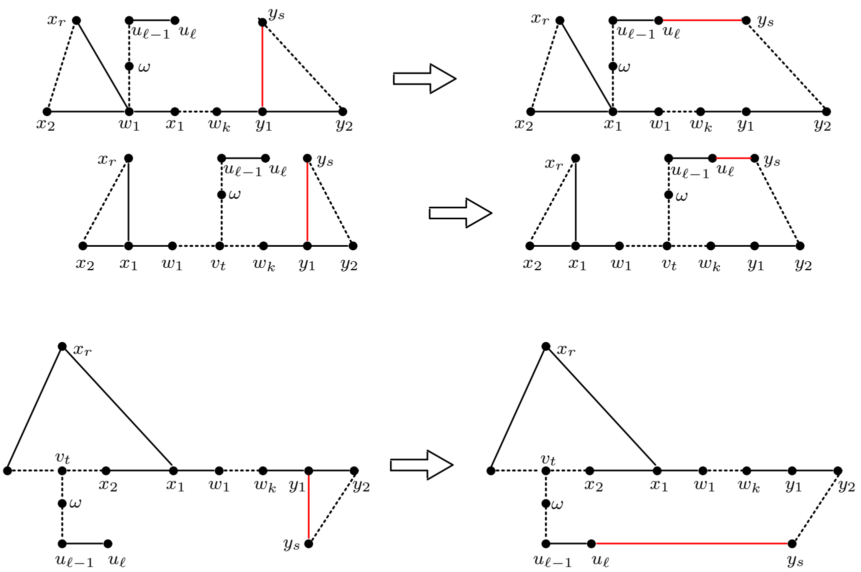


Figure 6. **The transformations used in proof of Lemma 3, case $\omega \notin V_{cycle} \cup V(P)$.**

Lemma 4. If $G \in \mathcal{B}_n^{\Delta}$ is a graph with the maximum value of the harmonic index where $\Delta \geq 4$, then $|V(C_1) \cap V(C_2)| = 1$.

Proof. By Lemma 3, we have $|V(C_1) \cap V(C_2)| \geq 1$. Assume, to the contrary, $|V(C_1) \cap V(C_2)| \geq 2$. Since G is bicyclic, C_1 and C_2 intersect each other in a path. Let $Q := x_1 x_2 \ldots x_k$ be the longest path belonging to $V(C_1) \cap V(C_2)$. We may assume that $x_i = y_i$ for $i = 1, 2 \ldots, k$. Since $\delta \geq 4$, there is a pendant path $(\omega =) u_0 u_1 \ldots u_\ell$ $(\ell \geq 1)$ beginning at ω. We consider four cases.

Case 1. $\omega \in \{x_1, x_k\}$.
Assume without loss of generality that $\omega = x_1$. Since G is a simple graph, we may assume that $s > k$. Let G' be the graph obtained from G by removing the edge $y_k y_{k+1}$ and adding the edge $u_\ell y_{k+1}$ (see Figure 7). We show that $H(G) < H(G')$. Assume that $S = \{x_k x_{k-1}, x_k x_{k+1}, x_k y_{k+1}, u_\ell u_{\ell-1}\}$ and $S' = \{x_k x_{k-1}, x_k x_{k+1}, u_\ell y_{k+1}, u_\ell u_{\ell-1}\}$. We consider the following cases.

- $r \geq k+1$, $k \geq 3$ and $\ell \geq 2$.
  Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{4}{4}) - (\frac{3}{5} + \frac{1}{3}) > 0$.
- $r \geq k+1$, $k \geq 3$ and $\ell = 1$.
  Then $\frac{1}{2}(H(G') - H(G)) = (\frac{3}{4} + \frac{1}{\Delta+2}) - (\frac{3}{5} + \frac{1}{\Delta+1}) > 0$ as desired.
- $r \geq k+1$, $k = 2$ and $\ell \geq 2$.
  It is easy to see that $\frac{1}{2}(H(G') - H(G)) = (\frac{3}{4} + \frac{1}{\Delta+2}) - (\frac{2}{5} + \frac{1}{3} + \frac{1}{\Delta+3}) > 0$ as desired.
- $r \geq k+1$, $k = 2$ and $\ell = 1$.
  Then clearly $\frac{1}{2}(H(G') - H(G)) = (\frac{2}{4} + \frac{2}{\Delta+2}) - (\frac{2}{5} + \frac{1}{\Delta+3} + \frac{1}{\Delta+1}) > 0$.
- $r = k$ and $\ell \geq 2$.
  Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{3}{4} + \frac{1}{\Delta+2}) - (\frac{2}{5} + \frac{1}{3} + \frac{1}{\Delta+3}) > 0$.
- $r = k$ and $\ell = 1$.
  Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{2}{4} + \frac{2}{\Delta+2}) - (\frac{2}{5} + \frac{1}{\Delta+3} + \frac{1}{\Delta+1}) > 0$ as desired.

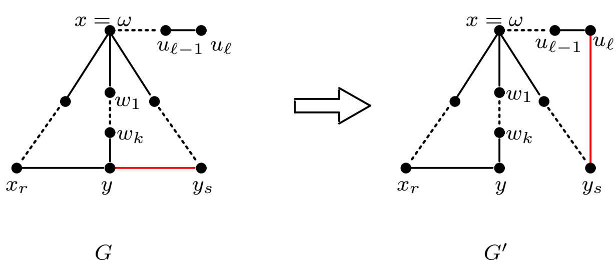


Figure 7. **The transformation used in proof of Lemma 4, case** $\omega \in \{x_1, x_k\}$.

Case 2. $\omega \in V(Q) - \{x_1, x_k\}$.
Suppose $\omega = x_t$ where $2 \leq t \leq k-1$. As Case 1, we may assume that $s \geq k+1$. Since $\Delta(G) \geq 4$, there are two pendant paths $(\omega =)u_0 u_1 \ldots u_\ell$ and $(\omega =)u'_0 u'_1 \ldots u'_{\ell'}$ $(\ell, \ell' \geq 1)$ beginning at ω. We conclude from Corollary 3 that $d(x_1) = d(x_k) = 3$. Let G' be the graph obtained from $G - \{x_1 x_r, x_k y_{k+1}\}$ by adding two edges $u_\ell x_r$ and $u'_{\ell'} y_s$ (see Figure 8). We show that $H(G) < H(G')$. Let $S = \{x_1 x_2, x_1 x_r, x_1 y_s, x_k x_{k-1}, x_k x_{k+1}, x_1 y_{k+1}, u_\ell u_{\ell-1}, u'_\ell u'_{\ell'-1}\}$, $S' = \{x_1 x_2, u_\ell x_r, x_1 y_s, x_k x_{k-1}, x_k x_{k+1}, u'_{\ell'} y_{k+1}, u_\ell u_{\ell-1}, u'_\ell u'_{\ell'-1}\}$ and $A = \sum_{e \in E-S} h_\omega(e)$. We distinguish the following subcases.

