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Strong alliances in graphs

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Abstract: For any simple connected undirected graph $G = (V, E)$, a *defensive alliance* is a subset S of V satisfying the condition that every vertex $v \in S$ has at most one more neighbour in $V - S$ than it has in S. The minimum cardinality of any defensive alliance in G is called the *alliance number* of G, and is denoted by $a_d(G)$. In this paper, we introduce a new type of alliance number called the *k-strong alliance number* and its varieties. The bounds for 1-strong alliance number in terms of different graphical parameters are determined and the characterizations of graphs with 1-strong alliance number 1, 2, and n are obtained.

Keywords: Alliances, Defensive alliances, Secure sets, Strong alliances

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1. Introduction

All the graphs considered in this article here are undirected, finite, connected and simple. We use the standard terminology, the terms not defined here may be found in [2]. Let $G = (V, E)$ be a connected graph and $v \in V$. Then $\Delta(G)$ is the maximum degree of a vertex in G, $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$.
The concept of an *alliance* and few of its variants was introduced by P. Kristiansen, S. T. Hedetniemi, and S. M. Hedetniemi in [10] and [9]. A *defensive alliance* in G is a subset S of V such that $|N[x] \cap S| \geq |N[x] - S|$, for all $x \in S$. For each vertex $x \in S$, the vertices of $N[x] - S$ are termed as attackers of x and those of $N[x] \cap S$ as defenders

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of x. An alliance S is a *minimal* (or *critical*) *alliance* if none of its proper subsets is an alliance of the same type. The minimum cardinality of a defensive alliance in G is called the defensive alliance number of G and is denoted by $a_d(G)$.
Alliances are formed to join forces if one or more of them are attacked. In any *defensive alliance* S of G each vertex $v \in S$ has at most one more neighbor in $V - S$ than it has in S. Further for any integer $k \in \{-\Delta(G), \ldots, \Delta(G)\}$, a nonempty subset $S \subset V$ is a *defensive k-alliance* in G whenever $|N[x] \cap S| - |N[x] - S| \geq k$, for all $x \in S$. The minimum cardinality of a defensive k-alliance in G is denoted by $a_k(G)$ and called the defensive k-alliance number of G. If k is a positive integer, then every vertex in a defensive k-alliance has at least k more defenders than its attackers. This generalization of defensive alliances was presented by Shafique and Dutton in [12, 13]. Till date, several other varieties of alliances have been introduced and studied. The related works can be found in [1, 3–5, 7–11].
In many practical situations like decision taken by board of directors of a company or no-confidence motion against the ruling government, 50% and above majority is required. Suppose a ruling party has 275 members in its alliance group in the house of 500 members. Even if one or two members leave the alliance, ruling party continues to win the majority. But if 25 people leave the alliance, then it has to yield to the opposite force. Therefore this ruling alliance has 24 surplus members in it. The situation like this may also appear in the board of directors of a company and other places. To deal with these situations, we introduce a new type of alliance called *k-strong defensive alliances* in graphs.
A defensive alliance S of a nontrivial graph G of order at least $k+1$ is said to be a *k-strong defensive alliance* if S remains a defensive alliance in G by the omission of at most k vertices from it. The minimum cardinality of a k-strong defensive alliance in G is called the *k-strong alliance number* of G and is denoted by $a^k(G)$. In particular, a 1-strong defensive alliance S of G is a defensive alliance in G with the property that, for each vertex $v \in S$, $S - \{v\}$ is also a defensive alliance in G. The k-strong alliance number of G is a natural generalization of alliance number, since $a_d(G) = a^0(G)$.
For the graph G of Figure 1, $S_1 = \{x, y, z\}$ is a 2-strong defensive alliance, but not a defensive 2-alliance. Also the set $S = \{t, u, v, w\}$ is a defensive 3-alliance, but it is not a 2-strong defensive alliance in G. Further, it is easy to see that $a^1(G) = 2, a_1(G) = 3$; $a^2(G) = 3, a_2(G) = 4$; $a^3(G) = 7, a_3(G) = 4$.
The invariant $a^k(G)$ introduced in the paper is incomparable with $a_k(G)$ in general. In fact, $a_1(C_3) = 2 < 3 = a^1(C_3)$; $a_1(K_{1,4}) = 3 > 2 = a^1(K_{1,4})$; and $a_1(K_4) = 3 = a^1(K_4)$.
We recall the following theorems for immediate reference.

Theorem 1 (P.Kristiansen, S.T.Hedetniemi, and S.M.Hedetniemi [10]). *The subgraph induced by a minimal defensive alliance of a connected graph G is connected.*

Theorem 2 (P.Kristiansen, S.T.Hedetniemi, and S.M.Hedetniemi [10]). *For any graph G of order* $n \geq 2$, $a_d(G) \leq \lceil \frac{n}{2} \rceil$.

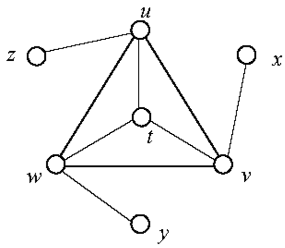

Figure 1. **The graph** G.

