

## Some results on a supergraph of the comaximal ideal graph of a commutative ring

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**Abstract:** The rings considered in this article are commutative with identity which admit at least two maximal ideals. We denote the set of all maximal ideals of a ring  $R$  by  $Max(R)$  and we denote the Jacobson radical of  $R$  by  $J(R)$ . Let  $R$  be a ring such that  $|Max(R)| \geq 2$ . Let  $\mathbb{I}(R)$  denote the set of all proper ideals of  $R$ . In this article, we associate an undirected graph denoted by  $\mathbb{INC}(R)$  with a subcollection of ideals of  $R$  whose vertex set is  $\{I \in \mathbb{I}(R) | I \not\subseteq J(R)\}$  and two distinct vertices  $I_1, I_2$  are adjacent in  $\mathbb{INC}(R)$  if and only if  $I_1 \not\subseteq I_2$  and  $I_2 \not\subseteq I_1$  (that is,  $I_1$  and  $I_2$  are not comparable under the inclusion relation). The aim of this article is to investigate the interplay between the graph-theoretic properties of  $\mathbb{INC}(R)$  and the ring-theoretic properties of  $R$ .

**Keywords:** Chained ring, diameter of a graph, bipartite graph, split graph, complemented graph

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### 1. Introduction

The rings considered in this article are commutative with identity which admit at least two maximal ideals. Let  $R$  be a ring. We denote the set of all maximal ideals of  $R$  by  $Max(R)$  and we denote the Jacobson radical of  $R$  by  $J(R)$ . As in [8], we denote the collection of all proper ideals of  $R$  by  $\mathbb{I}(R)$ . For a set  $A$ , we denote the cardinality of  $A$  by the notation  $|A|$ . This article is motivated by the interesting theorems proved by M. Ye and T. Wu in [17]. Let  $R$  be a ring with  $|Max(R)| \geq 2$ . Inspired by the research work done on the comaximal graph of a ring in [10, 12–16] and the research work done on the annihilating-ideal graph of a ring in [8, 9]. M. Ye and T. Wu in [17], introduced and investigated an undirected graph associated with  $R$  whose vertex set

equals  $\{I \in \mathbb{I}(R) \mid I \not\subseteq J(R)\}$  and distinct vertices  $I_1$  and  $I_2$  are joined by an edge if and only if  $I_1 + I_2 = R$ . M. Ye and T. Wu called the graph introduced and studied by them in [17] as the *comaximal ideal graph* of  $R$  and denoted it by the notation  $\mathcal{C}(R)$ . For a ring  $R$ , we denote the set of all units of  $R$  by  $U(R)$  and the set of all nonunits of  $R$  by  $NU(R)$ .

This article is also motivated by the inspiring theorems proved on cozero-divisor graph of a commutative ring by M. Afkhami and K. Khashyarmansh in [1–3]. Let  $R$  be a ring. Recall from [1] that the *cozero-divisor graph* of  $R$  denoted by  $\Gamma'(R)$  is an undirected graph whose vertex set is  $NU(R) \setminus \{0\}$  and distinct vertices  $a, b$  are joined by an edge if and only if  $a \notin Rb$  and  $b \notin Ra$ . That is,  $a, b$  are joined by an edge if and only if  $Ra$  and  $Rb$  are not comparable under the inclusion relation.

Let  $R$  be a ring with  $|Max(R)| \geq 2$ . Motivated by the research work on the comaximal ideal graph of a commutative ring in [17] and by the research work on the cozero-divisor graph of a ring in [1–3], in this article, we introduce an undirected graph structure associated with  $R$ , denoted by  $\mathbb{INC}(R)$  whose vertex set equals  $\{I \in \mathbb{I}(R) \mid I \not\subseteq J(R)\}$  and distinct vertices  $I_1, I_2$  are joined by an edge if and only if  $I_1 \not\subseteq I_2$  and  $I_2 \not\subseteq I_1$ . That is,  $I_1$  and  $I_2$  are joined by an edge if and only if  $I_1$  and  $I_2$  are not comparable under the inclusion relation. The aim of this article is to study the interplay between the graph-theoretic properties of  $\mathbb{INC}(R)$  and the ring-theoretic properties of  $R$ .

The graphs considered in this article are undirected and simple. Let  $G = (V, E)$  be a graph. We denote the vertex set of  $G$  by  $V(G)$  and the edge set of  $G$  by  $E(G)$ . If  $H$  is a subgraph of  $G$ , then we say that  $G$  is a *supergraph* of  $H$ . A subgraph  $H$  of  $G$  is said to be a *spanning subgraph* of  $G$  if  $V(H) = V(G)$ . Let  $R$  be a ring with  $|Max(R)| \geq 2$ . Observe that  $V(\mathcal{C}(R)) = V(\mathbb{INC}(R)) = \{I \in \mathbb{I}(R) \mid I \not\subseteq J(R)\}$ . Let  $I_1, I_2 \in V(\mathcal{C}(R))$  be such that  $I_1 \neq I_2$ . If there is an edge of  $\mathcal{C}(R)$  joining  $I_1$  and  $I_2$ , then  $I_1 + I_2 = R$ . Since  $I_1, I_2 \in \mathbb{I}(R)$ , it follows that  $I_1 \not\subseteq I_2$  and  $I_2 \not\subseteq I_1$ . Hence, there is an edge of  $\mathbb{INC}(R)$  joining  $I_1$  and  $I_2$ . Therefore,  $\mathcal{C}(R)$  is a spanning subgraph of  $\mathbb{INC}(R)$ . In this article, we study the influence of some graph parameters of  $\mathbb{INC}(R)$  on the structure of the ring  $R$ .

It is useful to recall the following definitions from graph theory before we describe the results that are proved in this article on  $\mathbb{INC}(R)$ , where  $R$  is a ring with  $|Max(R)| \geq 2$ . Let  $G = (V, E)$  be a graph. Let  $a, b \in V$  with  $a \neq b$ . Recall that the *distance* between  $a$  and  $b$ , denoted by  $d(a, b)$  is defined as the length of a shortest path in  $G$  if there exists such a path in  $G$ ; otherwise, we define  $d(a, b) = \infty$ . We define  $d(a, a) = 0$ . The *diameter* of  $G$ , denoted by  $diam(G)$  is defined as  $diam(G) = \sup\{d(a, b) \mid a, b \in V\}$  [6]. A graph  $G = (V, E)$  is said to be *connected* if for any distinct  $a, b \in V$ , there exists a path in  $G$  between  $a$  and  $b$ . Let  $G = (V, E)$  be a connected graph. Let  $a \in V$ . Then the *eccentricity* of  $a$ , denoted by  $e(a)$  is defined as  $e(a) = \sup\{d(a, b) \mid b \in V\}$ . The *radius* of  $G$ , denoted by  $r(G)$  is defined as  $r(G) = \min\{e(a) \mid a \in V\}$  [6]. Let  $G = (V, E)$  be a graph. Recall from [[6], page 159] that the *girth* of  $G$ , denoted by  $girth(G)$  is defined as the length of a shortest cycle in  $G$  if  $G$  admits at least one cycle. If  $G$  does not admit any cycle, then we set  $girth(G) = \infty$ . A simple graph  $G = (V, E)$  is said to be *complete* if every pair of distinct vertices of  $G$  are adjacent

in  $G$  [[6], Definition 1.1.11]. Recall from [[6], Definition 1.2.2] that a *clique* of  $G$  is a complete subgraph of  $G$ . A subset  $S$  of  $G$  is said to be an *independent set* if no two members of  $S$  are adjacent in  $G$ .

A graph  $G = (V, E)$  is said to be *bipartite* if  $V$  can be partitioned into nonempty subsets  $V_1$  and  $V_2$  such that each edge of  $G$  has one end in  $V_1$  and the other in  $V_2$ . A bipartite graph with vertex partition  $V_1$  and  $V_2$  is said to be *complete* if each element of  $V_1$  is adjacent to every element of  $V_2$ . A complete bipartite graph with vertex partition  $V_1$  and  $V_2$  is called a *star* if either  $|V_1| = 1$  or  $|V_2| = 1$  [[6], Definition 1.1.12]. Let  $I$  be an ideal of a ring  $R$ . As in [15], we denote  $\{\mathfrak{m} \in \text{Max}(R) \mid \mathfrak{m} \supseteq I\}$  by  $M(I)$ . The Krull dimension of a ring  $R$  is simply referred to as the dimension of  $R$  and is denoted by the notation  $\dim R$ . A ring which has only one maximal ideal is referred to as a *quasilocal* ring. A ring which has only a finite number of maximal ideals is referred to as a *semiquasilocal* ring. A Noetherian quasilocal (respectively, semiquasilocal) ring is referred to as a *local* (respectively, *semilocal*) ring. Recall from [[11], page 184] that a ring  $R$  is said to be a *chained ring* if the set of ideals of  $R$  is linearly ordered by inclusion. Whenever a set  $A$  is a subset of a set  $B$  and  $A \neq B$ , then we denote it symbolically by the notation  $A \subset B$ . For  $n \in \mathbb{N}$  with  $n \geq 2$ , we denote the ring of integers modulo  $n$  by  $\mathbb{Z}_n$ .

Let  $R$  be a ring such that  $|\text{Max}(R)| \geq 2$ . In Section 2 of this article, some basic properties of  $\text{INC}(R)$  are proved. It is proved in Lemma 1 that  $\text{INC}(R)$  is connected and  $\text{diam}(\text{INC}(R)) \leq 2$ . If  $|\text{Max}(R)| \geq 3$ , then it is shown that  $\text{diam}(\text{INC}(R)) = r(\text{INC}(R)) = 2$  (see, Lemmas 1 and 2). Let  $R$  be a ring with  $\text{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ . We denote  $\{I \in V(\text{INC}(R)) \mid M(I) = \{\mathfrak{m}_i\}\}$  by  $V_i$  for each  $i \in \{1, 2\}$ . It is proved in Lemma 3 that  $\text{diam}(\text{INC}(R)) = r(\text{INC}(R)) = 2$  if  $|V_i| \geq 2$  for each  $i \in \{1, 2\}$ . It is shown in Proposition 1 that  $\text{INC}(R)$  is complete if and only if  $R \cong F_1 \times F_2$  as rings, where  $F_i$  is a field for each  $i \in \{1, 2\}$ . It is observed in Lemma 4 that if  $|\text{Max}(R)| \geq 3$ , then  $\mathcal{C}(R) \neq \text{INC}(R)$ . The rest of Section 2 is devoted to characterizing rings  $R$  with  $|\text{Max}(R)| = 2$  such that  $\mathcal{C}(R) = \text{INC}(R)$ . If  $|\text{Max}(R)| = 2$ , then we know from (3)  $\Rightarrow$  (1) of [[17], Theorem 4.5] that  $\mathcal{C}(R)$  is a complete bipartite graph. Hence, we focus on characterizing rings  $R$  such that  $\text{INC}(R)$  is a bipartite graph. If  $\text{INC}(R)$  is bipartite, then it is verified in Proposition 2 that  $\text{INC}(R) = \mathcal{C}(R)$  is a complete bipartite graph. Let  $R = R_1 \times R_2$ , where  $(R_i, \mathfrak{m}_i)$  is a quasilocal ring for each  $i \in \{1, 2\}$ . It is shown in Proposition 3 that  $\text{INC}(R)$  is a bipartite graph if and only if  $R_i$  is a chained ring for each  $i \in \{1, 2\}$ . For a ring  $R$  with  $|\text{Max}(R)| \geq 2$ , it is proved in Proposition 4 that  $\text{INC}(R)$  is a star graph if and only if  $R \cong R_1 \times R_2$  as rings, where  $R_i$  is a chained ring for each  $i \in \{1, 2\}$  with  $R_i$  a field for at least one  $i \in \{1, 2\}$ . If  $\dim R = 0$ , then it is shown in Proposition 5 that  $\text{INC}(R)$  is a bipartite graph if and only if  $R \cong R_1 \times R_2$  as rings, where  $R_i$  is a zero-dimensional chained ring for each  $i \in \{1, 2\}$ . In Example 1 (respectively, in Example 2), an example is provided to illustrate that (i)  $\Rightarrow$  (ii) of Proposition 5 can fail to hold if the hypothesis  $\dim R = 0$  is omitted in Proposition 5. For a ring  $R$  with  $|\text{Max}(R)| \geq 2$ , it is verified in Proposition 6 that  $\text{girth}(\text{INC}(R)) \in \{3, 4, \infty\}$ . Moreover, it is proved in Proposition 6 that  $\text{girth}(\text{INC}(R)) = \infty$  if and only if  $R \cong R_1 \times R_2$  as rings, where  $R_1$  is a field and  $R_2$  is a chained ring.

Let  $G = (V, E)$  be a graph. Recall that  $G$  is a *split graph* if  $V$  is the disjoint union of two nonempty subsets  $K$  and  $S$  such that the subgraph of  $G$  induced on  $K$  is complete and  $S$  is an independent set of  $G$ . In [12], M. I. Jinnah and S.C. Mathew classified rings  $R$  such that the comaximal graph of  $R$  is a split graph. In Section 3 of this article, we try to characterize rings  $R$  with  $|Max(R)| \geq 2$  such that  $\text{INC}(R)$  is a split graph. It is proved in Proposition 7 that if  $\text{INC}(R)$  is a split graph, then  $|Max(R)| = 2$ . Let  $R$  be a ring such that  $|Max(R)| = 2$ . It is shown in Theorem 2 that  $\text{INC}(R)$  is a split graph if and only if  $R \cong R_1 \times R_2$  as rings, where  $R_i$  is a quasilocal ring for each  $i \in \{1, 2\}$  with  $R_i$  is a field for at least one  $i \in \{1, 2\}$  and if  $R_i$  is not a field for some  $i \in \{1, 2\}$ , then either  $R_i$  is a chained ring or  $\mathbb{I}(R_i) = W_1 \cup W_2$  satisfying the property that  $|W_k| \geq 2$  for each  $k \in \{1, 2\}$  such that  $W_1$  is a chain under the inclusion relation and no two distinct members of  $W_2$  are comparable under the inclusion relation. Some examples are given in Example 4 to illustrate Theorem 2.