1. $k = 3$, $r = k$ and $\ell = \ell' = 1$.
   Then we have $\frac{1}{2}H(G) = A + \frac{2}{5} + \frac{1}{6} + \frac{2}{\Delta+1} + \frac{2}{\Delta+3} < A + \frac{3}{4} + \frac{4}{\Delta+2} = \frac{1}{2}H(G')$ as desired.

2. $k = 3$, $r = k$ and $\ell, \ell' \geq 2$.
   It is easy to see that $\frac{1}{2}H(G) = A + \frac{2}{5} + \frac{1}{6} + \frac{2}{3} + \frac{2}{\Delta+3} < A + \frac{5}{4} + \frac{2}{\Delta+2} = \frac{1}{2}H(G')$.

3. $k = 3$, $r = k$, $\ell = 1$ and $\ell' \geq 2$ (the case $k = 3$, $r = 3$, $\ell \geq 2$ and $\ell' = 1$ is similar).
   It is easy to see that $\frac{1}{2}(H(G') - H(G)) = (\frac{4}{4} + \frac{3}{\Delta+2}) - (\frac{2}{5} + \frac{1}{6} + \frac{1}{3} + \frac{1}{\Delta+1} + \frac{2}{\Delta+3}) > 0$.

4. $k = 3$, $r \geq k+1$ and $\ell = \ell' = 1$.
   It is not hard to see that $\frac{1}{2}(H(G') - H(G)) = (\frac{4}{4} + \frac{4}{\Delta+2}) - (\frac{4}{5} + \frac{2}{\Delta+1} + \frac{2}{\Delta+3}) > 0$.

5. $k = 3$, $r \geq k+1$ and $\ell, \ell' \geq 2$.
   Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4} + \frac{2}{\Delta+2}) - (\frac{4}{5} + \frac{2}{3} + \frac{2}{\Delta+3}) > 0$ as desired.

6. $k = 3$, $r \geq k+1$, $\ell = 1$ and $\ell' \geq 2$ (the case $k = 3$, $r \geq 4$, $\ell \geq 2$ and $\ell' = 1$ is similar).
   It is not hard to see that $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{3}{\Delta+2}) - (\frac{4}{5} + \frac{1}{3} + \frac{1}{\Delta+1} + \frac{2}{\Delta+3}) > 0$.

7. $k \geq 4$, $r = k$ and $\ell = \ell' = 1$.
   We may assume without loss of generality that $\omega \neq x_2$. If $\omega \neq x_{k-1}$, then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{2}{\Delta+2}) - (\frac{4}{5} + \frac{1}{6} + \frac{2}{\Delta+1}) > 0$, and if $\omega = x_{k-1}$, then we obtain $\frac{1}{2}(H(G') - H(G)) = (\frac{4}{4} + \frac{3}{\Delta+2}) - (\frac{3}{5} + \frac{1}{6} + \frac{2}{\Delta+1} + \frac{1}{\Delta+3}) > 0$

8. $k \geq 4$, $r = k$ and $\ell, \ell' \geq 2$.
   Assume without loss of generality that $\omega \neq x_2$. If $\omega \neq x_{k-1}$, then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{7}{4}) - (\frac{4}{5} + \frac{1}{6} + \frac{2}{3}) > 0$, and if $\omega = x_{k-1}$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4} + \frac{1}{\Delta+2}) - (\frac{3}{5} + \frac{1}{6} + \frac{2}{3} + \frac{1}{\Delta+3}) > 0$ as desired.

9. $k \geq 4$, $r = k$, $\ell = 1$ and $\ell' \geq 2$ (the case $k = 3$, $r = 3$, $\ell \geq 2$ and $\ell' = 1$ is similar).
   Suppose without loss of generality that $\omega \neq x_2$. If $\omega \neq x_{k-1}$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4} + \frac{1}{\Delta+2}0 - (\frac{4}{5} + \frac{1}{6} + \frac{1}{3} + \frac{1}{\Delta+1}) > 0$, and if $\omega = x_{k-1}$, then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{2}{\Delta+2}) - (\frac{3}{5} + \frac{1}{6} + \frac{1}{3} + \frac{1}{\Delta+1} + \frac{1}{\Delta+3}) > 0$.

10. $k \geq 4$, $r \geq k+1$ and $\ell = \ell' = 1$.
    Suppose without loss of generality that $\omega \neq x_2$. If $\omega \neq x_{k-1}$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4} + \frac{2}{\Delta+2}) - (\frac{6}{5} + \frac{2}{\Delta+1}) > 0$, and if $\omega = x_{k-1}$ then $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{3}{\Delta+2}) - (\frac{5}{5} + \frac{1}{\Delta+3} + \frac{2}{\Delta+1}) > 0$ as desired.

11. $k \geq 4$, $r \geq k+1$ and $\ell, \ell' \geq 2$.
    Assume that $\omega \neq x_2$. If $\omega \neq x_{k-1}$, then $\frac{1}{2}H(G) = A + \frac{6}{5} + \frac{2}{3} < A + \frac{8}{4} = \frac{1}{2}H(G')$. If $\omega = x_{k-1}$, then we have $\frac{1}{2}H(G) = A + \frac{5}{5} + \frac{2}{3} + \frac{1}{\Delta+3} < A + \frac{7}{4} + \frac{1}{\Delta+2} = \frac{1}{2}H(G')$.

12. $k \geq 4$, $r \geq k+1$, $\ell = 1$ and $\ell' \geq 2$ (the case $k = 3$, $r \geq 4$, $\ell \geq 2$ and $\ell' = 1$ is similar).