If S is any defensive alliance of G, then for each vertex $v \in S$, S should contain at least $\lfloor \frac{\deg v}{2} \rfloor$ neighbors of v. Thus;

Theorem 3. *If S is any defensive alliance set of the graph G and $v \in S$, then*

$$|S| \geq 1 + \left\lfloor \frac{\deg(v)}{2} \right\rfloor = \left\lceil \frac{1 + \deg(v)}{2} \right\rceil \geq \left\lceil \frac{1 + \delta(G)}{2} \right\rceil$$

2. 1-Strong Alliances in Graphs

It is clear from the definition that 1-strong alliance number is defined for non-trivial graphs. For any graph G of order 2, $a^1(G) = 2$. If G is a disconnected graph with components $G_1, G_2, \ldots, G_k$ such that $\delta(G_i) \geq 2$ for every component G_i except possible for one i, then $a^1(G) = \min\{a^1(G_i) : 1 \leq i \leq k\}$. But, if at least two components, say G_i and G_j, contains pendent vertices, then a 2-element set S' containing pendent vertices one each from G_i and G_j is a minimal 1-strong defensive alliance. Also, S' may have lesser cardinality of every minimal 1-strong defensive of each component G_i, for $1 \leq i \leq k$. Therefore, here onwards we consider simple connected graphs of order at least 3.
The following is a direct consequence of the definition.

Proposition 1. *For any positive integer k and a graph G of order at least $k+1$, $a^k(G) \geq a^{k-1}(G) \geq a_d(G) \geq 1$.*

We now see that $a^1(G)$ need not be equal to $a_d(G) + 1$. In fact, for the cycle C_4, on 4 vertices, $a_d(C_4) = 2$, but $a^1(C_4) = 4$. In [9], Hedetniemi and Kristiansen proved that the problem of finding the alliance number of a graph is NP-complete. Thus the problem of finding 1-strong alliance number of a graph is also NP-complete. For any graph $G(V, E)$ of order at least 3, the set $S = V - \{v\}$ is a defensive alliance. Thus the following proposition is trivial.

Proposition 2. *For any graph G on n (≥ 3) vertices, $2 \leq a^1(G) \leq n$.*

Proposition 3. *For any connected non-trivial graph G, $a^1(G) = 2$ if and only if G has at least two pendant vertices.*

Proof. Let G be a graph with $a^1(G) = 2$. Then there exists a defensive alliance $S = \{u, v\}$, such that both the subsets $S_1 = \{u\}$ and $S_2 = \{v\}$ are defensive alliances of G. This is possible only if $deg_G(u) = 1$ and $deg_G(v) = 1$. Hence u and v are pendant vertices in G. The converse is trivial. □

The above Propositions 2 and 3 imply the following:

Corollary 1. *For any graph G of order at least 3 having at most one pendant vertex, $a^1(G) \geq 3$.*

Corollary 2. *For any tree T, $a^1(T) = 2$.*

Proposition 4. *For any connected graph G, $a^1(G) = 3$ if and only if $a^1(G) \neq 2$ and there exist three vertices each of degree at most three that induce the graph K_3.*

Proof. Let G be a graph with $a^1(G) = 3$. Let $S = \{u, v, w\}$ be a 1-strong defensive alliance in G. Then, by Proposition 3, G can have at most one pendant vertex. Without loss of generality, we assume $\deg(v) \geq 2$ and $\deg(w) \geq 2$. Since S is a 1-strong defensive alliance in G, $S_1 = S - \{u\} = \{v, w\}$, $S_2 = S - \{v\} = \{u, w\}$, and $S_3 = S - \{w\} = \{u, v\}$ are defensive alliances of G. So, by Theorem 1, $\langle S_1 \rangle$, $\langle S_2 \rangle$, and $\langle S_3 \rangle$ are connected and hence each is isomorphic to K_2. Thus u, v, w are mutually adjacent. Further if $deg(u) \geq 4$, then $\left|N[u] - (S - \{v\})\right| \geq 3 > 2 = \left|N[u] \cap (S - \{v\})\right|$, a contradiction to the fact that S is a 1-strong defensive alliance. The other cases follow similarly.
Conversely, let $a^1(G) \neq 2$ and G has three mutually adjacent vertices, say u, v, w, each of degree at most three. Then by Proposition 2, $a^1(G) \geq 3$ and G has at most one pendant vertex. The set $S = \{u, v, w\}$ is a 1-strong defensive alliance in G. □

3. Bounds and Characterizations

In this section, we give bound for a 1-strong alliance number of the graph in terms of graphical parameters. Also, we characterize the graphs having 1-strong alliance number n. First we estimate the bounds for 1-strong alliance number of a graph in terms of degree of a vertex.

Lemma 1. *For a graph G, $a^1(G) \geq \left\lceil \frac{\delta(G)+1}{2} \right\rceil + 1$.*

Proof. Let S be a 1-strong defensive alliance of minimum cardinality in G and $v \in S$. Then $S' = S - \{v\}$ is defensive alliance of G. Therefore, by Theorem 3, we get $|S| = 1 + |S'| \geq 1 + \left\lceil \frac{\delta(G)+1}{2} \right\rceil$. □

Remark 1. For any 1-strong defensive alliance S of a graph G and $v \in S$, $|S| \geq \left\lceil \frac{\deg(v)+1}{2} \right\rceil + 1$ or equivalently $\deg(v) \leq 2|S| - 3$.