Let  $G = (V, E)$  be a graph. Recall from [4] that two distinct vertices  $u, v$  of  $G$  are said to be *orthogonal*, written  $u \perp v$  if  $u$  and  $v$  are adjacent in  $G$  and there is no vertex of  $G$  which is adjacent to both  $u$  and  $v$  in  $G$ ; that is, the edge  $u - v$  is not the edge of any triangle in  $G$ . Let  $u \in V$ . A vertex  $v$  of  $G$  is said to be a *complement* of  $u$  if  $u \perp v$  [4]. Moreover, recall from [4] that  $G$  is *complemented* if each vertex of  $G$  admits a complement in  $G$ . Furthermore,  $G$  is said to be *uniquely complemented* if  $G$  is complemented and whenever the vertices  $u, v, w$  of  $G$  are such that  $u \perp v$  and  $u \perp w$ , then a vertex  $x$  of  $G$  is adjacent to  $v$  in  $G$  if and only if  $x$  is adjacent to  $w$  in  $G$ . Let  $R$  be a ring which is not an integral domain. The authors of [4] determined in Section 3 of [4] rings  $R$  such that  $\Gamma(R)$  is complemented or uniquely complemented, where  $\Gamma(R)$  is the zero-divisor graph of  $R$ . Let  $R$  be a ring with  $|Max(R)| \geq 2$ . In [[15], Proposition 3.11], it was shown that the subgraph of the comaximal graph of  $R$  induced on  $NU(R) \setminus J(R)$  is complemented if and only if  $\dim(\frac{R}{J(R)}) = 0$ . In Section 4 of this article, we try to characterize rings  $R$  with  $|Max(R)| \geq 2$  such that  $\text{INC}(R)$  is complemented. It is proved in Lemma 10 that if  $\text{INC}(R)$  is complemented, then  $|Max(R)| \leq 3$ . Let  $R$  be a ring with  $|Max(R)| = 2$ . Let  $Max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ . Let  $V_1 = \{I \in \mathbb{I}(R) \mid M(I) = \{\mathfrak{m}_1\}\}$  and let  $V_2 = \{J \in \mathbb{I}(R) \mid M(J) = \{\mathfrak{m}_2\}\}$ . If  $|V_i| = 1$  for each  $i \in \{1, 2\}$ , (it is noted in Remark 3 that this can happen if and only if  $R \cong F_1 \times F_2$  as rings, where  $F_i$  is a field for each  $i \in \{1, 2\}$ ) then it is clear that  $\text{INC}(R)$  is a complete graph on two vertices and hence,  $\text{INC}(R)$  is complemented. Suppose that  $|V_1| = 1$  and  $|V_2| \geq 2$ . Then it is shown in Proposition 9 that  $\text{INC}(R)$  is complemented if and only if  $R \cong R_1 \times R_2$  as rings, where  $R_1$  is a chained ring which is not a field and  $R_2$  is a field. Suppose that  $|V_i| \geq 2$  for each  $i \in \{1, 2\}$ . It is proved in Proposition 10 that  $\text{INC}(R)$  is complemented if and only if  $\text{INC}(R)$  is a complete bipartite graph. We are not able to characterize rings  $R$  with  $|Max(R)| = 2$  such that  $|V_i| \geq 2$  for each  $i \in \{1, 2\}$  and  $\text{INC}(R)$  is complemented. However, if  $\dim R = 0$ , it is shown in Proposition 11 that  $R$  has the above mentioned properties if and only if  $R \cong R_1 \times R_2$  as rings, where  $R_i$  is a chained ring which is not a field for each  $i \in \{1, 2\}$ . In Remark 4, an example is mentioned to illustrate that the hypothesis  $\dim R = 0$  cannot be omitted in Proposition 11. Let  $R$  be a ring such that  $|Max(R)| = 3$ . It is proved in Theorem 3 that  $\text{INC}(R)$  is complemented if and only if  $R \cong F_1 \times F_2 \times F_3$

as rings, where  $F_i$  is a field for each  $i \in \{1, 2, 3\}$ .

## 2. Basic properties of $\text{INC}(R)$

In this section we investigate some basic properties of  $\text{INC}(R)$ .

**Lemma 1.** Let  $R$  be a ring such that  $|\text{Max}(R)| \geq 2$ . Then  $\text{INC}(R)$  is connected and  $\text{diam}(\text{INC}(R)) \leq 2$ . If  $|\text{Max}(R)| \geq 3$ , then  $\text{diam}(\text{INC}(R)) = 2$ .

*Proof.* Let  $I_1, I_2 \in V(\text{INC}(R))$  be such that  $I_1 \neq I_2$ . Suppose that  $I_1$  and  $I_2$  are not adjacent in  $\text{INC}(R)$ . Then either  $I_1 \subset I_2$  or  $I_2 \subset I_1$ . Without loss of generality, we can assume that  $I_1 \subset I_2$ . Since  $I_1 \in V(\text{INC}(R))$ , there exists  $\mathfrak{m} \in \text{Max}(R)$  such that  $I_1 \not\subseteq \mathfrak{m}$ . As  $I_1 \subset I_2$ , it follows that  $I_2 \not\subseteq \mathfrak{m}$ . Thus  $I_i + \mathfrak{m} = R$  for each  $i \in \{1, 2\}$  and so,  $I_1 - \mathfrak{m} - I_2$  is a path in  $\mathcal{C}(R)$  and hence, it is a path in  $\text{INC}(R)$ . This proves that  $\text{INC}(R)$  is connected and  $\text{diam}(\text{INC}(R)) \leq 2$ .

Suppose that  $|\text{Max}(R)| \geq 3$ . Let  $\{\mathfrak{m}_i \mid i \in \{1, 2, 3\}\} \subseteq \text{Max}(R)$ . Note that  $\mathfrak{m}_1, \mathfrak{m}_1 \cap \mathfrak{m}_2 \in V(\text{INC}(R))$  and as  $\mathfrak{m}_1 \cap \mathfrak{m}_2 \subset \mathfrak{m}_1$ , it follows that  $\mathfrak{m}_1$  and  $\mathfrak{m}_1 \cap \mathfrak{m}_2$  are not adjacent in  $\text{INC}(R)$ . Hence, we obtain that  $\text{diam}(\text{INC}(R)) \geq 2$  and so,  $\text{diam}(\text{INC}(R)) = 2$ .  $\square$

**Lemma 2.** Let  $R$  be a ring such that  $|\text{Max}(R)| \geq 3$ . Then  $r(\text{INC}(R)) = 2$ .

*Proof.* We know from Lemma 1 that  $\text{INC}(R)$  is connected and  $\text{diam}(\text{INC}(R)) = 2$ . Let  $I \in V(\text{INC}(R))$ . Then  $I \subseteq \mathfrak{m}$  for some  $\mathfrak{m} \in \text{Max}(R)$ . We consider the following cases.

**Case 1.**  $I = \mathfrak{m}$ .

Let  $\mathfrak{m}' \in \text{Max}(R)$  be such that  $\mathfrak{m}' \neq \mathfrak{m}$ . As  $|\text{Max}(R)| \geq 3$ ,  $\mathfrak{m} \cap \mathfrak{m}' \in V(\text{INC}(R))$ . From  $\mathfrak{m} \cap \mathfrak{m}' \subset \mathfrak{m}$ , we obtain that  $\mathfrak{m}$  and  $\mathfrak{m} \cap \mathfrak{m}'$  are not adjacent in  $\text{INC}(R)$ . Hence,  $d(\mathfrak{m}, \mathfrak{m} \cap \mathfrak{m}') \geq 2$  in  $\text{INC}(R)$ . This shows that  $e(\mathfrak{m}) \geq 2$  in  $\text{INC}(R)$ .

**Case 2.**  $I \subset \mathfrak{m}$ .

Now,  $I$  and  $\mathfrak{m}$  are not adjacent in  $\text{INC}(R)$  and so,  $d(I, \mathfrak{m}) \geq 2$  in  $\text{INC}(R)$ . Hence,  $e(I) \geq 2$  in  $\text{INC}(R)$ .

This proves that  $e(I) \geq 2$  in  $\text{INC}(R)$  for any  $I \in V(\text{INC}(R))$  and from  $\text{diam}(\text{INC}(R)) = 2$ , it follows that  $e(I) = 2$  for each  $I \in \text{INC}(R)$ . Therefore,  $r(\text{INC}(R)) = 2$ .  $\square$

**Lemma 3.** Let  $R$  be a ring such that  $|\text{Max}(R)| = 2$ . Let  $\{\mathfrak{m}_1, \mathfrak{m}_2\}$  denote the set of all maximal ideals of  $R$ . Let  $V_1 = \{I \in V(\text{INC}(R)) \mid M(I) = \{\mathfrak{m}_1\}\}$  and let  $V_2 = \{J \in V(\text{INC}(R)) \mid M(J) = \{\mathfrak{m}_2\}\}$ . If  $|V_i| \geq 2$  for each  $i \in \{1, 2\}$ , then  $\text{diam}(\text{INC}(R)) = r(\text{INC}(R)) = 2$ .

*Proof.* We know from Lemma 1 that  $\text{INC}(R)$  is connected and  $\text{diam}(\text{INC}(R)) \leq 2$ . Suppose that  $|V_i| \geq 2$  for each  $i \in \{1, 2\}$ . Note that  $\mathfrak{m}_i \in V_i$  for each  $i \in \{1, 2\}$ .

Observe that there exist  $I_1 \in V_1$  such that  $I_1 \neq \mathfrak{m}_1$  and  $I_2 \in V_2$  such that  $I_2 \neq \mathfrak{m}_2$ . It is clear that  $\mathfrak{m}_1$  and any  $I \in V_1 \setminus \{\mathfrak{m}_1\}$  are not adjacent in  $\mathbb{INC}(R)$ . Therefore,  $e(A) \geq 2$  in  $\mathbb{INC}(R)$  for any  $A \in V_1$ . If  $J$  is any element of  $V_2$  with  $J \neq \mathfrak{m}_2$ , then  $J$  and  $\mathfrak{m}_2$  are not adjacent in  $\mathbb{INC}(R)$ . Hence,  $e(B) \geq 2$  in  $\mathbb{INC}(R)$  for any  $B \in V_2$ . As  $V(\mathbb{INC}(R)) = V_1 \cup V_2$  and  $\text{diam}(\mathbb{INC}(R)) \leq 2$ , we obtain that  $e(A) = 2$  for any  $A \in V(\mathbb{INC}(R))$ . Therefore, we obtain that  $\text{diam}(\mathbb{INC}(R)) = r(\mathbb{INC}(R)) = 2$ .  $\square$

Let  $R$  be a ring such that  $|\text{Max}(R)| \geq 2$ . In Proposition 1, we characterize such rings  $R$  whose  $\mathbb{INC}$  graph is complete.

**Proposition 1.** Let  $R$  be a ring such that  $|\text{Max}(R)| \geq 2$ . The following statements are equivalent:

(i)  $\mathbb{INC}(R)$  is complete.

(ii)  $R \cong F_1 \times F_2$  as rings, where  $F_i$  is a field for each  $i \in \{1, 2\}$ .

*Proof.* (i)  $\Rightarrow$  (ii) We are assuming that  $\mathbb{INC}(R)$  is complete. Hence, we obtain from Lemma 1 that  $|\text{Max}(R)| = 2$ . Let  $\{\mathfrak{m}_1, \mathfrak{m}_2\}$  denote the set of all maximal ideals of  $R$ . Let  $V_1, V_2$  be as in the statement of Lemma 3. Note that  $\mathfrak{m}_i \in V_i$  for each  $i \in \{1, 2\}$ . Let  $i \in \{1, 2\}$ . Let  $I \in V_i$ . We claim that  $I = \mathfrak{m}_i$ . Suppose that  $I \neq \mathfrak{m}_i$ . Then  $I \subset \mathfrak{m}_i$  and so,  $I$  and  $\mathfrak{m}_i$  are not adjacent in  $\mathbb{INC}(R)$ . This is in contradiction to the assumption that  $\mathbb{INC}(R)$  is complete. Therefore,  $I = \mathfrak{m}_i$  and this shows that  $|V_i| = 1$  for each  $i \in \{1, 2\}$ . Note that  $V_i = \{\mathfrak{m}_i\}$  for each  $i \in \{1, 2\}$ . Let  $a \in \mathfrak{m}_1 \setminus \mathfrak{m}_2$ . Then  $Ra \in V_1$  and so,  $Ra = \mathfrak{m}_1$ . Let  $b \in \mathfrak{m}_2 \setminus \mathfrak{m}_1$ . Then  $Rb \in V_2$  and so,  $\mathfrak{m}_2 = Rb$ . Now, for each  $i \in \{1, 2\}$ ,  $\mathfrak{m}_i^2 \in V_i$  and hence,  $\mathfrak{m}_i = \mathfrak{m}_i^2$ . Therefore,  $\mathfrak{m}_1 = Ra = Ra^2$  and  $\mathfrak{m}_2 = Rb = Rb^2$ . Thus we get that  $Rab = Ra^2b^2$ . This implies that  $ab = ra^2b^2$  for some  $r \in R$  and hence,  $ab(1 - rab) = 0$ . Since  $ab \in \mathfrak{m}_1 \cap \mathfrak{m}_2 = J(R)$ , we obtain that  $1 - rab \in U(R)$  and so,  $ab = 0$ . Hence,  $\mathfrak{m}_1\mathfrak{m}_2 = Rab = (0)$ . As  $\mathfrak{m}_1 + \mathfrak{m}_2 = R$ , it follows from [5, Proposition 1.10(i)] that  $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \mathfrak{m}_1\mathfrak{m}_2$ . Therefore,  $\mathfrak{m}_1 \cap \mathfrak{m}_2 = (0)$ . Now, it follows from the Chinese remainder theorem [[5], Proposition 1.10(ii) and (iii)] that  $R \cong \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2}$ , as desired.