Let as above $\omega \neq x_2$. If $\omega \neq x_{k-1}$, then $\frac{1}{2}(H(G')-H(G)) = (\frac{6}{4}+\frac{1}{\Delta+2})-(\frac{6}{5}+\frac{1}{3}+\frac{1}{\Delta+1}) > 0$, and if $\omega = x_{k-1}$, then $\frac{1}{2}(H(G')-H(G)) = (\frac{6}{4}+\frac{2}{\Delta+2})-(\frac{5}{5}+\frac{1}{3}+\frac{1}{\Delta+1}+\frac{1}{\Delta+3}) > 0$ as desired.

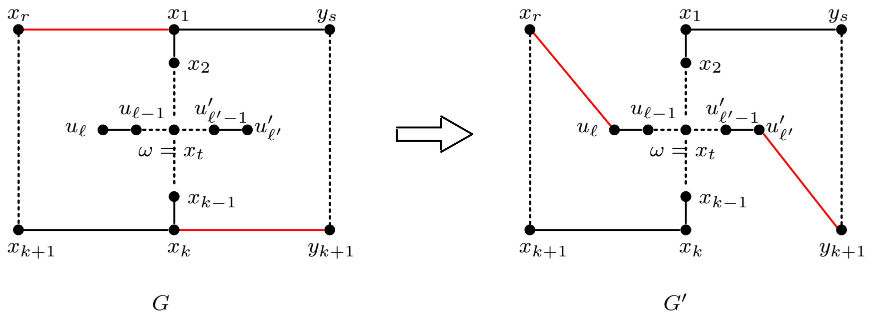


Figure 8. **The transformation used in proof of Lemma 4, case $\omega \in V(P)-\{x_1, x_k\}$.**

Case 3. $\omega \in V_{cycle} - V(Q)$ (see Figure 9).
Assume without loss of generality that $\omega \in V(C_1)-V(C_2)$. As Case 2, there are two pendant paths $(\omega =)u_0u_1\ldots u_\ell$ and $(\omega =)u'_0u'_1\ldots u'_{\ell'}$ $(\ell, \ell' \geq 1)$ beginning at ω and that $d(x_1) = d(x_k) = 3$. Since G is a simple graph, we may assume that $s > k$. Let G' be the graph obtained from $G - \{x_1y_s, x_ky_{k+1}\}$ by adding two edges $u_\ell y_s$ and $u'_{\ell'}y_{k+1}$. We prove that $H(G) < H(G')$. Let $S = \{x_1x_2, x_1x_r, x_1y_s, x_kx_{k-1}, x_kx_{k+1}, x_ky_{k+1}, u_\ell u_{\ell-1}, u'_{\ell'}u'_{\ell'-1}\}$, $S' = \{x_1x_2, x_1x_r, u_\ell y_s, x_kx_{k-1}, x_kx_{k+1}, u'_{\ell'}y_{k+1}, u_\ell u_{\ell-1}, u'_{\ell'}u'_{\ell'-1}\}$ and $A = \sum_{e\in E-S} h_\omega(e)$. Consider the following subcases.

1. $r = k+1$, $k = 2$ and $\ell = \ell' = 1$.
   Then we have $\frac{1}{2}H(G) = A + \frac{2}{5} + \frac{1}{6} + \frac{2}{\Delta+1} + \frac{2}{\Delta+3} < A + \frac{3}{4} + \frac{4}{\Delta+2} = \frac{1}{2}H(G')$ as desired.

2. $r = k+1$, $k = 2$ and $\ell, \ell' \geq 2$.
   Then $\frac{1}{2}(H(G')-H(G)) = (\frac{5}{4}+\frac{2}{\Delta+2})-(\frac{2}{5}+\frac{1}{6}+\frac{2}{3}+\frac{2}{\Delta+3}) > 0$ and we are done.

3. $r = k+1$, $k = 2$, $\ell = 1$ and $\ell' \geq 2$ (the case $r = k+1$, $k = 2$, $\ell \geq 2$ and $\ell' = 1$ is similar).
   Clearly, we have $\frac{1}{2}(H(G')-H(G)) = (\frac{4}{4}+\frac{3}{\Delta+2})-(\frac{2}{5}+\frac{1}{6}+\frac{1}{3}+\frac{1}{\Delta+1}+\frac{2}{\Delta+3}) > 0$.

4. $r \geq k+2$, $k = 2$ and $\ell = \ell' = 1$.
   We may assume that $\omega \neq x_r$. If $\omega \neq x_{k+1}$, then $\frac{1}{2}(H(G')-H(G)) = (\frac{5}{4}+\frac{2}{\Delta+2})-(\frac{4}{5}+\frac{1}{6}+\frac{2}{\Delta+1}) > 0$, and if $\omega = x_{k+1}$ then we have $\frac{1}{2}(H(G')-H(G)) = (\frac{4}{4}+\frac{3}{\Delta+2})-(\frac{3}{5}+\frac{1}{6}+\frac{2}{\Delta+1}+\frac{1}{\Delta+3}) > 0$.

5. $r \geq k+2$, $k = 2$ and $\ell, \ell' \geq 2$.
   Assume that $\omega \neq x_r$. If $\omega \neq x_{k+1}$, then $\frac{1}{2}(H(G')-H(G)) = (\frac{7}{4})-(\frac{4}{5}+\frac{1}{6}+\frac{2}{3}) > 0$,

and if $\omega = x_{k+1}$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4} + \frac{1}{\Delta+2}) - (\frac{3}{5} + \frac{1}{6} + \frac{2}{3} + \frac{1}{\Delta+3}) > 0$ as desired.

6. $r \geq k+2$, $k = 2$, $\ell = 1$ and $\ell' \geq 2$ (the case $r = k+1$, $k = 2$, $\ell \geq 2$ and $\ell' = 1$ is similar).
Suppose that $\omega \neq x_r$. If $\omega \neq x_{k+1}$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4} + \frac{1}{\Delta+2}0 - (\frac{4}{5} + \frac{1}{6} + \frac{1}{3} + \frac{1}{\Delta+1}) > 0$ and if $\omega = x_{k+1}$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{2}{\Delta+2}) - (\frac{3}{5} + \frac{1}{6} + \frac{1}{3} + \frac{1}{\Delta+1} + \frac{1}{\Delta+3}) > 0$ as desired.