Lemma 2 (P.Kristiansen, S.T.Hedetniemi, and S.M.Hedetniemi [9]). *If S is any defensive alliance in a graph G with minimum cardinality, then $\langle S \rangle$ is connected.*

We obtain a similar result for the 1-strong defensive alliance with minimum cardinality in a graph G. In view of Proposition 3, we see that if S is a 1-strong defensive alliance with cardinality 2, then $\langle S \rangle$ is not connected. Thus we consider only the graphs having at most one pendant vertex for further discussions.

Proposition 5. *If a graph G of order n (≥ 3) has a pendant vertex, then $a^1(G) \leq n-1$.*

Proof. If G has more than one pendent vertex, then $a^1(G) = 2 \leq n - 1$. We now suppose that G has exactly one pendent vertex v. Let $a^1(G) = n$. Then every $(n-1)$-element subset of V is not a 1-strong defensive alliance. Let $S = V - \{v\}$. Then S contains some $x \in S$ such that $S - \{x\}$ is not a defensive alliance. This is possible only if x is adjacent to some y in S such that y is adjacent to v and $\deg_G(y) = 2$. Let $S' = S - \{y\}$. Then clearly S' is a defensive alliance. But then, as S' cannot be a 1-strong defensive alliance (since $|S'| < a^1(G)$), there is a vertex $z \in S'$ with $\deg_G(z) = 2$ and that is adjacent to x and $\deg_G(x) = 2$. Then $S'' = S' - \{x\}$ is a defensive alliance. Continuing this we end up with a set containing only one vertex u, which is a defensive alliance in G. Then this vertex u should be a pendent vertex, a contradiction to the fact that G has only one pendent vertex. □

If S_1 and S_2 are any two disjoint defensive alliances in G, then $S_1 \cup S_2$ is also a defensive alliance in G. Moreover, if S_1 and S_2 are any two 1-strong defensive alliances in G, then for any $x \in S_1$ and $y \in S_2$ the subsets $S_1 - \{x\}$ and $S_2 - \{y\}$ are defensive alliances of G. So, for any vertex $z \in S_1 \cup S_2$, we observe $(S_1 \cup S_2) - \{z\} = (S_1 - \{z\}) \cup S_2$ if $z \in S_1$, otherwise $(S_1 \cup S_2) - \{z\} = S_1 \cup (S_2 - \{z\})$. Thus $(S_1 \cup S_2) - \{z\}$ is a defensive alliance in G. Thus we have proved the following:

Lemma 3. *If S_1 and S_2 are any two disjoint 1-strong defensive alliances in G, then so is $S_1 \cup S_2$.*

Now we prove the following result similar to Lemma 2.

Theorem 4. *Let G be a graph with $\delta(G) \geq 2$. If S is any 1-strong defensive alliance with minimum cardinality in G, then the graph $\langle S \rangle$ is connected and has no pendant vertex.*

Proof. Let G be a graph with $\delta(G) \geq 2$ and S be a 1-strong defensive alliance with minimum cardinality in G. We first prove that $\langle S \rangle$ is connected. Suppose to contrary that $\langle S \rangle$ is disconnected. Without loss of generality we may assume $\langle S \rangle$ has exactly two components say $\langle S_1 \rangle$ and $\langle S_2 \rangle$ (result follows by induction hypothesis if $\langle S \rangle$ has more than two components). Then $S_1 \cap S_2 = \phi$ and $S_1 \cup S_2 = S$. Since $S = S_1 \cup S_2$ is a 1-strong defensive alliance in G, we have for every $x \in S$, $S - \{x\}$ is a defensive alliance. Since $S_1 \cap S_2 = \phi$, for every $x \in S$, either $x \in S_1$ or $x \in S_2$. Thus both $S_1 - \{x\}$ and $S_2 - \{x\}$ are defensive alliance in G. Hence both S_1 and S_2 are 1-strong defensive alliances in G. But we have $|S_1| < |S|$ and $|S_2| < |S|$, a contradiction to the fact that S is a 1-strong defensive alliance in G with minimum cardinality. Therefore, $\langle S \rangle$ is connected.
Now to prove the induced subgraph $\langle S \rangle$ has no pendant vertex, we use method of contradiction. Suppose that the graph $\langle S \rangle$ has a pendant vertex v adjacent to the vertex w in S. Since $v \in S \subseteq V(G)$ and $\delta(G) \geq 2$, the vertex v is not a pendant vertex of G, therefore $\deg(v) \geq 2$ in G. But we have $N[v] \cap S = \{v, w\}$ and hence $|N[v] \cap (S - \{w\})| = \{v\}$. So $|N[v] \cap (S - \{w\})| = 1$ and as $\deg(v) \geq 2$ in G, $|N[v] - (S - \{w\})| \geq 2$. Therefore, $|N[v] \cap (S - \{w\})| < |N[v] - (S - \{w\})|$, a contradiction to the fact that S is a 1-strong defensive alliance in G. □

Let G be a graph of order at least 3. If $\delta(G) \geq 2$, then every vertex of G lies on some cycle of G. Hence by using Theorem 4, we state the following corollary.

Corollary 3. *Let G be a graph with $\delta(G) \geq 2$. If S is any 1-strong defensive alliance in G with minimum cardinality, then every vertex of S lies on some cycle of $\langle S \rangle$.*

The *girth* of a graph G, denoted by $g(G)$ (or simply g), is the length of a shortest cycle (if any) in G. If G is a connected graph with $\delta(G) \geq 2$, then G contains a cycle and hence the girth is at least 3. Thus, in general for any graph with at most one pendent vertex, S may include the pendent vertex or not, so by above discussions, the following corollary is straightforward.