(ii)  $\Rightarrow$  (i) We are assuming that  $R \cong F_1 \times F_2$  as rings, where  $F_1$  and  $F_2$  are fields. Let us denote the ring  $F_1 \times F_2$  by  $T$ . Note that  $V(\mathbb{INC}(T)) = \{(0) \times F_2, F_1 \times (0)\}$  and  $(0) \times F_2$  and  $F_1 \times (0)$  are adjacent in  $\mathcal{C}(T)$  and so, they are adjacent in  $\mathbb{INC}(T)$ . This proves that  $\mathbb{INC}(T)$  is complete and therefore, we obtain that  $\mathbb{INC}(R)$  is complete.  $\square$

Let  $R$  be a ring such that  $|\text{Max}(R)| \geq 2$ . We want to determine such rings  $R$  which satisfies  $\mathcal{C}(R) = \mathbb{INC}(R)$ .

**Lemma 4.** Let  $R$  be a ring such that  $|\text{Max}(R)| \geq 3$ . Then  $\mathcal{C}(R) \neq \mathbb{INC}(R)$ .

*Proof.* Let  $\{\mathfrak{m}_i \mid i \in \{1, 2, 3\}\} \subseteq \text{Max}(R)$ . Let  $I_1 = \mathfrak{m}_1 \cap \mathfrak{m}_2$  and let  $I_2 = \mathfrak{m}_1 \cap \mathfrak{m}_3$ . Note that  $I_1, I_2 \in V(\mathcal{C}(R)) = V(\mathbb{INC}(R))$ ,  $I_1 \neq I_2$ , and as  $I_1 + I_2 \subseteq \mathfrak{m}_1$ , it follows that  $I_1$  and  $I_2$  are not adjacent in  $\mathcal{C}(R)$ . It is clear that  $I_1 \not\subseteq I_2$  and  $I_2 \not\subseteq I_1$ . Hence,  $I_1$  and  $I_2$  are adjacent in  $\mathbb{INC}(R)$ . This proves that  $\mathcal{C}(R) \neq \mathbb{INC}(R)$ .  $\square$

**Remark 1.** Let  $R$  be a ring such that  $|Max(R)| \geq 2$ . If  $\mathcal{C}(R) = \mathbb{INC}(R)$ , then we know from Lemma 4 that  $|Max(R)| = 2$ . If  $|Max(R)| = 2$ , then it follows from (3)  $\Rightarrow$  (2) of [[17], Theorem 4.5] that  $\mathcal{C}(R)$  is a bipartite graph. Thus if  $\mathcal{C}(R) = \mathbb{INC}(R)$ , then  $\mathbb{INC}(R)$  is necessarily a bipartite graph.

Motivated by the results proved on  $\mathcal{C}(R)$  in Section 4 of [17], we next try to characterize rings  $R$  with  $|Max(R)| \geq 2$  such that  $\mathbb{INC}(R)$  is bipartite.

**Lemma 5.** Let  $R$  be a ring such that  $|Max(R)| \geq 2$ . If  $\mathbb{INC}(R)$  is bipartite, then  $|Max(R)| = 2$ .

*Proof.* It is already noted in the introduction that  $\mathcal{C}(R)$  is a spanning subgraph of  $\mathbb{INC}(R)$ . Thus if  $\mathbb{INC}(R)$  is bipartite, then  $\mathcal{C}(R)$  is also a bipartite graph. Hence, we obtain from (2)  $\Rightarrow$  (3) of [[17], Theorem 4.5] that  $|Max(R)| = 2$ .  $\square$

We use Observation 1 in the proof of Proposition 2. As this observation is easy to prove, we omit its proof.

**Observation 1.** Let  $H$  be a spanning subgraph of a graph  $G = (V, E)$ . Suppose that  $H$  is a complete bipartite graph. If  $G$  is a bipartite graph, then  $H = G$ .

**Proposition 2.** Let  $R$  be a ring with  $|Max(R)| \geq 2$ . The following statements are equivalent:

- (i)  $\mathbb{INC}(R)$  is a bipartite graph.
- (ii)  $\mathbb{INC}(R) = \mathcal{C}(R)$  is a complete bipartite graph.
- (iii)  $\mathbb{INC}(R)$  is a complete bipartite graph.

*Proof.* (i)  $\Rightarrow$  (ii) Assume that  $\mathbb{INC}(R)$  is a bipartite graph. Then we know from Lemma 5 that  $|Max(R)| = 2$ . Hence, we obtain from (3)  $\Rightarrow$  (1) of [[17], Theorem 4.5] that  $\mathcal{C}(R)$  is a complete bipartite graph. As  $\mathcal{C}(R)$  is a spanning subgraph of  $\mathbb{INC}(R)$ , we obtain from Observation 1 that  $\mathbb{INC}(R) = \mathcal{C}(R)$ . Therefore,  $\mathbb{INC}(R) = \mathcal{C}(R)$  is a complete bipartite graph.

(ii)  $\Rightarrow$  (iii) This is clear.

(iii)  $\Rightarrow$  (i) This is clear.  $\square$

**Proposition 3.** Let  $(R_i, \mathfrak{m}_i)$  be a quasilocal ring for each  $i \in \{1, 2\}$  and let  $R = R_1 \times R_2$ . The following statements are equivalent:

- (i)  $\mathbb{INC}(R)$  is a bipartite graph.
- (ii)  $\mathbb{INC}(R) = \mathcal{C}(R)$  is a complete bipartite graph.
- (iii)  $R_i$  is a chained ring for each  $i \in \{1, 2\}$ .

*Proof.* Note that  $|Max(R)| = 2$  and  $Max(R) = \{\mathfrak{M}_1 = \mathfrak{m}_1 \times R_2, \mathfrak{M}_2 = R_1 \times \mathfrak{m}_2\}$ . Observe that  $\mathcal{C}(R)$  is a complete bipartite graph with vertex partition  $V_1$



and  $V_2$ , where  $V_1 = \{I \times R_2 \mid I \text{ is a proper ideal of } R_1\}$  and  $V_2 = \{R_1 \times J \mid J \text{ is a proper ideal of } R_2\}$ .

(i)  $\Rightarrow$  (ii) This follows from (i)  $\Rightarrow$  (ii) of Proposition 2.

(ii)  $\Rightarrow$  (iii) Let  $I_1, I_2$  be distinct proper ideals of  $R_1$ . Now,  $A_i = I_i \times R_2 \in V_1$  for each  $i \in \{1, 2\}$  and  $A_1 \neq A_2$ . Hence,  $A_1$  and  $A_2$  are not adjacent in  $\mathbb{INC}(R)$ . Therefore, either  $A_1 = I_1 \times R_2 \subset A_2 = I_2 \times R_2$  or  $A_2 \subset A_1$ . This implies that either  $I_1 \subset I_2$  or  $I_2 \subset I_1$ . This shows that  $R_1$  is a chained ring. Similarly, using the fact that no two distinct elements of  $V_2$  are adjacent in  $\mathbb{INC}(R)$ , it can be shown that  $R_2$  is a chained ring.

(iii)  $\Rightarrow$  (i) Assume that  $R_i$  is a chained ring for each  $i \in \{1, 2\}$ . Note that if  $A_1, A_2$  are any two distinct members of  $V_1$ , then  $A_i = I_i \times R_2$  for some proper ideal  $I_i$  of  $R_1$  for each  $i \in \{1, 2\}$ . It is clear that  $I_1 \neq I_2$ . Since  $R_1$  is a chained ring, it follows that either  $I_1 \subset I_2$  or  $I_2 \subset I_1$  and so, either  $A_1 \subset A_2$  or  $A_2 \subset A_1$ . Hence,  $A_1$  and  $A_2$  are not adjacent in  $\mathbb{INC}(R)$ . Similarly, using the hypothesis that  $R_2$  is a chained ring, it can be shown that no distinct members of  $V_2$  are adjacent in  $\mathbb{INC}(R)$ . Let  $A \in V_1$  and  $B \in V_2$ . Then  $A$  and  $B$  are adjacent in  $\mathcal{C}(R)$  and so, they are adjacent in  $\mathbb{INC}(R)$ . Hence, it follows that  $\mathbb{INC}(R)$  is a complete bipartite graph with vertex partition  $V_1$  and  $V_2$ .  $\square$

**Corollary 1.** Let  $(R_i, \mathfrak{m}_i)$  be a quasilocal ring for each  $i \in \{1, 2\}$ . Let  $R = R_1 \times R_2$ . Then the following statements are equivalent:

(i)  $\mathbb{INC}(R)$  is a star graph.

(ii)  $\mathcal{C}(R) = \mathbb{INC}(R)$  is a star graph.

(iii)  $R_i$  is a chained ring for each  $i \in \{1, 2\}$  with  $R_i$  is a field for at least one  $i \in \{1, 2\}$ .

*Proof.* Note that  $\mathcal{C}(R)$  is a complete bipartite graph with vertex partition  $V_1$  and  $V_2$ , where  $V_1 = \{I \times R_2 \mid I \text{ is a proper ideal of } R_1\}$  and  $V_2 = \{R_1 \times J \mid J \text{ is a proper ideal of } R_2\}$ .

(i)  $\Rightarrow$  (ii) Since any star graph is a bipartite graph, it follows from (i)  $\Rightarrow$  (ii) of Proposition 3 that  $\mathcal{C}(R) = \mathbb{INC}(R)$  is a star graph.

(ii)  $\Rightarrow$  (iii) We know from (ii)  $\Rightarrow$  (iii) of Proposition 3 that  $R_i$  is a chained ring for each  $i \in \{1, 2\}$ . Since  $\mathcal{C}(R)$  is a star graph, it follows that  $|V_i| = 1$  for at least one  $i \in \{1, 2\}$ . Without loss of generality, we can assume that  $|V_1| = 1$ . Then we obtain that  $(0)$  is the only proper ideal of  $R_1$  and so,  $R_1$  is a field.

(iii)  $\Rightarrow$  (i) It is shown in (iii)  $\Rightarrow$  (i) of Proposition 3 that  $\mathbb{INC}(R)$  is a complete bipartite graph with vertex partition  $V_1$  and  $V_2$ . Without loss of generality, we can assume that  $R_1$  is a field. Hence,  $|V_1| = 1$  and so,  $\mathbb{INC}(R)$  is a star graph.  $\square$

Let  $R$  be a ring such that  $|\text{Max}(R)| \geq 2$ . In Proposition 4, we characterize such rings  $R$  whose  $\mathbb{INC}$  graph is a star graph.

**Proposition 4.** Let  $R$  be a ring such that  $|\text{Max}(R)| \geq 2$ . Then the following statements are equivalent:



(i)  $\mathbb{INC}(R)$  is a star graph.

(ii)  $R \cong R_1 \times R_2$  as rings, where  $R_i$  is a chained ring for each  $i \in \{1, 2\}$  with  $R_i$  is a field for at least one  $i \in \{1, 2\}$ .

*Proof.* (i)  $\Rightarrow$  (ii) We know from the proof of (i)  $\Rightarrow$  (ii) of Proposition 2 that  $|Max(R)| = 2$  and  $\mathcal{C}(R) = \mathbb{INC}(R)$ . Let  $\{\mathfrak{m}_1, \mathfrak{m}_2\}$  denote the set of all maximal ideals of  $R$  and let  $V_1, V_2$  be as in the statement of Lemma 3. As  $\mathcal{C}(R)$  is a star graph with vertex partition  $V_1$  and  $V_2$ , we can assume without loss of generality that  $|V_1| = 1$ . Therefore,  $V_1 = \{\mathfrak{m}_1\}$ . Let  $a \in \mathfrak{m}_1 \setminus \mathfrak{m}_2$ . Then  $Ra, Ra^2 \in V_1$  and so,  $\mathfrak{m}_1 = Ra = Ra^2$ . Let  $r \in R$  be such that  $a = ra^2$ . Then  $e = ra$  is a nontrivial idempotent element of  $R$ . Hence, the mapping  $f : R \rightarrow Re \times R(1 - e)$  defined by  $f(x) = (xe, x(1 - e))$  is an isomorphism of rings. Let us denote the ring  $Re$  by  $R_1$  and  $R(1 - e)$  by  $R_2$ . Since  $|Max(R)| = 2$ , it follows that  $R_1$  and  $R_2$  are quasilocal rings. Let us denote the ring  $R_1 \times R_2$  by  $T$ . As  $\mathbb{INC}(T)$  is a star graph, it follows from (i)  $\Rightarrow$  (iii) of Corollary 1 that  $R_i$  is a chained ring for each  $i \in \{1, 2\}$  with  $R_i$  is a field for at least one  $i \in \{1, 2\}$ .