7. $r = k+1$, $k \geq 3$ and $\ell = \ell' = 1$.
It is easy to see that $\frac{1}{2}H(G) = A + \frac{4}{5} + \frac{2}{\Delta+1} + \frac{2}{\Delta+3} < A + \frac{4}{4} + \frac{4}{\Delta+2} = \frac{1}{2}H(G')$.

8. $r = k+1$, $k \geq 3$ and $\ell, \ell' \geq 2$.
Then $\frac{1}{2}H(G) = A + \frac{4}{5} + \frac{2}{3} + \frac{2}{\Delta+3} < A + \frac{6}{4} + \frac{2}{\Delta+2} = \frac{1}{2}H(G')$ and we are done.

9. $r = k+1$, $k \geq 3$, $\ell = 1$ and $\ell' \geq 2$ (the case $r = k+1$, $k \geq 3$, $\ell \geq 2$ and $\ell' = 1$ is similar).
Then we have $\frac{1}{2}H(G) = A + \frac{4}{5} + \frac{1}{3} + \frac{1}{\Delta+1} + \frac{2}{\Delta+3} < A + \frac{5}{4} + \frac{3}{\Delta+2} = \frac{1}{2}H(G')$.

10. $r \geq k+2$, $k \geq 3$ and $\ell = \ell' = 1$.
Then we can assume that $\omega \neq x_r$. If $\omega \neq x_{k+1}$, then $\frac{1}{2}H(G) = A + \frac{6}{5} + \frac{2}{\Delta+1} < A + \frac{6}{4} + \frac{2}{\Delta+2} = \frac{1}{2}H(G')$ and if $\omega = x_{k+1}$, then $\frac{1}{2}H(G) = A + \frac{5}{5} + \frac{2}{\Delta+1} + \frac{1}{\Delta+3} < A + \frac{5}{4} + \frac{3}{\Delta+2} = \frac{1}{2}H(G')$.

11. $r \geq k+2$, $k \geq 3$ and $\ell, \ell' \geq 2$.
Assume that $\omega \neq x_r$. If $\omega \neq x_{k+1}$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{8}{4}) - (\frac{6}{5} + \frac{2}{3}) > 0$, and if $\omega = x_{k+1}$ then $\frac{1}{2}(H(G') - H(G)) = (\frac{7}{4} + \frac{1}{\Delta+2}) - (\frac{5}{5} + \frac{2}{3} + \frac{1}{\Delta+3}) > 0$ as desired.

12. $r \geq k+2$, $k \geq 3$, $\ell = 1$ and $\ell' \geq 2$ (the case $r = k+1$, $k \geq 3$, $\ell \geq 2$ and $\ell' = 1$ is similar).
Suppose that $\omega \neq x_r$. If $\omega \neq x_{k+1}$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{7}{4} + \frac{1}{\Delta+2}) - (\frac{6}{5} + \frac{1}{3} + \frac{1}{\Delta+1}) > 0$. And if $\omega = x_{k+1}$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4} + \frac{2}{\Delta+2}) - (\frac{5}{5} + \frac{1}{3} + \frac{1}{\Delta+1} + \frac{1}{\Delta+3}) > 0$ as desired.

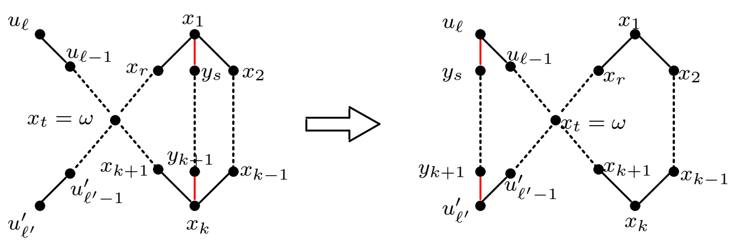


Figure 9. **The transformation used in proof of Lemma 4, Case $\omega \in V_{cycle} - V(P)$.**

Case 4. $\omega \notin V_{cycle} \cup V(Q)$.
Let $\omega v_1 \dots v_t$ be a shortest path between ω and $V(C_1) \cap V(C_2)$ and let $(\omega =)u_0 u_1 \dots u_\ell$

and $(\omega =)u'_0u'_1 \dots u'_{\ell'}$ $(\ell, \ell' \geq 1)$ be two pendant paths beginning at ω. We consider three subcases.

Subcase 4.1. $v_t \in \{x_1, x_k\}$.
We may assume that $v_t = x_1$. We deduce from Corollary 3 that there is no pendant path beginning at x_1 or x_k and so $d(x_1) = 4$ and $d(x_k) = 3$. Since G is a simple graph, we can assume that $s > k$. Let G' be the graph obtained from $G - x_ky_{k+1}$ by adding the edge $u_\ell y_{k+1}$ (see Figure 10 (a)). We show that $H(G') > H(G)$. Suppose $S = \{u_\ell u_{\ell-1}, x_kx_{k-1}, x_kx_{k+1}, x_ky_{k+1}\}$, $S' = \{u_\ell u_{\ell-1}, x_kx_{k-1}, x_kx_{k+1}, u_\ell y_{k+1}\}$ and $A = \sum_{e\in E-S} h_\omega(e)$. We distinguish the following.

1. $k = 2$ and $\ell = 1$.
   Then we have $\frac{1}{2}H(G) = A + \frac{2}{5} + \frac{1}{7} + \frac{1}{\Delta+1} < A + \frac{2}{4} + \frac{1}{6} + \frac{1}{\Delta+2} = \frac{1}{2}H(G')$ as desired.

2. $k = 2$ and $\ell \geq 2$.
   It is easy to see that $\frac{1}{2}H(G) = A + \frac{2}{5} + \frac{1}{7} + \frac{1}{3} < A + \frac{3}{4} + \frac{1}{6} = \frac{1}{2}H(G')$.

3. $k \geq 3$, $r = k$ and $\ell = 1$.
   Then we have $\frac{1}{2}(H(G')-H(G)) = (\frac{2}{4}+\frac{1}{6}+\frac{1}{\Delta+2})-(\frac{2}{5}+\frac{1}{7}+\frac{1}{\Delta+1}) > 0$ as desired.