Corollary 4. *For any graph G with at most one pendant vertex, $a^1(G) \geq g$, where g is the girth of G.*

Theorem 5. *For a graph G having at most one pendant vertex, with $\triangle(G) \leq 3$ and girth g, $a^1(G) = g$.*

Proof. Let G be a graph having at most one pendant vertex with $\triangle(G) \leq 3$ and girth g. From Corollary 4, $a^1(G) \not< g$. Let $v_1, v_2, \ldots, v_g$ be the vertices of the cycle of length g in G. Consider $S = \{v_1, v_2, \ldots, v_g\}$. Then for each $v_j \in S$, $\deg_G(v_j) \leq 3$ implies that $|N[v_j] \cap S| = 3$ and $|N[v_j] - S| \leq 1$. Hence S is a defensive alliance in G, and for any $v_k \in S$, the set $S' = S - \{v_k\}$ is also a defensive alliance (since $|N[v_k] \cap S| = 3, |N[v_k] - S| \leq 1$ if v_k is not adjacent to v_j in $\langle S \rangle$, and $|N[v_k] \cap S| =$

$2, |N[v_k] - S| \leq 2$ if v_k is adjacent to v_j in $\langle S \rangle$). Therefore, S is a 1-strong alliance in G and hence $a^1(G) = |S| = g$. □

Corollary 5. *For any cubic graph G with girth g, $a^1(G) = g$.*

One of the very famous cubic graphs is the Petersen graph, which is a graph of order 10, size 15, and girth 5. Therefore, for the Petersen graph, the 1-strong alliance number is 5. The generalized Petersen graph is denoted by $GP(n,k)$ where $n \geq 3$ is a graph with vertex set $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edge set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i = 1, 2, \ldots, n\}$, where subscripts are taken modulo n and $k < n/2$. The vertices $\{u_1, u_2, \ldots, u_n\}$ are called the inner polygon vertices and the vertices $\{v_1, v_2, \ldots, v_n\}$ are called the outer polygon vertices. In particular, if $k = 1$, then the generalized Petersen graph $GP(n,1)$ which is nothing but the cartesian product $C_n \Box P_2$. The graph $GP(n,1)$ is a cubic graph of girth 4. Therefore $a^1(GP(n,1)) = 4$.

Theorem 6. *For integers $n \geq 5$ and $2 \leq k < \frac{n}{2}$,*

$$a^1(GP(n,k)) = \begin{cases} \min\left\{k+3, \frac{n}{k}\right\}, & \textit{if } k \textit{ divides } n \\ \min\left\{k+3, \left\lfloor \frac{n}{k} \right\rfloor + 3\right\}, & \textit{otherwise} \end{cases}$$

Proof. The graph $GP(n,k)$ is cubic and hence by Theorem 5, $a^1(GP(n,k)) = g$, the girth of the graph. Suppose k divides n, then there will be only three types of chordless cycles in $GP(n,k)$ of lengths n, $k+3$, and $\frac{n}{k}$. Since $2 \leq k < \frac{n}{2}$, we get $n \geq k+3$ and hence girth of $GP(n,k)$ is $\min\left\{k+3, \frac{n}{k}\right\}$. Suppose if k does not divide n, then there will be only three types of chordless cycles in $GP(n,k)$ of lengths $n, k+3$, and $\left\lfloor \frac{n}{k} \right\rfloor + 3$. Since $n \geq k+3$, the girth of $GP(n,k)$ must be $\min\left\{k+3, \left\lfloor \frac{n}{k} \right\rfloor + 3\right\}$. □

Remark 2. Let $G = (V, E)$ be any graph with $\delta(G) \geq 2$. If G has a cycle C_k $(k \geq 3)$ such that degree of each vertex in C_k is at most 3, then similar to the proof of Theorem 5, we see that $S = V(C_k)$ is a 1-strong defensive alliance and hence $a^1(G) \leq k$.

Theorem 7. *For any graph G of order $n \geq 4$ and $\delta(G) \geq 3$,*

$$a^1(G) \leq n - \left\lfloor \frac{\delta(G) - 1}{2} \right\rfloor$$

Proof. Let G be a graph of order $n \geq 4$ and v be a vertex of smallest degree in G. Let $v_1, v_2, \ldots, v_{\delta(G)}$ be the vertices which are adjacent to v in G. Consider the set $S = V - \{v, v_1, v_2, \ldots, v_{\lfloor \frac{\delta(G)-1}{2} \rfloor - 1}\}$. Let x be any arbitrary vertex in S and $\bar{S} = V - S$. Then $|N[x] \cap \bar{S}| \leq \lfloor \frac{\delta(G)-1}{2} \rfloor < \lfloor \frac{\delta(G)}{2} \rfloor$ and $|N[x] \cap S| = |N[x]| - |N[x] \cap \bar{S}| \geq (1+\delta(G)) - \lfloor \frac{\delta(G)-1}{2} \rfloor \geq 1 + \lceil \frac{\delta(G)}{2} \rceil$. So S is a defensive alliance in G. Also, $S' = S - \{x\}$ is a defensive alliance in G. In fact, for any $x' \in S'$, $|N[x'] \cap \overline{S'}| \leq \lfloor \frac{\delta(G)-1}{2} \rfloor + 1$ and $|N[x'] \cap S'| = |N[x']| - |N[x'] \cap \bar{S'}| \geq (1 + \delta(G)) - (\lfloor \frac{\delta(G)-1}{2} \rfloor + 1) \geq \delta(G) - \lfloor \frac{\delta(G)-1}{2} \rfloor \geq$