(ii)  $\Rightarrow$  (i) Let us denote the ring  $R_1 \times R_2$  by  $T$ . We know from (iii)  $\Rightarrow$  (i) of Corollary 1 that  $\mathbb{INC}(T)$  is a star graph. It follows from  $R \cong T$  as rings that  $\mathbb{INC}(R)$  is a star graph. □

Let  $R$  be a ring and let  $\mathfrak{m} \in Max(R)$ . Let  $f : R \rightarrow R_{\mathfrak{m}}$  denote the ring homomorphism given by  $f(r) = \frac{r}{1}$ . For any ideal  $I$  of  $R$ ,  $f^{-1}(I_{\mathfrak{m}})$  is called the *saturation of  $I$  with respect to the multiplicatively closed set  $R \setminus \mathfrak{m}$*  and is denoted by the notation  $S_{\mathfrak{m}}(I)$ . It is well-known that for any ideal  $I$  of  $R$ ,  $I = \bigcap_{\mathfrak{m} \in Max(R)} S_{\mathfrak{m}}(I)$ . Let  $R$  be a ring with  $|Max(R)| \geq 2$ . Suppose that  $dim R = 0$ . In Proposition 5, we characterize such rings  $R$  whose  $\mathbb{INC}$  graph is a bipartite graph.

**Remark 2.** Let  $R$  be a semiquasilocal ring with  $|Max(R)| = n \geq 2$ . Suppose that  $dim R = 0$ . Then  $R \cong R_1 \times R_2 \times \dots \times R_n$  as rings, where  $(R_i, \mathfrak{n}_i)$  is a quasilocal ring for each  $i \in \{1, 2, \dots, n\}$ .

*Proof.* Let  $\{\mathfrak{m}_i \mid i \in \{1, 2, \dots, n\}\}$  denote the set of all maximal ideals of  $R$ . Let  $i \in \{1, 2, \dots, n\}$ . Since  $dim R = 0$ , it follows that  $\sqrt{(0)_{\mathfrak{m}_i}} = (\mathfrak{m}_i)_{\mathfrak{m}_i}$  and so,  $\sqrt{S_{\mathfrak{m}_i}((0))} = \mathfrak{m}_i$ . Hence, we obtain from [[5], Proposition 4.2] that  $S_{\mathfrak{m}_i}((0))$  is a  $\mathfrak{m}_i$ -primary ideal of  $R$ . Let us denote the ideal  $S_{\mathfrak{m}_i}((0))$  by  $\mathfrak{q}_i$  for each  $i \in \{1, 2, \dots, n\}$ . Observe that  $(0) = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_n$ . From  $\mathfrak{m}_i + \mathfrak{m}_j = R$  for all distinct  $i, j \in \{1, 2, \dots, n\}$ , we obtain from [[5], Proposition 1.16] that  $\mathfrak{q}_i + \mathfrak{q}_j = R$ . It now follows from the Chinese remainder theorem [[5], Proposition 1.10(ii) and (iii)] that  $R \cong \frac{R}{\mathfrak{q}_1} \times \frac{R}{\mathfrak{q}_2} \times \dots \times \frac{R}{\mathfrak{q}_n}$ . For each  $i$  with  $1 \leq i \leq n$ , let us denote the ring  $\frac{R}{\mathfrak{q}_i}$  by  $R_i$ . Note that  $R_i$  is quasilocal with  $\mathfrak{n}_i = \frac{\mathfrak{m}_i}{\mathfrak{q}_i}$  as its unique maximal ideal and  $R \cong R_1 \times R_2 \times \dots \times R_n$  as rings. □

**Proposition 5.** Let  $R$  be a ring such that  $|Max(R)| \geq 2$  and let  $dim R = 0$ . Then the following statements are equivalent:

(i)  $\mathbb{INC}(R)$  is a bipartite graph.

(ii)  $R \cong R_1 \times R_2$  as rings, where  $(R_i, \mathfrak{n}_i)$  is a chained ring with  $\dim R_i = 0$  for each  $i \in \{1, 2\}$ .

*Proof.* (i)  $\Rightarrow$  (ii) Assume that  $\text{INC}(R)$  is a bipartite graph. Then we know from Lemma 5 that  $|\text{Max}(R)| = 2$ . Since  $\dim R = 0$ , we know from Remark 2 that  $R \cong R_1 \times R_2$  as rings, where  $(R_i, \mathfrak{n}_i)$  is a quasilocal ring for each  $i \in \{1, 2\}$ . Let us denote the ring  $R_1 \times R_2$  by  $T$ . Since  $R \cong T$  as rings, we get that  $\text{INC}(T)$  is a bipartite graph. Hence, we obtain from (i)  $\Rightarrow$  (iii) of Proposition 3 that  $R_i$  is a chained ring for each  $i \in \{1, 2\}$ . Since  $\dim R = 0$ , it is clear that  $\dim R_i = 0$  for each  $i \in \{1, 2\}$ .

(ii)  $\Rightarrow$  (i) This follows from (iii)  $\Rightarrow$  (i) of Proposition 3.  $\square$

We provide an example in Example 1 to illustrate that (i)  $\Rightarrow$  (ii) of Proposition 5 can fail to hold if the hypothesis  $\dim R = 0$  is omitted.

**Lemma 6.** Let  $R$  be a principal ideal domain with  $|\text{Max}(R)| \geq 2$ . The following statements are equivalent:

(i)  $\mathcal{C}(R) = \text{INC}(R)$ .

(ii)  $|\text{Max}(R)| = 2$ .

*Proof.* (i)  $\Rightarrow$  (ii) Assume that  $\mathcal{C}(R) = \text{INC}(R)$ . Then we obtain from Lemma 4 that  $|\text{Max}(R)| = 2$ . (For this part of the proof, we do not need the assumption that  $R$  is a principal ideal domain.)

(ii)  $\Rightarrow$  (i) Assume that  $R$  is a principal ideal domain with  $|\text{Max}(R)| = 2$ . Let  $\{\mathfrak{m}_1 = Rp, \mathfrak{m}_2 = Rq\}$  denote the set of all maximal ideals of  $R$ . It is already noted in the introduction that for any ring  $T$  with  $|\text{Max}(T)| \geq 2$ ,  $\mathcal{C}(T)$  is a spanning subgraph of  $\text{INC}(T)$ . Observe that  $V(\text{INC}(R)) = V_1 \cup V_2$ , where  $V_1 = \{I \in \mathbb{I}(R) \mid I \subseteq Rp \text{ but } I \not\subseteq Rq\}$  and  $V_2 = \{J \in \mathbb{I}(R) \mid J \subseteq Rq \text{ but } J \not\subseteq Rp\}$ . Let  $I_1, I_2 \in V(\text{INC}(R))$  be such that  $I_1$  and  $I_2$  are adjacent in  $\text{INC}(R)$ . We assert that  $I_1 + I_2 = R$ . Suppose that  $I_1 + I_2 \neq R$ . Then either  $I_1 + I_2 \subseteq Rp$  or  $I_1 + I_2 \subseteq Rq$ . Without loss of generality, we can assume that  $I_1 + I_2 \subseteq Rp$ . Note that  $I_i \in V_1$  for each  $i \in \{1, 2\}$ . Hence,  $I_1 = Rp^n$  and  $I_2 = Rp^m$  for some  $n, m \in \mathbb{N}$ . Therefore,  $I_1$  and  $I_2$  are comparable under the inclusion relation and so,  $I_1$  and  $I_2$  are not adjacent in  $\text{INC}(R)$ . This is in contradiction to the assumption that  $I_1$  and  $I_2$  are adjacent in  $\text{INC}(R)$ . Therefore,  $I_1 + I_2 = R$  and so,  $I_1$  and  $I_2$  are adjacent in  $\mathcal{C}(R)$ . This proves that  $\mathcal{C}(R) = \text{INC}(R)$ .  $\square$

Example 1 mentioned below was mentioned in [[17], Example 4.10] to illustrate that [[17], Proposition 4.7] can fail to hold if the hypothesis  $R$  satisfies d.c.c on principal ideals is omitted.

**Example 1.** Let  $p, q$  be distinct prime numbers. Let  $R = S^{-1}\mathbb{Z}$ , where  $S = \mathbb{Z} \setminus (p\mathbb{Z} \cup q\mathbb{Z})$ . Then  $\text{INC}(R)$  is a complete bipartite graph but the statement (ii) of Proposition 5 does not hold.

*Proof.* Note that  $R$  is a principal ideal domain with  $|Max(R)| = 2$  and  $\{Rp, Rq\}$  is the set of all maximal ideals of  $R$ . Hence, we obtain from (ii)  $\Rightarrow$  (i) of Lemma 6 that  $\mathcal{C}(R) = \mathbb{INC}(R)$ . We know from (3)  $\Rightarrow$  (1) of [[17], Theorem 4.5] that  $\mathcal{C}(R)$  is a complete bipartite graph. Therefore,  $\mathbb{INC}(R)$  is a complete bipartite graph. As  $R$  is an integral domain,  $R$  has no nontrivial idempotent element. Hence, the statement (ii) of Proposition 5 does not hold.  $\square$

In Example 2, we provide another example to illustrate that (i)  $\Rightarrow$  (ii) of Proposition 5 can fail to hold if the hypothesis  $dim R = 0$  is omitted.

**Example 2.** Let  $V = \mathbb{Q}[[X]]$  be the power series ring in one variable  $X$  over  $\mathbb{Q}$ . Let us denote  $VX$  by  $\mathfrak{m}$ . Let  $T = R + \mathfrak{m}$ , where  $R$  is as in Example 1. Then  $\mathbb{INC}(T)$  is a complete bipartite graph but the statement (ii) of Proposition 5 does not hold.

*Proof.* Observe that  $V = \mathbb{Q} + \mathfrak{m}$  is a discrete valuation ring. We know from [[7], Theorem 2.1(c)] that each ideal of  $T$  compares with  $\mathfrak{m}$  under inclusion. As  $Max(R) = \{Rp, Rq\}$ , it follows from [[7], Theorem 2.1(d)] that  $Max(T) = \{\mathfrak{m}_1 = Rp + \mathfrak{m}, \mathfrak{m}_2 = Rq + \mathfrak{m}\}$ . Let  $V_1 = \{I \in \mathbb{I}(T) \mid M(I) = \{\mathfrak{m}_1\}\}$  and let  $V_2 = \{J \in \mathbb{I}(T) \mid M(J) = \{\mathfrak{m}_2\}\}$ . Observe that  $V(\mathbb{INC}(T)) = V_1 \cup V_2$ . Let  $I_1, I_2 \in V_1$  be such that  $I_1 \neq I_2$ . Let  $i \in \{1, 2\}$ . It is not hard to verify that  $I_i = A_i + \mathfrak{m}$  for some  $A_i \in \mathbb{I}(R) \setminus \{(0)\}$  such that  $M(A_i) = \{Rp\}$ . It is clear that  $A_1 \neq A_2$ . Observe that there exist distinct  $n, m \in \mathbb{N}$  such that  $I_1 = Rp^n + \mathfrak{m}$  and  $I_2 = Rp^m + \mathfrak{m}$ . Hence,  $I_1$  and  $I_2$  are comparable under the inclusion relation and so, they are not adjacent in  $\mathbb{INC}(T)$ . Similarly, if  $J_1, J_2$  are any two distinct members of  $V_2$ , then  $J_i = B_i + \mathfrak{m}$  for some distinct  $B_1, B_2 \in \mathbb{I}(R) \setminus \{(0)\}$  such that  $M(B_i) = \{Rq\}$  for each  $i \in \{1, 2\}$ . Hence, there exist distinct  $k, t \in \mathbb{N}$  such that  $J_1 = Rq^k + \mathfrak{m}$  and  $J_2 = Rq^t + \mathfrak{m}$ . Therefore, it follows that  $J_1$  and  $J_2$  are comparable under the inclusion relation and so, they are not adjacent in  $\mathbb{INC}(T)$ . If  $I \in V_1$  and  $J \in V_2$ , then  $I + J = T$  and so, they are adjacent in  $\mathbb{INC}(T)$ . This shows that  $\mathbb{INC}(T)$  is a complete bipartite graph. We know from [[7], Theorem 2.1(f)] that  $dim T = dim V + dim R = 1 + 1 = 2$ . Indeed, it follows from [[7], Theorem 2.1 (c), (d), (e)] and the fact that  $(0)$  and  $\mathfrak{m}$  are the only prime ideals of  $V$  that  $\{(0), \mathfrak{m}, Rp + \mathfrak{m}, Rq + \mathfrak{m}\}$  is the set of all prime ideals of  $T$ . Hence,  $(0) \subset \mathfrak{m}$ ,  $(0) \subset \mathfrak{m} \subset Rp + \mathfrak{m}$ , and  $(0) \subset \mathfrak{m} \subset Rq + \mathfrak{m}$  are the only chains of prime ideals of  $T$  of positive length and so,  $dim T = 2$ . Since  $T$  is an integral domain, we obtain that  $T$  has no nontrivial idempotent. Hence, the statement (ii) of Proposition 5 does not hold.  $\square$

In Proposition 6, we determine  $girth(\mathbb{INC}(R))$ , where  $R$  is a ring with  $|Max(R)| \geq 2$  and moreover, we characterize such rings  $R$  which satisfies  $girth(\mathbb{INC}(R)) = \infty$ .

**Proposition 6.** Let  $R$  be a ring with  $|Max(R)| \geq 2$ . Then  $girth(\mathbb{INC}(R)) \in \{3, 4, \infty\}$ . Moreover,  $girth(\mathbb{INC}(R)) = \infty$  if and only if  $R \cong R_1 \times R_2$  as rings, where  $R_1$  is a field and  $R_2$  is a chained ring.

*Proof.* Suppose that  $|Max(R)| \geq 3$ . Let  $\{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\} \subseteq Max(R)$ . Note that  $\mathfrak{m}_1 - \mathfrak{m}_2 - \mathfrak{m}_3 - \mathfrak{m}_1$  is a cycle of length three in  $\mathcal{C}(R)$  and hence, a cycle of length three in  $\mathbb{INC}(R)$ . Therefore,  $girth(\mathbb{INC}(R)) = 3$ .