4. $k \geq 3$, $r \geq k+1$ and $\ell = 1$.
   Clearly, we have $\frac{1}{2}(H(G') - H(G)) = (\frac{3}{4} + \frac{1}{\Delta+2}) - (\frac{3}{5} + \frac{1}{\Delta+1}) > 0$ as desired.

5. $k \geq 3$, $r = k$ and $\ell \geq 2$.
   It is easy to verify that $\frac{1}{2}(H(G') - H(G)) = (\frac{3}{4} + \frac{1}{6}) - (\frac{2}{5} + \frac{1}{7} + \frac{1}{3}) > 0$.

6. $k \geq 3$, $r \geq k+1$ and $\ell \geq 2$.
   Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{4}{4}) - (\frac{3}{5} + \frac{1}{3}) > 0$ as desired.

Subcase 4.2. $v_t \in V(Q) - \{x_1, x_k\}$.
As Subcase 4.1, we can see that $d(x_1) = d(x_k) = 3$ and we may assume that $s > k$. Let G' be the graph obtained from $G - x_ky_{k+1}$ by adding the edge $u_\ell y_{k+1}$ (see Figure 10 (b)). As Subcase 4.1, we can show that $H(G') > H(G)$.

Subcase 4.3. $v_t \in V_{cycle} - V(Q)$.
Suppose without loss of generality that $v_t \in V(C_1)$. As Subcase 4.1, we can see that $d(x_1) = d(x_k) = d(v_t) = 3$. Since G is a simple graph, we must have $k \geq 3$ or $s > k$. Assume without loss of generality that $s > k$. Let G' be the graph obtained from $G - \{x_ky_{k+1}, x_1y_s\}$ by adding two edges $u_\ell y_s, u'_{\ell'}y_{k+1}$ (see Figure 10 (c)). We show that $H(G') > H(G)$. Suppose

$$S = \{x_1x_2, x_1x_r, x_1y_s, x_kx_{k-1}, x_kx_{k+1}, x_ky_{k+1}, u_\ell u_{\ell-1}, u'_{\ell'}u'_{\ell'-1}\},$$

$$S' = \{x_1x_2, x_1x_r, x_kx_{k-1}, x_kx_{k+1}, u_\ell y_s, u'_{\ell'}y_{k+1}, u_\ell u_{\ell-1}, u'_{\ell'}u'_{\ell'-1}\}$$

and $A = \sum_{e\in E-S} h_\omega(e)$. We distinguish the following.

1. $k = 2$, $r = k+1$ and $\ell = \ell' = 1$.
   Then we have $\frac{1}{2}H(G) = A + \frac{2}{5} + \frac{3}{6} + \frac{2}{\Delta+1} < A + \frac{3}{4} + \frac{2}{5} + \frac{2}{\Delta+2} = \frac{1}{2}H(G')$ as desired.

2. $k = 2$, $r = k+1$ and $\ell, \ell' \geq 2$.
   It is easy to see that $\frac{1}{2}H(G) = A + \frac{2}{5} + \frac{3}{6} + \frac{2}{3} < A + \frac{5}{4} + \frac{2}{5} = \frac{1}{2}H(G')$ as desired.

3. $k = 2$, $r = k+1$ and $\ell = 1, \ell' \geq 2$ (the case $k = 2$, $r = k+1$ and $\ell \geq 1, \ell' = 1$ is similar).
   Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{4}{4} + \frac{2}{5} + \frac{1}{\Delta+2}) - (\frac{2}{5} + \frac{3}{6} + \frac{1}{3} + \frac{2}{\Delta+1}) > 0$.

4. $k = 2$, $r \geq k+2$ and $\ell = \ell' = 1$.
   We may assume without loss of generality that $v_t \neq x_r$. If $v_t \neq x_{k+1}$, then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{2}{\Delta+2}) - (\frac{4}{5} + \frac{1}{6} + \frac{2}{\Delta+1}) > 0$, and if $v_t = x_{k+1}$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{4}{4} + \frac{1}{5} + \frac{2}{\Delta+2}) - (\frac{3}{5} + \frac{2}{6} + \frac{2}{\Delta+1}) > 0$ as desired.

5. $k = 2$, $r \geq k+2$ and $\ell, \ell' \geq 2$.
   As above, we may assume that $v_t \neq x_r$. If $v_t \neq x_{k+1}$, then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{7}{4}) - (\frac{4}{5} + \frac{1}{6} + \frac{2}{3}) > 0$, and if $v_t = x_{k+1}$, then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4} + \frac{1}{5}) - (\frac{3}{5} + \frac{2}{6} + \frac{2}{3}) > 0$ as desired.

6. $k = 2$, $r \geq k+2$ and $\ell = 1, \ell' \geq 2$ (the case $k = 2$, $r \geq k+2$ and $\ell \geq 1, \ell' = 1$ is similar).
   We may assume that $v_t \neq x_r$. If $v_t \neq x_{k+1}$, then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{1}{5} + \frac{1}{\Delta+2}) - (\frac{4}{5} + \frac{1}{6} + \frac{1}{3} + \frac{1}{\Delta+1}) > 0$, and if $v_t = x_{k+1}$ then $\frac{1}{2}(H(G') - H(G) = (\frac{4}{4} + \frac{2}{5} + \frac{1}{\Delta+2}) - (\frac{3}{5} + \frac{2}{6} + \frac{1}{3} + \frac{1}{\Delta+1}) > 0$ as desired.

7. $k \geq 3$, $r = k+1$ and $\ell = \ell' = 1$.
   Then we have $\frac{1}{2}H(G) = A + \frac{4}{5} + \frac{2}{6} + \frac{2}{\Delta+1} < A + \frac{4}{4} + \frac{2}{5} + \frac{2}{\Delta+2} = \frac{1}{2}H(G')$ as desired.

8. $k \geq 3$, $r = k+1$ and $\ell, \ell' \geq 2$.
   It is easy to verify that $\frac{1}{2}H(G) = A + \frac{4}{5} + \frac{2}{6} + \frac{2}{3} < A + \frac{6}{4} + \frac{2}{5} = \frac{1}{2}H(G')$ as desired.

9. $k \geq 3$, $r = k+1$ and $\ell = 1, \ell' \geq 2$ (the case $k = 2$, $r = k+1$ and $\ell \geq 1, \ell' = 1$ is similar).
   Then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{2}{5} + \frac{1}{\Delta+2}) - (\frac{4}{5} + \frac{2}{6} + \frac{1}{3} + \frac{2}{\Delta+1}) > 0$.