$\lfloor\frac{\delta(G)-1}{2}\rfloor+1$. Therefore, S is a 1-strong defensive alliance in G. Thus $a^1(G) \leq |S| = n - \lfloor\frac{\delta(G)-1}{2}\rfloor$. □

The upper bound in Theorem 7 is tight for the graph $G = P_3 + \overline{K}_2$ (by Theorem 15). Now we characterize the graphs of order n having 1-strong alliance number n. A *nonseparable graph* is connected, nontrivial, and has no cutvertex. A *block* of a graph is its maximal nonseparable subgraph. If a graph G is nonseparable, then G itself is a block. If we take the blocks of G as the family F of sets, then the intersection graph of F is the *block graph* of G, denoted by $B(G)$. The blocks of G corresponds to the vertices of $B(G)$ and two of these vertices in $B(G)$ are adjacent whenever the corresponding blocks in G contain a common cutvertex.
For an integer $n \geq 3$, let Γ_n be the class of graphs such that $G \in \Gamma_n$ if and only if G is a graph of order n with every block of G is a cycle and block graph of G is a tree. In 1963, Frank Harary [6] had obtained a characterization of block graphs in the form of the following theorem:

Theorem 8 (F. Harary [6]). *A graph H is a block graph of some graph if and only if every block of H is complete.*

Remark 3. The block graph of a graph G is a tree if and only if each cutvertex of G lies on exactly two blocks of G.

Theorem 9. *Let G be a graph of order $n \geq 3$. Then $a^1(G) = n$ if and only if $G \in \Gamma_n$.*

Proof. Let $G \in \Gamma_n$ and $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ be the blocks of G. We prove the result by induction on k, the number of blocks. For $k = 1$, $G \cong C_n$ and by Theorem 4, $a^1(G) = n$. For $k = 2$, let $V(C_{n_1}) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(C_{n_2}) = \{u_1, u_2, \ldots, u_{n_2}\}$ such that $v_1 = u_1$ is the common cutvertex. Then $\deg(v_1) = \deg(u_1) = 4$ and $\deg(v_i) = \deg(u_j) = 2$ for all i, j with $2 \leq i \leq n_1$ and $2 \leq j \leq n_2$. Let S be a 1-strong defensive alliance in G. Then without loss of generality, we may assume $S \cap V(C_{n_1}) \neq \phi$. Now by Theorem 4, $v_i \in S$ for each i, $1 \leq i \leq n_1$. In particular, $v_1 = u_1 \in S$. Since $\deg(u_1) = 4$ and S is a 1-strong defensive alliance in G, at least one of the vertices in $V(C_{n_2})$ which is adjacent to u_1 must be in S. Again by Theorem 4, $u_j \in S$ for each j, $1 \leq j \leq n_2$. Therefore, S contains every vertex of G and hence $a^1(G) = n$ in this case.
Assume that the result holds for any $k \geq 2$. Consider a graph $G \in \Gamma_n$ having $k+1$ blocks $C_{n_1}, C_{n_2}, \ldots, C_{n_{k+1}}$. Let G' be the graph obtained from G by deleting vertices and edges which are only in $C_{n_{k+1}}$. Then by inductive assumption, $a^1(G') = n - n_{k+1} + 1$ and no proper subset of $V(G')$ is a 1-strong defensive alliance in G', as well as in G. Also by Theorem 4, no proper subset of $V(C_{n_{k+1}})$ is a 1-strong defensive alliance in G. Let S be a 1-strong defensive alliance in G. Then $S \cap V(G') \neq \phi$ or $S \cap V(C_{n_{k+1}}) \neq \phi$. If $S \cap V(C_{n_{k+1}}) \neq \phi$, then by Theorem 4, $V(C_{n_k}) \subseteq S$, which implies that a common cutvertex, say w, in S. If $S \cap V(G') \neq \phi$, then by inductive assumption, $a^1(G') = n - n_{k+1} + 1$ and hence $V(G') \subseteq S$. Again which implies that