Suppose that  $|Max(R)| = 2$ . Let  $\{\mathfrak{m}_1, \mathfrak{m}_2\}$  denote the set of all maximal ideals of  $R$ . Let  $V_1, V_2$  be as in the statement of Lemma 3. We consider the following cases.

**Case 1.**  $|V_i| \geq 2$  for each  $i \in \{1, 2\}$ .

Let  $I \in V_1 \setminus \{\mathfrak{m}_1\}$  and let  $J \in V_2 \setminus \{\mathfrak{m}_2\}$ . Observe that  $I + \mathfrak{m}_2 = I + J = R$  and so,  $\mathfrak{m}_1 - \mathfrak{m}_2 - I - J - \mathfrak{m}_1$  is a cycle of length four in  $\mathcal{C}(R)$  and hence, a cycle of length four in  $\mathbb{INC}(R)$ . Therefore,  $girth(\mathbb{INC}(R)) \leq 4$ .

**Case 2.**  $|V_i| = 1$  for at least one  $i \in \{1, 2\}$ .

In such a case, it follows as in the proof of (i)  $\Rightarrow$  (ii) of Proposition 4 that there exist quasilocal rings  $R_1$  and  $R_2$  such that at least one between  $R_1$  and  $R_2$  is a field and  $R \cong R_1 \times R_2$  as rings. Without loss of generality, we can assume that  $R_1$  is a field. Let us denote the ring  $R_1 \times R_2$  by  $T$ . Note that  $\mathbb{INC}(T)$  contains a cycle if and only if there are at least two distinct nonzero proper ideals  $J_1$  and  $J_2$  of  $R_2$  such that  $J_1$  and  $J_2$  are not comparable under the inclusion relation. Hence,  $(0) \times R_2 - R_1 \times J_1 - R_1 \times J_2 - (0) \times R_2$  is a cycle of length three in  $\mathbb{INC}(T)$ . From  $R \cong T$  as rings, it follows that  $girth(\mathbb{INC}(R)) = girth(\mathbb{INC}(T)) = 3$ . Observe that  $\mathbb{INC}(T)$  (equivalently,  $\mathbb{INC}(R)$ ) does not contain any cycle if and only if the set of ideals of  $R_2$  is linearly ordered by inclusion, that is,  $R_2$  is a chained ring.

It is clear from the above discussion that  $girth(\mathbb{INC}(R)) \in \{3, 4, \infty\}$  and  $girth(\mathbb{INC}(R)) = \infty$  if and only if  $R \cong R_1 \times R_2$  as rings, where  $R_1$  is a field and  $R_2$  is a chained ring. Now, it is clear that  $girth$  of any star graph equals  $\infty$ . It follows from (ii)  $\Rightarrow$  (i) of Proposition 4 that if  $girth(\mathbb{INC}(R)) = \infty$ , then  $\mathbb{INC}(R)$  is a star graph. □

In Example 3, we provide some examples to illustrate Proposition 6.

**Example 3.** (i) Let  $T = \mathbb{Z}_2[X, Y]$  be the polynomial ring in two variables  $X, Y$  over  $\mathbb{Z}_2$ . Let  $\mathfrak{m} = TX + TY$ . Let  $R = F \times \frac{T}{\mathfrak{m}^2}$ , where  $F$  is a field. Then  $girth(\mathbb{INC}(R)) = 3$ .

(ii) Let  $p, q$  be distinct prime numbers and let  $R = S^{-1}\mathbb{Z}$ , where  $S = \mathbb{Z} \setminus (p\mathbb{Z} \cup q\mathbb{Z})$ . Then  $girth(\mathbb{INC}(R)) = 4$ .

(iii) Let  $p$  be a prime number and let  $R = F \times \mathbb{Z}_{p^2}$ , where  $F$  is a field. Then  $girth(\mathbb{INC}(R)) = \infty$ .

*Proof.* (i) Note that  $\frac{T}{\mathfrak{m}^2}$  is a local ring with  $\frac{\mathfrak{m}}{\mathfrak{m}^2}$  as its unique maximal ideal. It is clear that  $|Max(R)| = 2$  and  $Max(R) = \{(0) \times \frac{T}{\mathfrak{m}^2}, F \times \frac{\mathfrak{m}}{\mathfrak{m}^2}\}$ . It is convenient to denote  $\frac{T}{\mathfrak{m}^2}$  by  $T_1$ ,  $X + \mathfrak{m}^2$  by  $x$ , and  $Y + \mathfrak{m}^2$  by  $y$ . Note that  $(0) \times T_1 - F \times T_1x - F \times T_1y - (0) \times T_1$  is a cycle of length three in  $\mathbb{INC}(R)$  and so,  $girth(\mathbb{INC}(R)) = 3$ .

(ii) We know from Example 1 that  $\mathbb{INC}(R)$  is a complete bipartite graph with vertex partition  $V_1$  and  $V_2$ , where  $V_1 = \{Rp^n \mid n \in \mathbb{N}\}$  and  $V_2 = \{Rq^n \mid n \in \mathbb{N}\}$ . As  $|V_i| \geq 2$  for each  $i \in \{1, 2\}$ , we obtain that  $girth(\mathbb{INC}(R)) = 4$ .

(iii) We know from [[5], Example (1), page 94] that  $\mathbb{Z}_{p\mathbb{Z}}$  is a discrete valuation ring and so, it is a chained ring. It follows from the moreover part of Proposition 6 that  $girth(\mathbb{INC}(R)) = \infty$ . □

### 3. When is $\mathbb{INC}(R)$ a split graph?

The aim of this section is to characterize rings  $R$  with  $|Max(R)| \geq 2$  such that  $\mathbb{INC}(R)$  is a split graph. Let  $R$  be a ring such that  $|Max(R)| \geq 2$ . Note that  $\mathbb{INC}(R)$  is a split graph if and only if there exist nonempty subsets  $K, S$  of  $V(\mathbb{INC}(R))$  such that  $V(\mathbb{INC}(R)) = K \cup S$ ,  $K \cap S = \emptyset$ , satisfying the property that the subgraph of  $\mathbb{INC}(R)$  induced on  $K$  is a clique and  $S$  is an independent set of  $\mathbb{INC}(R)$ . Throughout this section, whenever we consider rings  $R$  with  $\mathbb{INC}(R)$  is a split graph, we use  $K$  and  $S$  with the above mentioned properties.

Let  $R$  be a ring with  $|Max(R)| \geq 2$ . In Proposition 7, we determine a necessary condition on  $|Max(R)|$  in order that  $\mathbb{INC}(R)$  is a split graph. In Theorem 2, we characterize such rings  $R$  whose  $\mathbb{INC}$  graph is a split graph.

**Lemma 7.** Let  $R$  be a ring with  $|Max(R)| \geq 3$ . If  $\mathbb{INC}(R)$  is a split graph with  $V(\mathbb{INC}(R)) = K \cup S$ , then  $Max(R) = K$ .

*Proof.* As distinct maximal ideals of  $R$  are not comparable under the inclusion relation, it follows that distinct maximal ideals of  $R$  are adjacent in  $\mathbb{INC}(R)$ . Since  $S$  is an independent set of  $\mathbb{INC}(R)$ , we obtain that  $|S \cap Max(R)| \leq 1$ . By hypothesis,  $|Max(R)| \geq 3$ . Hence, there exist distinct  $\mathfrak{m}_1, \mathfrak{m}_2 \in Max(R)$  such that  $\mathfrak{m}_i \in K$  for each  $i \in \{1, 2\}$ . It follows from  $|Max(R)| \geq 3$  that  $\mathfrak{m}_1 \cap \mathfrak{m}_2 \in V(\mathbb{INC}(R)) = K \cup S$ . Since  $\mathfrak{m}_1 \cap \mathfrak{m}_2 \subset \mathfrak{m}_1$ , we get that  $\mathfrak{m}_1$  and  $\mathfrak{m}_1 \cap \mathfrak{m}_2$  are not adjacent in  $\mathbb{INC}(R)$ . As  $\mathfrak{m}_1 \in K$ , it follows that  $\mathfrak{m}_1 \cap \mathfrak{m}_2 \in S$ . Let  $\mathfrak{m} \in Max(R) \setminus \{\mathfrak{m}_1, \mathfrak{m}_2\}$ . It is clear that  $\mathfrak{m} \not\subseteq \mathfrak{m}_1 \cap \mathfrak{m}_2$  and from [[5], Proposition 1.11(ii)], it follows that  $\mathfrak{m}_1 \cap \mathfrak{m}_2 \not\subseteq \mathfrak{m}$ . This shows that  $\mathfrak{m}$  and  $\mathfrak{m}_1 \cap \mathfrak{m}_2$  are adjacent in  $\mathbb{INC}(R)$ . Since  $\mathfrak{m}_1 \cap \mathfrak{m}_2 \in S$ , we obtain that  $\mathfrak{m} \in K$ . This proves that  $Max(R) \subseteq K$ . Let  $I \in K$ . Note that there exists  $\mathfrak{m} \in Max(R)$  such that  $I \subseteq \mathfrak{m}$ . If  $I \neq \mathfrak{m}$ , then  $I$  and  $\mathfrak{m}$  are not adjacent in  $\mathbb{INC}(R)$ . This is in contradiction to the fact that the subgraph of  $\mathbb{INC}(R)$  induced on  $K$  is complete. Therefore,  $I = \mathfrak{m} \in Max(R)$  and so, we obtain that  $Max(R) = K$ . □

**Proposition 7.** Let  $R$  be a ring such that  $|Max(R)| \geq 2$ . If  $\mathbb{INC}(R)$  is a split graph, then  $|Max(R)| = 2$ .

*Proof.* Let  $V(\mathbb{INC}(R)) = K \cup S$ . Suppose that  $|Max(R)| \geq 3$ . Then we know from Lemma 7 that  $Max(R) = K$ . Let  $\{\mathfrak{m}_i \mid i \in \{1, 2, 3\}\} \subseteq Max(R)$ . Now,  $\mathfrak{m}_i \in K$  for each  $i \in \{1, 2, 3\}$ . Let us denote  $\mathfrak{m}_1 \cap \mathfrak{m}_2$  by  $A$  and  $\mathfrak{m}_2 \cap \mathfrak{m}_3$  by  $B$ . From the assumption that  $|Max(R)| \geq 3$ , it is clear that  $A, B \in V(\mathbb{INC}(R))$ . As  $A \subset \mathfrak{m}_1$ , it follows that  $A$  and  $\mathfrak{m}_1$  are not adjacent in  $\mathbb{INC}(R)$  and so, from  $\mathfrak{m}_1 \in K$ , we get that  $A \in S$ . Similarly, as  $B \subset \mathfrak{m}_2$  and  $\mathfrak{m}_2 \in K$ , it follows that  $B \in S$ . Now, it follows

from [[5], Proposition 1.11(ii)] that  $A \not\subseteq \mathfrak{m}_3$  and  $B \not\subseteq \mathfrak{m}_1$ . Therefore, we obtain that  $A \not\subseteq B$  and  $B \not\subseteq A$ . Hence,  $A$  and  $B$  are adjacent in  $\text{INC}(R)$ . This is impossible since  $A, B \in S$ . Therefore,  $|\text{Max}(R)| \leq 2$  and so,  $|\text{Max}(R)| = 2$ .  $\square$

**Lemma 8.** Let  $R$  be a ring with  $|\text{Max}(R)| = 2$ . Let  $\{\mathfrak{m}_1, \mathfrak{m}_2\}$  denote the set of all maximal ideals of  $R$ . Let  $V_1, V_2$  be as in the statement of Lemma 3. If  $\text{INC}(R)$  is a split graph, then  $|V_i| = 1$  for at least one  $i \in \{1, 2\}$ .

*Proof.* Let  $V(\text{INC}(R)) = K \cup S$ . Since  $\mathfrak{m}_1$  and  $\mathfrak{m}_2$  are adjacent in  $\text{INC}(R)$ , it follows that  $|S \cap \text{Max}(R)| \leq 1$ . We consider the following cases.

**Case 1.**  $\text{Max}(R) \subseteq K$ .

Suppose that  $|V_i| \geq 2$  for each  $i \in \{1, 2\}$ . Let  $I \in V_1 \setminus \{\mathfrak{m}_1\}$  and let  $J \in V_2 \setminus \{\mathfrak{m}_2\}$ . Since  $I \subset \mathfrak{m}_1$ , it follows that  $I$  and  $\mathfrak{m}_1$  are not adjacent in  $\text{INC}(R)$  and so,  $I \in S$ . Similarly, since  $J \subset \mathfrak{m}_2$ , it follows that  $J \in S$ . As  $I + J = R$ , we obtain that  $I$  and  $J$  are adjacent in  $\mathcal{C}(R)$  and so, they are adjacent in  $\text{INC}(R)$ . This is impossible, since  $I, J \in S$ . Therefore,  $|V_i| = 1$  for at least one  $i \in \{1, 2\}$ .

**Case 2.**  $|S \cap \text{Max}(R)| = 1$ .

Without loss of generality, we can assume that  $\mathfrak{m}_1 \in S$ . Then  $\mathfrak{m}_2 \in K$ . We claim that  $|V_2| = 1$ . Suppose that  $|V_2| \geq 2$ . Let  $J \in V_2 \setminus \{\mathfrak{m}_2\}$ . Since  $J \subset \mathfrak{m}_2$  and  $\mathfrak{m}_2 \in K$ , we obtain that  $J \notin K$  and so,  $J \in S$ . As  $J + \mathfrak{m}_1 = R$ , it follows that  $J$  and  $\mathfrak{m}_1$  are adjacent in  $\text{INC}(R)$ . This is impossible, since  $J, \mathfrak{m}_1 \in S$ . Therefore,  $|V_2| = 1$ .