10. $k \geq 3$, $r \geq k+2$ and $\ell = \ell' = 1$.
    We can assume that $v_t \neq x_r$. If $v_t \neq x_{k+1}$, then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4} + \frac{2}{\Delta+2}) - (\frac{6}{5} + \frac{2}{\Delta+1}) > 0$, and if $v_t = x_{k+1}$ then $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{1}{5} + \frac{2}{\Delta+2}) - (\frac{5}{5} + \frac{1}{6} + \frac{2}{\Delta+1}) > 0$ as desired.

11. $k \geq 3$, $r \geq k+2$ and $\ell, \ell' \geq 2$.
    As above, we may assume that $v_t \neq x_r$. If $v_t \neq x_{k+1}$, then we have $\frac{1}{2}H(G) = A + \frac{6}{5} + \frac{2}{3} < A + \frac{8}{4} = \frac{1}{2}H(G')$. If $v_t = x_{k+1}$, then we have $\frac{1}{2}H(G) = A + \frac{5}{5} + \frac{1}{6} + \frac{2}{3} < A + \frac{7}{4} + \frac{1}{5} = \frac{1}{2}H(G')$.

12. $k \geq 3$, $r \geq k+2$ and $\ell = 1, \ell' \geq 2$ (the case $k = 2$, $r \geq k+2$ and $\ell \geq 1, \ell' = 1$ is similar).
    We may assume that $v_t \neq x_r$. If $v_t \neq x_{k+1}$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{7}{4} +$

$\frac{1}{\Delta+2}) - (\frac{6}{5} + \frac{1}{3} + \frac{1}{\Delta+1}) > 0$, and if $v_t = x_{k+1}$ then $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4} + \frac{1}{5} + \frac{1}{\Delta+2}) - (\frac{5}{5} + \frac{1}{6} + \frac{1}{3} + \frac{1}{\Delta+1}) > 0$ as desired.

This completes the proof. □

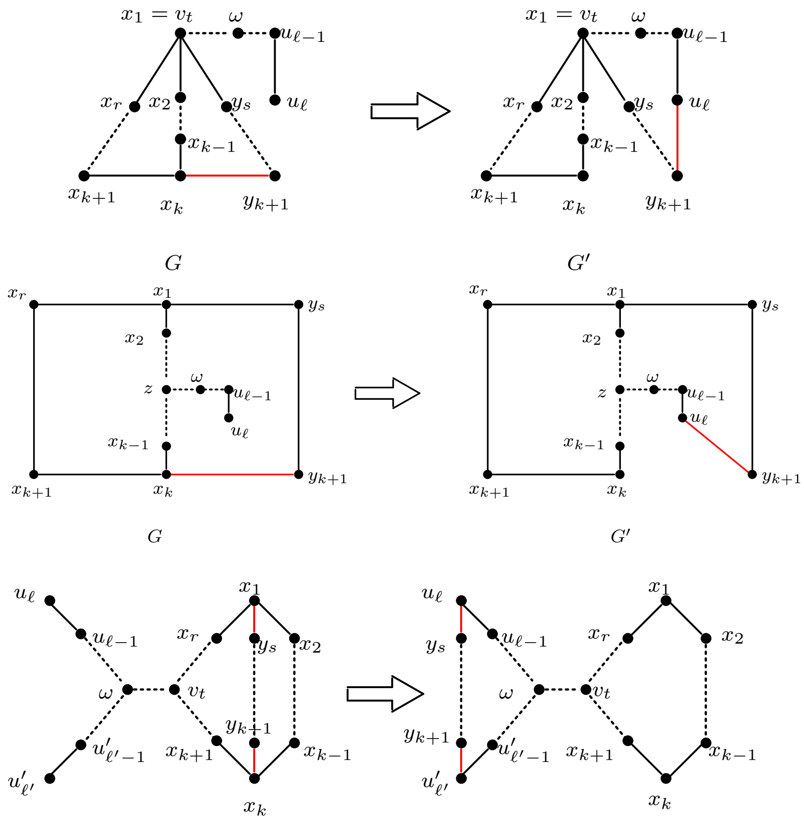


Figure 10. **The transformations used in proof of Lemma 4, Case $\omega \notin V_{cycle}$.**

Lemma 5. If $G \in \mathcal{B}_n^{\Delta}$ is a graph with the maximum value of the harmonic index and $\Delta \geq 4$, then $V(C_1) \cap V(C_2) = \{\omega\}$.

Proof. By Lemmas 3 and 4, we have $|V(C_1) \cap V(C_2)| = 1$. Let $V(C_1) \cap V(C_2) = \{x_1\}$. Suppose, to the contrary, that $\omega \neq x_1$. Since $\Delta(G) \geq 4$, there are two pendant paths $(\omega =)u_0u_1 \ldots u_\ell$ and $(\omega =)u'_0u'_1 \ldots u'_{\ell'}$ $(\ell, \ell' \geq 1)$ beginning at ω. It follows from Corollary 3 that there is no pendant path beginning at x_1 and so $d(x_1) = 4$ unless $x_1 = v_t$, and in this case $d(x_1) = 5$. We consider two cases.

Case 1. $\omega \in V_{cycle}$.

Suppose without loss of generality that $\omega \in V(C_1)$. Since $r \geq 3$, we may assume that $\omega \neq x_2$ (see Figure 11). Let G' be the graph obtained from $G - \{x_1y_2, x_1y_r\}$ by adding the edges $u_\ell y_2$ and $u'_{\ell'}y_s$. We show that $H(G') > H(G)$. We distinguish the following.

1. $\ell = \ell' = 1$.
   If $\omega \neq x_r$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{4}{4} + \frac{2}{\Delta+2}) - (\frac{4}{6} + \frac{2}{\Delta+1}) > 0$, and if $\omega = x_r$ then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{3}{4} + \frac{3}{\Delta+2}) - (\frac{3}{6} + \frac{1}{\Delta+4} + \frac{2}{\Delta+1}) > 0$ as desired.