$w \in S$. Thus in either case the cut-vertex common to G' and C_{k+1} must be in S, which implies S contains every vertex of G. Hence $a^1(G) = n$. Thus by principle of induction, the result holds for any graph $G \in \Gamma_n$.
Conversely, let $G = (V, E)$ be a graph of order $n \geq 3$ with $a^1(G) = n$. Then by Proposition 5, G has no pendant vertex. Hence degree of every vertex in G is at least 2. If $G \cong C_n$, then there is nothing to prove. Suppose $G \ncong C_n$, then G has a vertex of degree 3.
Claim 1: Every edge of G lies in some cycle of G.
If possible, let e be an edge in G which is not in any cycle of G. Then e is a bridge and $G - e$ has exactly two components. Let G_1 and G_2 be the components of $G - e$. Then both G_1 and G_2 are connected nontrivial graphs (because G has no pendant vertex). The set $S = V(G_1)$ is 1-strong defensive alliance in G_1, as well as in G. But $|V(G_1)| \leq n - 2$ implies $a^1(G) \leq n - 2$, which is a contradiction to the fact that $a^1(G) = n$. Hence the claim 1.
Claim 2 Every edge of G lies on exactly one cycle.
If possible, let us assume that there is an edge common to two cycles in G. Since G has no pendant vertex, if every vertex of G is of degree at least 3, then by Theorem 7, $a^1(G) \leq n - 1$, again a contradiction to the fact that $a^1(G) = n$. Thus there must be a vertex of degree 2 in G. Let v be the vertex of degree 2 in G and w_1, w_2 be the vertices which are adjacent to v in G. Then by the above Claim 1, it follows that w_1, v, and w_2 are on some cycle of G. Thus there is a path in G between w_1 and w_2 not containing v. Let $P : w_1 = v_1 - v_2 - \cdots - v_l = w_2$ be a $w_1 - w_2$ path in G not containing v. Let v_j be the first vertex of degree more than 2 in P (while tracing the path from w_1 to w_2) for some j, $1 \leq j \leq l$ (we note that such a vertex v_j exists in P because G is connected and $G \ncong C_n$). Let v_k be the first vertex of degree more than 2 in P while tracing P from w_2 to w_1 (v_k may be equal to v_j). Let $S = V - \{v, v_1, \ldots v_{j-1}, v_{k+1}, v_{k+2}, \ldots v_l\}$. Then we observe that for any vertex $x \in S - \{v_j, v_k\}$, $N[x] - S = \phi$ and $|N[v_j] - S| = |N[v_k] - S| = 1$. Since both v_j and v_k are of degree at least 3, it follows that the set S is a 1-strong defensive alliance in G. But $|S| \leq n - 1$ implies $a^1(G) \leq n - 1$, again a contradiction to the fact that $a^1(G) = n$. Therefore, every edge of G lies in exactly one cycle of G. Hence the Claim 2.
Claim 3: Every cut vertex of G lies on exactly two blocks of G.
If not, let us assume that v is a cutvertex of G common to the block $B_1, B_2, \ldots, B_k$, where $k \geq 3$. Then by the Claim 1 and Claim 2, each block is a cycle, hence $|V(B_i)| \geq 3$ for each i, $1 \leq i \leq k$. So, the set $S = V - (V(B_1) - \{v\})$ is a 1-strong defensive alliance in G with $|S| \leq n - 2$, a contraction to the fact that $a^1(G) = n$. Hence the Claim 3.
Now by the above three claims and by Remark 3, it follows $G \in \Gamma_n$. This completes the proof of the theorem. □

4. 1-Strong Alliance Number of Standard Graphs

Theorem 10. *For an integer $n \geq 3$, $a^1(K_n) = \lceil \frac{n}{2} \rceil + 1$ and $a^1(C_n) = n$.*

Proof. In K_n, any subset S containing $\lceil \frac{n}{2} \rceil + 1$ vertices of K_n is a 1-defensive alliance. Hence $a^1(K_n) \leq |S| = \lceil \frac{n}{2} \rceil + 1$. Therefore, by Theorem 1, $a^1(K_n) = \lceil \frac{n}{2} \rceil + 1$. Further by Theorem 9, $a^1(C_n) = n$. □

Theorem 11. *For integers $m, n \geq 2$, $a^1(K_{m,n}) = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + 2$.*

Proof. Let V_1 and V_2 be the vertex partition of $K_{m,n}$ with $|V_1| = m$, $|V_2| = n$ and every vertex in V_1 is adjacent to every vertex in V_2. Let S be a 1-strong defensive alliance in $K_{m,n}$ with minimum cardinality. For any vertex $v \in S$, we have either $v \in V_1$ or $v \in V_2$. Suppose $v \in V_1$ (in case of $v \in V_2$ we have a similar argument), then among the n vertices of V_2, adjacent to v, at least $\lfloor \frac{n}{2} \rfloor + 1$ vertices must be in S. Similarly, at least $\lfloor \frac{m}{2} \rfloor + 1$ vertices of V_1 must belong to S. Therefore, $a^1(K_{m,n}) \geq \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2$. The set S containing any $\lfloor \frac{m}{2} \rfloor + 1$ vertices of V_1 and any $\lfloor \frac{n}{2} \rfloor + 1$ vertices of V_2 is a 1-strong defensive alliance and hence $a^1(K_{m,n}) = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + 2$. □

Theorem 12. *For an integer $n \geq 3$, $a^1(W_{1,n}) = \lceil \frac{n+1}{2} \rceil + 1$.*