This proves that  $|V_i| = 1$  for at least one  $i \in \{1, 2\}$ .  $\square$

**Proposition 8.** Let  $(R_i, \mathfrak{m}_i)$  be a quasilocal ring for each  $i \in \{1, 2\}$  and let  $R = R_1 \times R_2$ .

The following statements are equivalent:

(i)  $\text{INC}(R)$  is a split graph.

(ii)  $R_i$  is a field for at least one  $i \in \{1, 2\}$  and if  $R_i$  is not a field for some  $i \in \{1, 2\}$ , then either  $R_i$  is a chained ring or  $\mathbb{I}(R_i) = W_1 \cup W_2$ , where  $|W_k| \geq 2$  for each  $k \in \{1, 2\}$  with the property that  $W_1 \cap W_2 = \emptyset$ ,  $W_1$  is a chain under the inclusion relation, and no two distinct members of  $W_2$  are comparable under the inclusion relation.

*Proof.* (i)  $\Rightarrow$  (ii) We are assuming that  $\text{INC}(R)$  is a split graph. Let  $V(\text{INC}(R)) = K \cup S$ . Note that  $V(\text{INC}(R)) = V_1 \cup V_2$ , where  $V_1 = \{I \times R_2 \mid I \text{ is a proper ideal of } R_1\}$  and  $V_2 = \{R_1 \times J \mid J \text{ is a proper ideal of } R_2\}$ . It follows from Lemma 8 that  $|V_i| = 1$  for at least one  $i \in \{1, 2\}$ . Without loss of generality, we can assume that  $|V_1| = 1$ . Hence, we obtain that  $R_1$  is a field. We can assume that  $R_2$  is not a field. Now,  $V(\text{INC}(R)) = V_1 \cup V_2 = K \cup S$ . We consider the following cases.

**Case 1.**  $\text{Max}(R) \subseteq K$ .

Note that  $(0) \times R_2$  and  $R_1 \times \mathfrak{m}_2 \in K$ . Let  $J_1, J_2$  be any two distinct proper ideals of  $R_2$ . We claim that  $J_1$  and  $J_2$  are comparable under the inclusion relation. This is clear if either  $J_1 = \mathfrak{m}_2$  or  $J_2 = \mathfrak{m}_2$ . Hence, we can assume that  $J_i \neq \mathfrak{m}_2$  for each  $i \in \{1, 2\}$ . As  $R_1 \times \mathfrak{m}_2 \in K$ , we obtain that  $R_1 \times J_i \in S$  for each  $i \in \{1, 2\}$ . Since  $S$  is an independent set of  $\text{INC}(R)$ , we obtain that  $R_1 \times J_1$  and  $R_1 \times J_2$  are not adjacent in

$\text{INC}(R)$ . Hence,  $J_1$  and  $J_2$  are comparable under the inclusion relation. This proves that  $R_2$  is a chained ring.

**Case 2.**  $|Max(R) \cap S| = 1$ .

If  $(0) \times R_2 \in S$ , then  $R_1 \times \mathfrak{m}_2, R_1 \times (0) \in K$ . This is impossible since  $R_1 \times \mathfrak{m}_2$  and  $R_1 \times (0)$  are not adjacent in  $\text{INC}(R)$ . Therefore,  $(0) \times R_2 \notin S$  and so,  $R_1 \times \mathfrak{m}_2 \in S$ . We can assume that  $R_2$  is not a chained ring. Let  $W_1 = \{J \in \mathbb{I}(R_2) \mid R_1 \times J \in S\}$  and let  $W_2 = \{J \in \mathbb{I}(R_2) \mid R_1 \times J \in K\}$ . Note that  $R_1 \times \mathfrak{m}_2 \in S$  and so,  $W_1 \neq \emptyset$ . Since  $R_2$  is not a chained ring by assumption, there exist proper ideals  $J_1, J_2$  of  $R_2$  such that they are not comparable under the inclusion relation. Let  $a \in J_1 \setminus J_2$  and let  $b \in J_2 \setminus J_1$ . Let  $A = R_2a, B = R_2b$ , and  $C = R_2(a + b)$ . It is clear that  $A \not\subseteq B$  and  $B \not\subseteq A$ . As  $C \not\subseteq J_1$  and  $C \not\subseteq J_2$ , we obtain that  $C \not\subseteq A$  and  $C \not\subseteq B$ . We claim that  $A \not\subseteq C$  and  $B \not\subseteq C$ . For if  $A \subseteq C$ , then  $a = y(a + b)$  for some  $y \in R_2$ . Suppose that  $y \in \mathfrak{m}_2$ . Then  $1 - y \in U(R_2)$  and from  $a(1 - y) = yb \in J_2$ , we get that  $a = (1 - y)^{-1}yb \in J_2$ . This is impossible. If  $y \in U(R_2)$ , then from  $a = y(a + b)$ , it follows that  $a + b = y^{-1}a \in J_1$ . This is impossible. Therefore,  $A \not\subseteq C$ . Similarly, it can be shown that  $B \not\subseteq C$ . Hence,  $R_1 \times A - R_1 \times B - R_1 \times C - R_1 \times A$  is a cycle of length 3 in  $\text{INC}(R)$ . As  $S$  is an independent set of  $\text{INC}(R)$ , it follows that at least two among  $R_1 \times A, R_1 \times B, R_1 \times C$  must be in  $K$ . Hence, at least two among  $A, B, C$  must be in  $W_2$  and so,  $|W_2| \geq 2$ . Observe that  $R_1 \times (0)$  must be in  $S$ . Thus  $R_1 \times \mathfrak{m}_2, R_1 \times (0) \in S$  and so,  $|W_1| \geq 2$ . It is clear that  $W_1 \cup W_2 \subseteq \mathbb{I}(R_2)$ . Let  $J \in \mathbb{I}(R_2)$ . Then  $R_1 \times J \in V_2 \subseteq K \cup S$ . If  $R_1 \times J \in S$ , then  $J \in W_1$  and if  $R_1 \times J \in K$ , then  $J \in W_2$ . This proves that  $\mathbb{I}(R_2) = W_1 \cup W_2$ . It follows from  $K \cap S = \emptyset$  that  $W_1 \cap W_2 = \emptyset$ .

(ii)  $\Rightarrow$  (i) If both  $R_1$  and  $R_2$  are fields, then  $\text{INC}(R)$  is a complete graph on two vertices and so,  $\text{INC}(R)$  is a split graph. We can assume that  $R_1$  is a field and  $R_2$  is not a field. If  $R_2$  is a chained ring, then we know from (ii)  $\Rightarrow$  (i) of Proposition 4 that  $\text{INC}(R)$  is a star graph and so,  $\text{INC}(R)$  is a split graph. Suppose that  $\mathbb{I}(R_2) = W_1 \cup W_2$ , where  $|W_i| \geq 2$  for each  $i \in \{1, 2\}$  satisfying the property that  $W_1 \cap W_2 = \emptyset$ ,  $W_1$  is a chain under the inclusion relation, and no two distinct members of  $W_2$  are comparable under the inclusion relation. Let  $K = \{(0) \times R_2, R_1 \times I \mid I \in W_2\}$  and let  $S = \{R_1 \times I \mid I \in W_1\}$ . It is clear that  $V(\text{INC}(R)) = K \cup S, K \neq \emptyset, S \neq \emptyset, K \cap S = \emptyset$ , the subgraph of  $\text{INC}(R)$  induced on  $K$  is a clique, and  $S$  is an independent set of  $\text{INC}(R)$ . Therefore,  $\text{INC}(R)$  is a split graph.  $\square$

**Theorem 2.** Let  $R$  be a ring with  $|Max(R)| \geq 2$ . The following statements are equivalent:

- (i)  $\text{INC}(R)$  is a split graph.
- (ii)  $R \cong R_1 \times R_2$  as rings, where  $R_1$  and  $R_2$  are quasilocal rings which satisfy the conditions mentioned in the statement (ii) of Proposition 8.

*Proof.* (i)  $\Rightarrow$  (ii) We are assuming that  $\text{INC}(R)$  is a split graph. Let  $V(\text{INC}(R)) = K \cup S$ . We know from Proposition 7 that  $|Max(R)| = 2$ . Let  $\{\mathfrak{m}_1, \mathfrak{m}_2\}$  denote the set of all maximal ideals of  $R$ . Let  $V_1, V_2$  be as in the statement of Lemma 3. It follows from Lemma 8 that  $|V_i| = 1$  for at least one  $i \in \{1, 2\}$ . Without loss of generality,



we can assume that  $|V_1| = 1$ . Now, it can be shown as in the proof of (i)  $\Rightarrow$  (ii) of Proposition 4 that there exist nonzero rings  $R_1$  and  $R_2$  such that  $R \cong R_1 \times R_2$  as rings. As  $|Max(R)| = 2$ , it is clear that  $R_1$  and  $R_2$  are quasilocal rings. Let us denote the ring  $R_1 \times R_2$  by  $T$ . Since  $\mathbb{INC}(T)$  is a split graph, we obtain from (i)  $\Rightarrow$  (ii) of Proposition 8 that the rings  $R_1, R_2$  satisfy the conditions mentioned in the statement (ii) of Proposition 8.

(ii)  $\Rightarrow$  (i) Assume that  $R \cong R_1 \times R_2$  as rings, where  $R_1$  and  $R_2$  are quasilocal rings and they satisfy the conditions mentioned in the statement (ii) of Proposition 8. Let us denote the ring  $R_1 \times R_2$  by  $T$ . We know from (ii)  $\Rightarrow$  (i) of Proposition 8 that  $\mathbb{INC}(T)$  is a split graph. Since  $R \cong T$  as rings, we obtain that  $\mathbb{INC}(R)$  is a split graph.  $\square$

We provide some examples in Example 4 to illustrate Theorem 2.

**Example 4.** (i) Let  $F$  be a field and let  $T = \mathbb{Z}_{p\mathbb{Z}}$ , where  $p$  is a prime number. Let  $R = F \times T$ . Then  $\mathbb{INC}(R)$  is a split graph.

(ii) Let  $T = \mathbb{Z}_2[X, Y]$  be the polynomial ring in two variables  $X, Y$  over  $\mathbb{Z}_2$  and let  $\mathfrak{m} = TX + TY$ . Let  $R = F \times \frac{T}{\mathfrak{m}^2}$ , where  $F$  is a field. Then  $\mathbb{INC}(R)$  is a split graph.

(iii) Let  $A = \mathbb{Z}_2[X, Y, Z]$  be the polynomial ring in three variables  $X, Y, Z$  over  $\mathbb{Z}_2$  and let  $\mathfrak{m} = AX + AY + AZ$ . Let  $R = F \times \frac{A}{\mathfrak{m}^2}$ , where  $F$  is a field. Then  $\mathbb{INC}(R)$  is not a split graph.

*Proof.* (i) We know from [[5], Example (1), page 94] that  $T = \mathbb{Z}_{p\mathbb{Z}}$  is a discrete valuation ring and so,  $T$  is a chained ring. Hence, we obtain from (ii)  $\Rightarrow$  (i) of Proposition 8 that  $\mathbb{INC}(R)$  is a split graph.

(ii) It is convenient to denote  $X + \mathfrak{m}^2$  by  $x$  and  $Y + \mathfrak{m}^2$  by  $y$ . It is clear that  $\frac{T}{\mathfrak{m}^2}$  is a local ring with unique maximal ideal  $\frac{\mathfrak{m}}{\mathfrak{m}^2}$ . Observe that  $\mathbb{I}(\frac{T}{\mathfrak{m}^2}) = W_1 \cup W_2$ , where  $W_1 = \{(0 + \mathfrak{m}^2), \frac{\mathfrak{m}}{\mathfrak{m}^2}\}$  and  $W_2 = \{\frac{T}{\mathfrak{m}^2}x, \frac{T}{\mathfrak{m}^2}y, \frac{T}{\mathfrak{m}^2}(x + y)\}$ . It is clear that  $W_1$  is a chain under the inclusion relation and no two distinct members of  $W_2$  are comparable under the inclusion relation. Hence, we obtain from (ii)  $\Rightarrow$  (i) of Proposition 8 that  $\mathbb{INC}(R)$  is a split graph.

(iii) It is convenient to denote  $X + \mathfrak{m}^2$  by  $x$ ,  $Y + \mathfrak{m}^2$  by  $y$ , and  $Z + \mathfrak{m}^2$  by  $z$ . It is clear that  $\frac{A}{\mathfrak{m}^2}$  is a local ring with  $\frac{\mathfrak{m}}{\mathfrak{m}^2}$  as its unique maximal ideal. It is convenient to denote  $\frac{A}{\mathfrak{m}^2}$  by  $A_1$  and  $\frac{\mathfrak{m}}{\mathfrak{m}^2}$  by  $\mathfrak{m}_1$ . Observe that  $\mathbb{I}(A_1) = \{(0 + \mathfrak{m}^2), A_1x, A_1y, A_1z, A_1(x + y), A_1(y + z), A_1(z + x), A_1(x + y + z), A_1x + A_1y, A_1y + A_1z, A_1z + A_1x, A_1x + A_1(y + z), A_1y + A_1(x + z), A_1z + A_1(x + y), \mathfrak{m}_1)\}$ . Note that  $A_1$  is not a field and is not a chained ring. Let  $W_1, W_2$  be subsets of  $\mathbb{I}(A_1)$  such that  $W_1$  is a chain under the inclusion relation and no two distinct members of  $W_2$  are comparable under the inclusion relation. We claim that  $\mathbb{I}(A_1) \neq W_1 \cup W_2$ . Suppose that  $\mathbb{I}(A_1) = W_1 \cup W_2$ . If  $A_1x \in W_1$ , then  $A_1(x + y + z), A_1y + A_1(x + z)$  must be in  $W_2$ . This is impossible since  $A_1(x + y + z) \subset A_1y + A_1(x + z)$ . Hence,  $A_1x \notin W_1$ . If  $A_1x \in W_2$ , then both  $A_1x + A_1y$  and  $A_1x + A_1z$  must be in  $W_1$ . This is impossible since  $A_1x + A_1y$  and  $A_1x + A_1z$  are not comparable under the inclusion relation. Therefore,  $\mathbb{I}(A_1) \neq W_1 \cup W_2$ . Hence, it follows from (i)  $\Rightarrow$  (ii) of Proposition 8 that  $\mathbb{INC}(R)$  is not a split graph.  $\square$

### 4. When is $\text{INC}(R)$ complemented?

Let  $R$  be a ring with  $|\text{Max}(R)| \geq 2$ . In this section, we try to characterize such rings  $R$  whose  $\text{INC}$  graph is complemented.