2. $\ell, \ell' \geq 2$.
   If $\omega \neq x_r$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4}) - (\frac{4}{6} + \frac{2}{3}) > 0$, and if $\omega = x_r$, then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{1}{\Delta+2}) - (\frac{3}{6} + \frac{1}{\Delta+4} + \frac{2}{3}) > 0$.

3. $\ell = 1$ and $\ell' \geq 2$ (the case $\ell \geq 2$ and $\ell' = 1$ is similar).
   If $\omega \neq x_r$, then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{1}{\Delta+2}) - (\frac{4}{6} + \frac{1}{3} + \frac{1}{\Delta+1}) > 0$, and if $\omega = x_r$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{4}{4} + \frac{2}{\Delta+2}) - (\frac{3}{6} + \frac{1}{3} + \frac{1}{\Delta+4} + \frac{1}{\Delta+1}) > 0$ as desired.

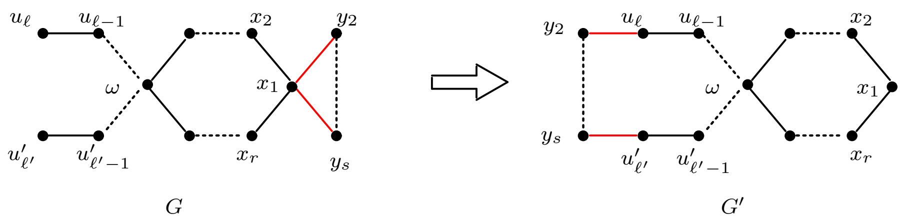


Figure 11. **The transformation used in proof of Lemma 5, Case $\omega \in V_{cycle}$.**

Case 2. $\omega \notin V_{cycle}$.
Let $\omega v_1 \ldots v_t$ be a shortest path between ω and $V(C_1) \cap V(C_2)$. We conclude from Theorem D that there is no pendant path beginning at v_t and this implies $d(v_t) = 5$ when $v_t = x_1$ and $d(v_t) = 3$ when $v_t \neq x_1$. Similarly, we have $d(x_1) = 4$ when $v_t \neq x_1$. We consider two subcases.

Subcase 2.1. $v_t \neq x_1$.
Suppose without loss of generality that $v_t \in V(C_1)$. Since $r \geq 3$, we may assume that $v_t \neq x_2$ (see Figure 12). Let G' be the graph obtained from $G - \{x_1y_2, x_1y_r\}$ by adding the edges $u_\ell y_2$ and $u'_{\ell'}y_s$. We show that $H(G') > H(G)$. We distinguish the following.

1. $\ell = \ell' = 1$.
   If $v_t \neq x_r$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{4}{4} + \frac{2}{\Delta+2}) - (\frac{4}{6} + \frac{2}{\Delta+1}) > 0$ and if $v_t = x_r$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{3}{4} + \frac{1}{5} + \frac{3}{\Delta+2}) - (\frac{3}{6} + \frac{1}{7} + \frac{2}{\Delta+1}) > 0$ as desired.

2. $\ell, \ell' \geq 2$.
   If $v_t \neq x_r$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{6}{4}) - (\frac{4}{6} + \frac{2}{3}) > 0$, and if $v_t = x_r$, then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{1}{5}) - (\frac{3}{6} + \frac{1}{7} + \frac{2}{3}) > 0$ as desired.

3. $\ell = 1$ and $\ell' \geq 2$ (the case $\ell \geq 2$ and $\ell' = 1$ is similar).
   If $v_t \neq x_r$, then we have $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{4} + \frac{1}{\Delta+2}) - (\frac{4}{6} + \frac{1}{3} + \frac{1}{\Delta+1}) > 0$, and if $v_t = x_r$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{4}{4} + \frac{1}{5} + \frac{1}{\Delta+2}) - (\frac{3}{6} + \frac{1}{3} + \frac{1}{7} + \frac{1}{\Delta+1}) > 0$ as desired.

Subcase 2.2. $v_t = x_1$ (see Figure 12 (b)).
Let G' be the graph obtained from $G - \{x_1 y_r\}$ by adding the edge $u_\ell y_s$. We show that $H(G') > H(G)$. We distinguish the following.

1. $\ell = 1$.
   If $\omega \neq v_{t-1}$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{6} + \frac{1}{\Delta+2}) - (\frac{5}{7} + \frac{1}{\Delta+1}) > 0$ and if $\omega = v_{t-1}$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{4}{6} + \frac{1}{\Delta+4} + \frac{1}{\Delta+2}) - (\frac{4}{7} + \frac{1}{\Delta+5} + \frac{1}{\Delta+1}) > 0$ as desired.

2. $\ell \geq 2$.
   If $\omega \neq v_{t-1}$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{5}{6} + \frac{1}{4}) - (\frac{5}{7} + \frac{1}{3}) > 0$ and if $\omega = v_{t-1}$, then $\frac{1}{2}(H(G') - H(G)) = (\frac{4}{6} + \frac{1}{\Delta+4} + \frac{1}{4}) - (\frac{4}{7} + \frac{1}{\Delta+5} + \frac{1}{3}) > 0$ as desired.

This completes the proof. □

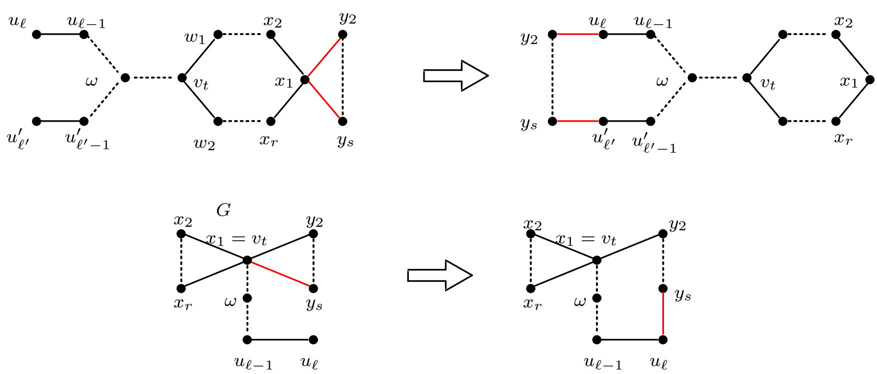


Figure 12. **The transformations used in proof of Lemma 5, Case $\omega \notin V_{cycle}$.**

Next result is an immediate consequence of Lemmas 2,3, 4 and 5.