Proof. Let v be the central vertex and $v_1, v_2, \ldots, v_n$ be the rim vertices of the wheel $W_{1,n}$. Let S be a 1-strong defensive alliance in $W_{1,n}$. If $v \in S$, then by Remark 1, we have $a^1(W_{1,n}) \geq \left\lceil \frac{\deg(v)+1}{2} \right\rceil + 1 = \lceil \frac{n+1}{2} \rceil + 1$. Else if $v \notin S$, then clearly S contains a rim vertex, say u. Let w_1 and w_2 be the rim vertices which are adjacent to u. Since S is a defensive alliance and $v \notin S$, w_1 or w_2 must be in S. We claim that both w_1 and w_2 must be in S. If not without loss of generality, we may assume that $w_1 \in S$ and $w_2 \notin S$. Then for the set $S' = S - \{w_1\}$, clearly $|N[u] \cap S'| = 1$ and $|N[u] - S'| = 3$. Hence S' is not a defensive alliance. Which implies that S is not a 1-strong defensive alliance, a contradiction. Thus the claim follows. Continuing the same argument for neighboring rim vertices, we see that S contains all rim vertices. Thus $|S| = n$ in this case. Therefore, $|a^1(W_{1,n})| \geq \min \left\{ \lceil \frac{n+1}{2} \rceil + 1, n \right\} = \lceil \frac{n+1}{2} \rceil + 1$.
On the other hand, it is easy to verify that the set $S = \{v, v_1, v_2, \ldots, v_{\lceil \frac{n+1}{2} \rceil}\}$ is a 1-strong defensive alliance in $W_{1,n}$. Therefore, $a^1(W_{1,n}) = \lceil \frac{n+1}{2} \rceil + 1$. □

Theorem 13. *For an integer $n \geq 4$, $a^1(K_n - e) = \lceil \frac{n}{2} \rceil + 1$.*

Proof. For $n = 4$, the result follows by Proposition 4. For $n \geq 5$, let $V(K_n - e) = \{v_1, v_2, \ldots, v_n\}$ and S be a 1-strong defensive alliance in $K_n - e$. Without loss of generality, we assume that v_1 and v_2 are nonadjacent in $K_n - e$. Then $\deg(v_1) = \deg(v_2) = n - 2$ and $\deg(v_i) = n - 1$ for $3 \leq i \leq n$. Since $K_n - e$ has no pendant vertex, by Corollary 1, $|S| \geq 3$ and hence there exists a vertex $v_k \in S$ for some

k, $3 \le k \le n$. But then, as $\deg(v_k) = n-1$, by Remark 1, it follows that $|S| \ge \left\lceil \frac{(n-1)+1}{2} \right\rceil + 1 = \lceil \frac{n}{2} \rceil + 1$. On the other hand, the set $S = \{v_2, v_3, v_4, v_5\}$ for $n = 5$ and the set $S = \{v_3, v_4, \ldots, v_{\lceil \frac{n}{2} \rceil + 3}\}$ for $n \ge 6$ is a 1-strong defensive alliance in $K_n - e$. Hence $a^1(K_n - e) = \lceil \frac{n}{2} \rceil + 1$. □

Let G_1 and G_2 be any two graphs having disjoint vertex set V_1 and V_2 and edge sets E_1 and E_2, respectively. Then the *join* of G_1 and G_2 is denoted by $G_1 + G_2$ and is the graph whose vertex set is $V_1 \cup V_2$ and consists of all the edges of G_1, G_2 and all the edges joining every vertex of V_1 with every vertex of V_2. In particular, $K_{m,n} = \overline{K}_m + \overline{K}_n$ and $W_{1,n} = C_n + K_1$. The graph $P_8 + \overline{K}_3$ is shown in Figure 2.

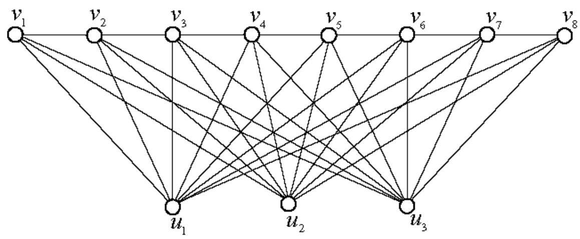

Figure 2. **The graph $P_8 + \overline{K}_3$.**

Theorem 14. *For any positive integer n, $a^1(P_n + K_1) = \lceil \frac{n+1}{2} \rceil + 1$.*

Proof. Let $G = P_n + K_1$. Let v be the vertex of K_1 and $v_1, v_2, \ldots, v_n$ be the vertices of P_n with v_i adjacent to v_{i+1}, for $1 \le i \le n-1$. Let S be a 1-strong defensive alliance in $P_n + K_1$. For $n = 1, 2$, the result follows by Corollary 2 and Theorem 10. When $n = 3$, the result follows by Theorem 13.

Let $n \ge 4$ and S be a 1-strong defensive alliance of G.

Claim: $v \in S$.

Let us suppose to contrary that $v \notin S$. Since $\delta(G) = 2$, by Lemma 1, we have $|S| \ge 3$ and hence $v_i \in S$ for some $i, 2 \le i \le n-1$. But then, $\{v_{i-1}, v_{i+1}\} \subseteq S$, otherwise $|N[v_i] \cap S'| = 1 < 3 = |N[v_i] - S'|$ where $S' = S - \{v_{i-1}, v_{i+1}\}$, a contradiction to the fact that S is a 1-strong defensive alliance. Thus, $S = \{v_1, v_2, \ldots, v_n\}$. Now for the set $S'' = S - \{v_2\}$, we get $|N[v_1] \cap S''| = 1 < 2 = |N[v_1] - S''|$, again a contradiction. Hence the claim follows. Thus, by Remark 1, $a^1(G) \ge \left\lceil \frac{\deg(v)+1}{2} \right\rceil + 1 = \lceil \frac{n+1}{2} \rceil + 1$. On the other hand, the set $S = \{v, v_1, v_2, \ldots, v_{\lceil \frac{n+1}{2} \rceil}\}$ is a 1-strong defensive alliance in G. Hence $a^1(P_n + K_1) = \lceil \frac{n+1}{2} \rceil + 1$. □