**Lemma 9.** Let  $R$  be a ring such that  $|\text{Max}(R)| \geq 2$ . Let  $I \in V(\text{INC}(R))$ . If  $J$  is a vertex in  $\text{INC}(R)$  such that  $I \perp J$  in  $\text{INC}(R)$ , then  $IJ \subseteq J(R)$ .

*Proof.* Now, by assumption  $I \perp J$  in  $\text{INC}(R)$ . Hence,  $I$  and  $J$  are adjacent in  $\text{INC}(R)$  and there is no  $A \in V(\text{INC}(R))$  which is adjacent to both  $I$  and  $J$  in  $\text{INC}(R)$ . Let  $\mathfrak{m} \in \text{Max}(R)$ . We claim that  $IJ \subseteq \mathfrak{m}$ . This is clear if  $\mathfrak{m} \in \{I, J\}$ . Hence, we can assume that  $\mathfrak{m} \notin \{I, J\}$ . Since  $I \perp J$ , either  $\mathfrak{m}$  is not adjacent to  $I$  or  $\mathfrak{m}$  is not adjacent to  $J$  in  $\text{INC}(R)$ . Hence, either  $I \subset \mathfrak{m}$  or  $J \subset \mathfrak{m}$ . Therefore,  $IJ \subset \mathfrak{m}$ . This is true for any  $\mathfrak{m} \in \text{Max}(R)$  and so,  $IJ \subseteq J(R)$ . □

**Lemma 10.** Let  $R$  be a ring such that  $|\text{Max}(R)| \geq 2$ . If  $\text{INC}(R)$  is complemented, then  $|\text{Max}(R)| \leq 3$ .

*Proof.* Assume that  $\text{INC}(R)$  is complemented. Suppose that  $|\text{Max}(R)| \geq 4$ . Let  $\{\mathfrak{m}_i \mid i \in \{1, 2, 3, 4\}\} \subseteq \text{Max}(R)$ . Note that  $\mathfrak{m}_1 \cap \mathfrak{m}_2 \in V(\text{INC}(R))$ . Let us denote  $\mathfrak{m}_1 \cap \mathfrak{m}_2$  by  $I$ . Since  $\text{INC}(R)$  is complemented, there exists  $J \in V(\text{INC}(R))$  such that  $I \perp J$  in  $\text{INC}(R)$ . We know from Lemma 9 that  $IJ \subseteq J(R)$ . It follows from [[5], Proposition 1.11(ii)] that  $I \not\subseteq \mathfrak{m}$  for any  $\mathfrak{m} \in \text{Max}(R) \setminus \{\mathfrak{m}_1, \mathfrak{m}_2\}$ . From  $IJ \subseteq J(R)$ , we obtain that  $J \subseteq \mathfrak{m}_3 \cap \mathfrak{m}_4$ . Since  $I$  and  $J$  are adjacent in  $\text{INC}(R)$ , we get that  $J \not\subseteq I$ . Hence, either  $J \not\subseteq \mathfrak{m}_1$  or  $J \not\subseteq \mathfrak{m}_2$ . Without loss of generality, we can assume that  $J \not\subseteq \mathfrak{m}_1$ . Consider the ideal  $A = \mathfrak{m}_1 \cap \mathfrak{m}_3$ . It is clear that  $A \in V(\text{INC}(R))$  and  $I = \mathfrak{m}_1 \cap \mathfrak{m}_2 \not\subseteq A = \mathfrak{m}_1 \cap \mathfrak{m}_3$  and  $A \not\subseteq I$ . Since  $A \not\subseteq \mathfrak{m}_4$ , we obtain that  $A \not\subseteq J$ . From  $J \not\subseteq \mathfrak{m}_1$ , it follows that  $J \not\subseteq A$ . Hence, we get that  $A$  is adjacent to both  $I$  and  $J$  in  $\text{INC}(R)$ . This is in contradiction to the assumption that  $I \perp J$  in  $\text{INC}(R)$ . Therefore,  $|\text{Max}(R)| \leq 3$ . □

Let  $R$  be a ring such that  $|\text{Max}(R)| = 2$ . We try to characterize such rings  $R$  whose  $\text{INC}$  graph is complemented.

**Remark 3.** Let  $R$  be a ring such that  $|\text{Max}(R)| = 2$ . Let  $\text{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ . Let  $V_1, V_2$  be as in the statement of Lemma 3. If  $|V_i| = 1$  for each  $i \in \{1, 2\}$ , then it is verified in the proof of (i)  $\Rightarrow$  (ii) of Proposition 1 that  $R \cong F_1 \times F_2$  as rings, where  $F_i$  is a field for each  $i \in \{1, 2\}$  and in such a case, it is observed in (ii)  $\Rightarrow$  (i) of Proposition 1 that  $\text{INC}(R)$  is a complete graph on two vertices. Hence,  $\text{INC}(R)$  is complemented. Thus in characterizing rings  $R$  with  $|\text{Max}(R)| = 2$  whose  $\text{INC}$  graph is complemented, we assume that  $|V_i| \geq 2$  for at least one  $i \in \{1, 2\}$ .

**Lemma 11.** Let  $R$  be a ring such that  $|\text{Max}(R)| = 2$ . Let  $\text{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ . Let  $V_1, V_2$  be as in the statement of Lemma 3. Suppose that  $|V_i| \geq 2$  for some  $i \in \{1, 2\}$ . If  $\text{INC}(R)$  is complemented, then any  $I_1, I_2 \in V_i$  are comparable under the inclusion relation.

*Proof.* Suppose that  $|V_1| \geq 2$ . Let  $I_1 \in V_1$ . We are assuming that  $\mathbb{INC}(R)$  is complemented. Hence, there exists  $J_1 \in V(\mathbb{INC}(R))$  such that  $I_1 \perp J_1$  in  $\mathbb{INC}(R)$ . We know from Lemma 9 that  $I_1 J_1 \subseteq J(R) = \mathfrak{m}_1 \cap \mathfrak{m}_2$ . From  $I_1 \not\subseteq \mathfrak{m}_2$ , we obtain that  $J_1 \subseteq \mathfrak{m}_2$ . Hence,  $M(J_1) = \{\mathfrak{m}_2\}$  and so,  $J_1 \in V_2$ . Let  $I_2 \in V_1$  be such that  $I_2 \neq I_1$ . Since  $I_2 + J_1 = R$ ,  $I_2$  and  $J_1$  are adjacent in  $\mathcal{C}(R)$  and hence, they are adjacent in  $\mathbb{INC}(R)$ . As  $I_1 \perp J_1$  in  $\mathbb{INC}(R)$ ,  $I_2$  and  $I_1$  cannot be adjacent in  $\mathbb{INC}(R)$ . Therefore,  $I_1$  and  $I_2$  are comparable under the inclusion relation. Similarly, if  $|V_2| \geq 2$ , it can be shown that any two members of  $V_2$  are comparable under the inclusion relation.  $\square$

**Proposition 9.** Let  $R$  be a ring such that  $|Max(R)| = 2$ . Let  $Max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ . Let  $V_1, V_2$  be as in the statement of Lemma 3. Suppose that  $|V_1| = 1$  and  $|V_2| \geq 2$ . The following statements are equivalent:

- (i)  $\mathbb{INC}(R)$  is complemented.
- (ii)  $R \cong R_1 \times R_2$  as rings, where  $R_1$  is a chained ring which is not a field and  $R_2$  is a field.

*Proof.* (i)  $\Rightarrow$  (ii) Let  $a \in \mathfrak{m}_1 \setminus \mathfrak{m}_2$ . It follows from  $|V_1| = 1$  that  $\mathfrak{m}_1 = Ra = \mathfrak{m}_1^2 = Ra^2$ . Hence, there exists a nontrivial idempotent  $e \in \mathfrak{m}_1$  such that  $\mathfrak{m}_1 = Re$ . Note that the mapping  $f : R \rightarrow Re \times R(1 - e)$  defined by  $f(r) = (re, r(1 - e))$  is an isomorphism of rings. Let us denote the ring  $Re$  by  $R_1$ ,  $R(1 - e)$  by  $R_2$ , and  $R_1 \times R_2$  by  $T$ . Observe that  $f(\mathfrak{m}_1) = R_1 \times (0)$  and as  $f(\mathfrak{m}_1) \in Max(T)$ , it follows that  $R_2$  is a field. Since  $R \cong T$  as rings, we obtain that  $|Max(T)| = 2$  and so,  $R_1$  is quasilocal. Let us denote the unique maximal ideal of  $R_1$  by  $\mathfrak{n}_1$ . It is clear that  $f(\mathfrak{m}_2) = \mathfrak{n}_1 \times R_2$ . Note that under the isomorphism  $f$ ,  $V_1$  is mapped onto  $W_1 = \{R_1 \times (0)\}$  and  $V_2$  is mapped onto  $W_2 = \{I \times R_2 \mid I \in \mathbb{I}(R_1)\}$ . We are assuming that  $\mathbb{INC}(R)$  is complemented. Therefore,  $\mathbb{INC}(T)$  is complemented. From  $|W_2| \geq 2$ , it follows from Lemma 11 that any two members of  $W_2$  are comparable under the inclusion relation. Hence, if  $I_1, I_2 \in \mathbb{I}(R_1)$ , then  $I_1$  and  $I_2$  are comparable under the inclusion relation. Therefore, we obtain that  $R_1$  is a chained ring and it follows from  $|W_2| \geq 2$  that  $R_1$  is not a field.

(ii)  $\Rightarrow$  (i) Assume that  $R \cong R_1 \times R_2$  as rings, where  $R_1$  is a chained ring which is not a field and  $R_2$  is a field. It follows from (ii)  $\Rightarrow$  (i) of Proposition 4 that  $\mathbb{INC}(R)$  is a star graph and so,  $\mathbb{INC}(R)$  is complemented.  $\square$

**Proposition 10.** Let  $R$  be a ring such that  $|Max(R)| = 2$ . Let  $\{\mathfrak{m}_1, \mathfrak{m}_2\}$  denote the set of all maximal ideals of  $R$ . Let  $V_1, V_2$  be as in the statement of Lemma 3. Suppose that  $|V_i| \geq 2$  for each  $i \in \{1, 2\}$ . Then the following statements are equivalent:

- (i)  $\mathbb{INC}(R)$  is complemented.
- (ii) Any two members of  $V_i$  are comparable under the inclusion relation for each  $i \in \{1, 2\}$ .
- (iii)  $\mathbb{INC}(R) = \mathcal{C}(R)$  is a complete bipartite graph.

*Proof.* (i)  $\Rightarrow$  (ii) We are assuming that  $\mathbb{INC}(R)$  is complemented. By hypothesis,  $|V_i| \geq 2$  for each  $i \in \{1, 2\}$ . Hence, we obtain from Lemma 11 that any two members of  $V_i$  are comparable under the inclusion relation for each  $i \in \{1, 2\}$ .

(ii)  $\Rightarrow$  (iii) Note that  $V(\mathbb{INC}(R)) = V_1 \cup V_2$ . Observe that  $V_1 \cap V_2 = \emptyset$ . It follows from (ii) that if  $I_1 - I_2$  is an edge of  $\mathbb{INC}(R)$ , then both  $I_1, I_2$  cannot be in the same  $V_i$  for any  $i \in \{1, 2\}$ . If  $I \in V_1$  and  $J \in V_2$ , then  $I + J = R$  and so,  $I$  and  $J$  are adjacent in  $\mathbb{INC}(R)$ . Therefore,  $\mathbb{INC}(R) = \mathcal{C}(R)$  is a complete bipartite graph.

(iii)  $\Rightarrow$  (i) This is clear. □

Let  $R$  be a ring with  $|Max(R)| = 2$  satisfying the hypothesis of Proposition 10. We are not able to characterize such rings  $R$  which satisfies the statement (ii) of Proposition 10. However, we mention one instance where the statement (ii) of Proposition 10 is satisfied. Let  $R_1, R_2$  be chained rings which are not fields and let  $R = R_1 \times R_2$ . Let  $i \in \{1, 2\}$  and let  $\mathfrak{m}_i$  denote the unique maximal ideal of  $R_i$ . Note that in this case,  $V_1 = \{I \times R_2 \mid I \in \mathbb{I}(R_1)\}$  and  $V_2 = \{R_1 \times J \mid J \in \mathbb{I}(R_2)\}$ . Since  $R_1$  and  $R_2$  are not fields, we obtain that  $|V_i| \geq 2$  for each  $i \in \{1, 2\}$ . As  $R_i$  is a chained ring for each  $i \in \{1, 2\}$ , we obtain that  $R$  satisfies the statement (ii) of Proposition 10. Therefore,  $\mathbb{INC}(R)$  is complemented. In Proposition 11, we characterize zero-dimensional rings  $R$  with  $|Max(R)| = 2$  such that  $\mathbb{INC}(R)$  is complemented.