Corollary 4. If $G \in \mathcal{B}_n^4$ is a graph with the maximum value of the harmonic index, then

$$H(G) \leq \frac{3n-1}{6},$$

with equality if and only if $G \in \widetilde{\mathcal{B}}_n^{(3)}$.

Corollary 5. If $G \in \mathcal{B}_n^\Delta$ ($\Delta \geq 4$) is a graph with the maximum value of the harmonic index, then $V(C_1) \cap V(C_2) = \{\omega\}$ and $d(v) \leq 2$ for each $v \in V(G) - \{\omega\}$.

Lemma 6. Let $\Delta \geq 5$ and $G \in \mathcal{B}_n^{\Delta}$ be a graph with the maximum value of the harmonic index. If there is an edge $e = uv$ belonging to either a cycle of length at least 4 or a pendant path such that $d(u) = d(v) = 2$, then ω is not a support vertex.

Proof. Assume there is an edge $e = uv$ belonging to either a cycle of length at least 4 or a pendant path such that $d(u) = d(v) = 2$. Suppose, to the contrary, that ω is adjacent to a leaf p. By Corollary 5, we have $V(C_1) \cap V(C_2) = \{\omega\}$. If e belongs to a pendant path, then let $d(v, \omega) > d(u, \omega)$, and if e belongs to a cycle of length at four, then let z be the neighbor of v different from u with degree 2 (this is possible because the length of C_1 is at least four). Assume G' is the graph obtained from $G - \{vu, vz\}$ by adding the edges uz and pv (see Figure 13). We show that $H(G') > H(G)$ which is a contradiction by the choice of G. We have $\frac{1}{2}(H(G') - H(G)) = (\frac{1}{4} + \frac{1}{3} + \frac{1}{\Delta+2}) - (\frac{2}{4} + \frac{1}{\Delta+1}) > 0$ and the proof is complete. □

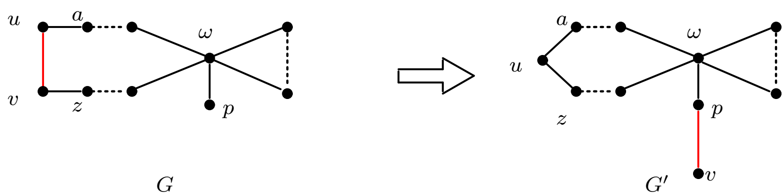


Figure 13. **The transformation used in proof of Lemma 6.**

Theorem 1. If $G \in \mathcal{B}_n^{\Delta}$ $(\Delta \geq 5)$ is a graph with the maximum value of the harmonic index, then

$$\frac{1}{2}H(G) = \begin{cases} \frac{2\Delta-n-3}{\Delta+1} + \frac{n-\Delta+3}{\Delta+2} + \frac{1}{2} + \frac{n-\Delta-1}{3} & \text{if } n \leq 2\Delta - 4 \\ \frac{\Delta}{\Delta+2} + \frac{\Delta-4}{3} + \frac{n-2\Delta+4}{4} & \text{if } n \geq 2\Delta - 3. \end{cases}$$

Proof. Let $G \in \mathcal{B}_n^{\Delta}$ $(\Delta \geq 5)$ be a graph with the maximum value of the harmonic index. By Corollary 5, $V(C_1) \cap V(C_2) = \{\omega\}$ and $d(v) \leq 2$ for each vertex $v \in V(G) - \{\omega\}$. First let ω be a support vertex. Then we conclude from Lemma 6 that C_1 and C_2 are triangle and that each pendant path has length at mots two. Suppose p is the number of pendant paths of length 2 beginning at ω. Then clearly $p \leq \Delta - 5$ and $n = 5 + \Delta - 4 + p$ yielding $n \leq 2\Delta - 4$. Now we have

$$\frac{1}{2}H(G) = \frac{1}{2} + \frac{p}{3} + \frac{\Delta - 4 - p}{\Delta + 1} + \frac{4 + p}{\Delta + 2} = \frac{2\Delta - n - 3}{\Delta + 1} + \frac{n - \Delta + 3}{\Delta + 2} + \frac{1}{2} + \frac{n - \Delta - 1}{3}.$$

Now assume ω is not a support vertex. Obviously, $n \geq 2\Delta - 3$. On the other have, we have $h_w(e) = \frac{1}{3}$ if e is a pendant edge, $h_w(e) = \frac{1}{\Delta+2}$ if e is incident to ω, and

$h_w(e) = \frac{1}{4}$ otherwise. Thus

$$\frac{1}{2}H(G) = \frac{\Delta}{\Delta+2} + \frac{\Delta-4}{3} + \frac{n-2\Delta+4}{4}$$

and the proof is complete. □

Considering the proof of Theorem 1, we introduce two family of bicyclic graphs. Suppose $\Delta \geq 5$. Let F_n^{Δ} be the set of bicyclic graphs of order $n \geq 6$ obtained from S_5^{++} by adding $\Delta - 4$ pendant paths of length at most two to the vertex x_1 of S_5^{++} with degree four, such that x_1 is a support vertex. Let E_n^{Δ} be the set of bicyclic graphs of order n obtained from a graph G in $\widetilde{\mathcal{B}}_{n'}^{(3)}$ $(n' \leq n - 2\Delta + 8)$ by adding $\Delta - 4$ pendant paths of length at least two to the vertex x_1 of G with degree four.
Next result is an immediate consequence of Theorem 1 and its proof.

Corollary 6. If $G \in \mathcal{B}_n^{\Delta}$ $(\Delta \geq 5)$, then

$$\frac{1}{2}H(G) \leq \begin{cases} \frac{2\Delta-n-3}{\Delta+1} + \frac{n-\Delta+3}{\Delta+2} + \frac{1}{2} + \frac{n-\Delta-1}{3} & \text{if } n \leq 2\Delta - 4 \\ \frac{\Delta}{\Delta+2} + \frac{\Delta-4}{3} + \frac{n-2\Delta+4}{4} & \text{if } n \geq 2\Delta - 3, \end{cases}$$

with equality if and only if $G \in F_n^{\Delta}$ if $n \leq 2\Delta - 4$ and $G \in E_n^{\Delta}$ if $n \geq 2\Delta - 3$.

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