Theorem 15. *For any positive integers n and m,*

$$a^1(P_n+\overline{K}_m)=\begin{cases}\lceil\frac{n+1}{2}\rceil+1 & \textit{if } m=1\\ 2 & \textit{if } m\geq 2 \textit{ and } n=1\\ \lceil\frac{m}{2}\rceil+2 & \textit{if } m\geq 2 \textit{ and } n=2\\ \lceil\frac{n+1}{2}\rceil+\lceil\frac{m+1}{2}\rceil & \textit{if } m\geq 2, n\geq 3\end{cases}$$

Proof. Let $G=P_n+\overline{K}_m$. The case $m=1$ follows by Theorem 14 and the case $m\geq 2, n=1$ follows by Corollary 2. If $m=2$ and $n=2$, then $P_2+\overline{K}_2\cong K_4-e$ and the result follows by Theorem 13. We now suppose that $m\geq 2$ and $n\geq 2$. Let $u_1,u_2,\ldots,u_m$ be the vertices of $\overline{K}_m$ and $v_1,v_2,\ldots,v_n$ be the vertices of P_n with v_i adjacent to v_{i+1}, for $1\leq i\leq n-1$. Let S be a 1-strong defensive alliance in G. Then, by Theorem 4, $v_i\in S$ for at least one i, $1\leq i\leq n$ (otherwise $\langle S\rangle$ is disconnected). We now show that $u_k\in S$ for some k, $1\leq k\leq m$. In fact, if $S\cap\{u_1,u_2,\ldots,u_m\}=\emptyset$, then certainly $m\leq 3$ (since $\deg_G(v_i)\geq m+2$) and $v_j\in S$, where $j=i+1$ if $i<n$ or $j=i-1$ if $i=n$. Now, for the set $S'=S-\{v_j\}$, we have $|N[v_i]-S'|\geq 3$ and $|N[v_i]\cap S'|\leq 2$, a contradiction to the fact that S is a 1-strong defensive alliance in G. For this vertex $u_k\in S$, $\deg_G(u_k)=n$, $N(u_k)=V(P_n)$, and hence by Remark 1, the set S should contain at least $\lceil\frac{n+1}{2}\rceil$ vertices of P_n.

Case 1: $n=2$.

In this case, $\{v_1,v_2\}\subseteq S$, $\deg_G(v_1)=\deg_G(v_2)=m+1$ and hence by Remark 1, the set S should contain at least $\lceil\frac{m}{2}\rceil+1$ vertices of G of which at least $\lceil\frac{m}{2}\rceil$ vertices in $\bar{K}_m$. Therefore, $|S|=|V(P_n)\cap S|+|V(\bar{K}_m)\cap S|\geq 2+\lceil\frac{m}{2}\rceil$.

On the other hand, let $S=\{v_1,v_2,u_1,u_2,\ldots,u_{\lceil\frac{m}{2}\rceil}\}$, then $|S|=2+\lceil\frac{m}{2}\rceil$ and for any vertex $x\in S$, the subset $S-\{x\}$ is a defensive alliance. Hence S is a 1-strong defensive alliance in G. Therefore, $a^1(P_n+\overline{K}_m)=2+\lceil\frac{m}{2}\rceil$.

Case 2: $n\geq 3$.

In this case, $\deg_G(v_i)\geq m+1$ for all v_i, $1\leq i\leq n$. If $\deg_G(v_i)=m+1$ for all $v_i\in S$, then clearly $S=\{v_1,v_n\}$ and $n=3$ (since S has at least $\lceil\frac{n+1}{2}\rceil$ vertices of P_n). The vertex v_2 is an attacker for v_1, therefore, similar than above, by Remark 1, S should contain at least $\lceil\frac{m}{2}\rceil+1$ vertices of $\bar{K}_m$ adjacent to v_1 (since $N(v_1)\cap S\cap V(P_n)=\emptyset$). Else if $\deg_G(v_i)=m+2$ for at least one vertex $v_i\in S$, then it is easy to see that S has a vertex v_j such that either $v_{j-1}\notin S$ or $v_{j+1}\notin S$ (v_j may be one of the end vertices). So by Remark 1, $|S|\geq\lceil\frac{\deg(v_j)+1}{2}\rceil=\lceil\frac{m+1}{2}\rceil+1$. This implies that S should contain at least $\lceil\frac{m+1}{2}\rceil$ neighbouring vertices of v_j in $\bar{K}_m$ (since $N(v_j)\cap V(P_n)=1$). Thus, in any case, $|S|=|V(P_n)\cap S|+|V(\bar{K}_m)\cap S|\not< \lceil\frac{n+1}{2}\rceil+\lceil\frac{m+1}{2}\rceil$

On the other hand, let $S=\{u_1,u_2,\ldots,u_{\lceil\frac{m+1}{2}\rceil},v_1,v_2,\ldots,v_{\lceil\frac{n+1}{2}\rceil}\}$, then $|S|=\lceil\frac{n+1}{2}\rceil+\lceil\frac{m+1}{2}\rceil$ and for any vertex $x\in S$, the subset $S-\{x\}$ is a defensive alliance. Hence S is a 1-strong defensive alliance in G. Therefore, $a^1(P_n+\overline{K}_m)=\lceil\frac{n+1}{2}\rceil+\lceil\frac{m+1}{2}\rceil$. □

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