**Proposition 11.** Let  $R$  be a ring with  $|Max(R)| = 2$ . Let  $dim R = 0$ . Let  $Max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$  and let  $V_1, V_2$  be as in the statement of Lemma 3. Suppose that  $|V_i| \geq 2$  for each  $i \in \{1, 2\}$ . Then the following statements are equivalent:

- (i)  $\mathbb{INC}(R)$  is complemented.
- (ii)  $R \cong R_1 \times R_2$  as rings, where  $R_i$  is a chained ring which is not a field for each  $i \in \{1, 2\}$ .

*Proof.* (i)  $\Rightarrow$  (ii) Since  $dim R = 0$  and  $|Max(R)| = 2$ , we obtain from Remark 2 that  $R \cong R_1 \times R_2$  as rings, where  $(R_i, \mathfrak{n}_i)$  is a quasilocal ring for each  $i \in \{1, 2\}$ . Let us denote the ring  $R_1 \times R_2$  by  $T$ . Note that under the isomorphism from  $R$  onto  $T$ ,  $V_1$  is mapped onto  $W_1 = \{I \times R_2 \mid I \in \mathbb{I}(R_1)\}$  and  $V_2$  is mapped onto  $W_2 = \{R_1 \times J \mid J \in \mathbb{I}(R_2)\}$ . Since  $R \cong T$  as rings, we obtain that  $\mathbb{INC}(T)$  is complemented. By hypothesis,  $|V_i| \geq 2$  for each  $i \in \{1, 2\}$  and so,  $|W_i| \geq 2$  for each  $i \in \{1, 2\}$ . Therefore,  $R_i$  is not a field for each  $i \in \{1, 2\}$ . Let  $i \in \{1, 2\}$ . We know from Lemma 11 that any two members of  $W_i$  are comparable under the inclusion relation and so, any two proper ideals of  $R_i$  are comparable under the inclusion relation. Therefore,  $R_i$  is a chained ring.

(ii)  $\Rightarrow$  (i) Let us denote the ring  $R_1 \times R_2$  by  $T$ . It follows from (iii)  $\Rightarrow$  (ii) of Proposition 3 that  $\mathbb{INC}(T)$  is a complete bipartite graph. Therefore,  $\mathbb{INC}(T)$  is complemented and so,  $\mathbb{INC}(R)$  is complemented. □

**Remark 4.** In this Remark, we mention an example to illustrate that (i)  $\Rightarrow$  (ii) of Proposition 11 can fail to hold if the hypothesis  $dim R = 0$  is omitted in Proposition 11. Let  $p, q$  be distinct prime numbers and let  $R = S^{-1}\mathbb{Z}$ , where  $S = \mathbb{Z} \setminus (p\mathbb{Z} \cup q\mathbb{Z})$ . Note that  $R$  is a principal ideal domain and  $Max(R) = \{pR, qR\}$ . It is verified in Example 1 that  $\mathcal{C}(R) = \mathbb{INC}(R)$  is a complete bipartite graph. Hence,  $\mathbb{INC}(R)$  is complemented. Since  $R$  is an integral domain, 0 and 1 are the only idempotent elements of  $R$ . Therefore, (ii) of Proposition 11 does not hold.

Let  $R$  be a ring with  $|Max(R)| = 3$ . In Theorem 3, we characterize such rings  $R$  whose  $\mathbb{INC}$  graph is complemented.

**Lemma 12.** Let  $R$  be a ring such that  $|Max(R)| = 3$ . Let  $\{\mathfrak{m}_i | i \in \{1, 2, 3\}\}$  denote the set of all maximal ideals of  $R$ . If  $\mathbb{INC}(R)$  is complemented, then  $\mathfrak{m}_i = \mathfrak{m}_i^2$  for each  $i \in \{1, 2, 3\}$ .

*Proof.* We are assuming that  $\mathbb{INC}(R)$  is complemented and  $Max(R) = \{\mathfrak{m}_i | i \in \{1, 2, 3\}\}$ . We claim that  $\mathfrak{m}_i = \mathfrak{m}_i^2$  for each  $i \in \{1, 2, 3\}$ . Since  $\mathbb{INC}(R)$  is complemented, there exists  $J \in V(\mathbb{INC}(R))$  such that  $\mathfrak{m}_1^2 \perp J$  in  $\mathbb{INC}(R)$ . It now follows from Lemma 9 that  $\mathfrak{m}_1^2 J \subseteq J(R) = \cap_{i=1}^3 \mathfrak{m}_i$ . This implies that  $J \subseteq \mathfrak{m}_2 \cap \mathfrak{m}_3$ . Let us denote the ideal  $\mathfrak{m}_1 \mathfrak{m}_3$  by  $A$ . It is clear that  $A \in V(\mathbb{INC}(R))$ . Observe that  $A \not\subseteq \mathfrak{m}_2$ , whereas  $J \subseteq \mathfrak{m}_2$  and so,  $A \not\subseteq J$ . Since  $J \not\subseteq J(R)$ , it follows that  $J \not\subseteq \mathfrak{m}_1$ . As  $A \subseteq \mathfrak{m}_1$ , we obtain that  $J \not\subseteq A$ . Hence,  $A$  and  $J$  are adjacent in  $\mathbb{INC}(R)$ . Since  $\mathfrak{m}_1^2 \not\subseteq \mathfrak{m}_3$ , whereas  $A \subseteq \mathfrak{m}_3$ , we obtain that  $\mathfrak{m}_1^2 \not\subseteq A$ . As  $\mathfrak{m}_1^2 \perp J$  in  $\mathbb{INC}(R)$ , it follows that  $\mathfrak{m}_1^2$  and  $A$  cannot be adjacent in  $\mathbb{INC}(R)$ . Therefore,  $A = \mathfrak{m}_1 \mathfrak{m}_3 \subseteq \mathfrak{m}_1^2$ . We know from [[5], Proposition 4.2] that  $\mathfrak{m}_1^2$  is a  $\mathfrak{m}_1$ -primary ideal of  $R$ . As  $\mathfrak{m}_3 \not\subseteq \mathfrak{m}_1 = \sqrt{\mathfrak{m}_1^2}$ , we get that  $\mathfrak{m}_1 \subseteq \mathfrak{m}_1^2$  and so,  $\mathfrak{m}_1 = \mathfrak{m}_1^2$ . Similarly, it can be shown that  $\mathfrak{m}_2 = \mathfrak{m}_2^2$  and  $\mathfrak{m}_3 = \mathfrak{m}_3^2$ .  $\square$

**Lemma 13.** Let  $R$  be a ring such that  $|Max(R)| = 3$ . Let  $\{\mathfrak{m}_i | i \in \{1, 2, 3\}\}$  denote the set of all maximal ideals of  $R$ . If  $\mathbb{INC}(R)$  is complemented, then  $R_{\mathfrak{m}_i}$  is a field for each  $i \in \{1, 2, 3\}$ .

*Proof.* We are assuming that  $\mathbb{INC}(R)$  is complemented. We first verify that  $R_{\mathfrak{m}_1}$  is a field. Since  $\mathfrak{m}_1 \not\subseteq \mathfrak{m}_2 \cup \mathfrak{m}_3$ , there exists  $a \in \mathfrak{m}_1 \setminus (\mathfrak{m}_2 \cup \mathfrak{m}_3)$ . Note that  $Ra \in V(\mathbb{INC}(R))$ . As  $\mathbb{INC}(R)$  is complemented, there exists  $J \in V(\mathbb{INC}(R))$  such that  $Ra \perp J$  in  $\mathbb{INC}(R)$ . We know from Lemma 9 that  $(Ra)J \subseteq J(R) = \cap_{i=1}^3 \mathfrak{m}_i$ . Hence, we obtain that  $J \subseteq \mathfrak{m}_2 \cap \mathfrak{m}_3$ . Let us denote the ideal  $\mathfrak{m}_1 \mathfrak{m}_2$  by  $A$ . It is clear that  $A \in V(\mathbb{INC}(R))$ . As  $J \subseteq \mathfrak{m}_3$  and  $A \not\subseteq \mathfrak{m}_3$ , we obtain that  $A \not\subseteq J$ . Since  $J \not\subseteq J(R)$ , it follows that  $J \not\subseteq \mathfrak{m}_1$  and so,  $J \not\subseteq \mathfrak{m}_1 \mathfrak{m}_2 = A$ . Hence,  $A$  and  $J$  are not comparable under the inclusion relation and therefore,  $A$  and  $J$  are adjacent in  $\mathbb{INC}(R)$ . As  $a \notin \mathfrak{m}_2$ , it follows that  $Ra \not\subseteq A$ . Since  $Ra \perp J$  in  $\mathbb{INC}(R)$ , we obtain that  $Ra$  and  $A$  cannot be adjacent in  $\mathbb{INC}(R)$ . Therefore,  $A = \mathfrak{m}_1 \mathfrak{m}_2 \subseteq Ra$ . This implies that  $(\mathfrak{m}_1 \mathfrak{m}_2)_{\mathfrak{m}_1} \subseteq (Ra)_{\mathfrak{m}_1} \subseteq (\mathfrak{m}_1)_{\mathfrak{m}_1}$ . From  $(\mathfrak{m}_2)_{\mathfrak{m}_1} = R_{\mathfrak{m}_1}$ , we get that  $(\mathfrak{m}_1)_{\mathfrak{m}_1} = R_{\mathfrak{m}_1}(\frac{a}{1})$ . We know from Lemma 12 that  $\mathfrak{m}_1 = \mathfrak{m}_1^2$ . Hence, we obtain that  $R_{\mathfrak{m}_1}(\frac{a}{1}) = (\mathfrak{m}_1)_{\mathfrak{m}_1} = (\mathfrak{m}_1^2)_{\mathfrak{m}_1} = R_{\mathfrak{m}_1}(\frac{a^2}{1})$ . Hence,  $\frac{a}{1} = \frac{r a^2}{s 1}$  for some  $r \in R$  and  $s \in R \setminus \mathfrak{m}_1$ . Therefore,  $\frac{a}{1}(\frac{1}{1} - \frac{ra}{s}) = \frac{0}{1}$ . Since  $R_{\mathfrak{m}_1}$  is quasilocal with  $(\mathfrak{m}_1)_{\mathfrak{m}_1}$  as its unique maximal ideal, it follows that  $\frac{1}{1} - \frac{ra}{s}$  is a unit in  $R_{\mathfrak{m}_1}$ , and so, we obtain that  $\frac{a}{1} = \frac{0}{1}$ . Therefore,  $(\mathfrak{m}_1)_{\mathfrak{m}_1} = (\frac{0}{1})$ . This proves that  $R_{\mathfrak{m}_1}$  is a field. Similarly, it can be shown that  $R_{\mathfrak{m}_i}$  is a field for each  $i \in \{2, 3\}$ .  $\square$

**Lemma 14.** Let  $R = F_1 \times F_2 \times F_3$ , where  $F_i$  is a field for each  $i \in \{1, 2, 3\}$ . Then  $\mathbb{INC}(R)$  is complemented.

*Proof.* Note that  $Max(R) = \{\mathfrak{m}_1 = (0) \times F_2 \times F_3, \mathfrak{m}_2 = F_1 \times (0) \times F_3, \mathfrak{m}_3 = F_1 \times F_2 \times (0)\}$ . It is clear that  $J(R) = (0) \times (0) \times (0)$  and  $V(\mathbb{INC}(R)) = \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_1 \cap \mathfrak{m}_2, \mathfrak{m}_2 \cap \mathfrak{m}_3, \mathfrak{m}_1 \cap \mathfrak{m}_3\}$ . It is clear that  $\mathfrak{m}_1 \perp (\mathfrak{m}_2 \cap \mathfrak{m}_3), \mathfrak{m}_2 \perp (\mathfrak{m}_1 \cap \mathfrak{m}_3)$ , and  $\mathfrak{m}_3 \perp (\mathfrak{m}_1 \cap \mathfrak{m}_2)$  in  $\mathbb{INC}(R)$ . This proves that  $\mathbb{INC}(R)$  is complemented.  $\square$

**Theorem 3.** Let  $R$  be a ring such that  $|Max(R)| = 3$ . The following statements are equivalent:

(i)  $\mathbb{INC}(R)$  is complemented.

(ii)  $R \cong F_1 \times F_2 \times F_3$  as rings, where  $F_i$  is a field for each  $i \in \{1, 2, 3\}$ .

*Proof.* (i)  $\Rightarrow$  (ii) Assume that  $\mathbb{INC}(R)$  is complemented. Let  $\{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\}$  denote the set of all maximal ideals of  $R$ . We know from Lemma 13 that  $R_{\mathfrak{m}_i}$  is a field for each  $i \in \{1, 2, 3\}$ . Hence,  $(J(R))_{\mathfrak{m}_i} = (\mathfrak{m}_i)_{\mathfrak{m}_i} = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$  for each  $i \in \{1, 2, 3\}$ . Therefore, we obtain from (iii)  $\Rightarrow$  (i) of [[5], Proposition 3.8] that  $J(R) = (0)$ . Thus  $\bigcap_{i=1}^3 \mathfrak{m}_i = (0)$ . As distinct maximal ideals of a ring are comaximal, it follows from the Chinese remainder theorem [[5], Proposition 1.10(ii) and (iii)] that  $R \cong \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2} \times \frac{R}{\mathfrak{m}_3}$ .

(ii)  $\Rightarrow$  (i) Let us denote the ring  $F_1 \times F_2 \times F_3$  by  $T$ . We know from Lemma 14 that  $\mathbb{INC}(T)$  is complemented. Since  $R \cong T$  as rings, we obtain that  $\mathbb{INC}(R)$  is complemented.  $\square$

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