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New skew equienergetic oriented graphs

Xiangxiang Liu, Ligong Wang*[*] and Cunxiang Duan

Department of Applied Mathematics, School of Science, Northwestern Polytechnical University
Xi'an, Shaanxi 710072, People's Republic of China
`xxliumath@163.com`, `lgwangmath@163.com`, `cxduanmath@163.com`

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Abstract: Let $S(G^\sigma)$ be the skew-adjacency matrix of the oriented graph G^σ, which is obtained from a simple undirected graph G by assigning an orientation σ to each of its edges. The skew energy of an oriented graph G^σ is defined as the sum of absolute values of all eigenvalues of $S(G^\sigma)$. Two oriented graphs are said to be skew equienergetic if their skew energies are equal. In this paper, we determine the skew spectra of some new oriented graphs. As applications, we give some new methods to construct new non-cospectral skew equienergetic oriented graphs.

Keywords: Oriented graph, Skew energy, Skew equienergetic

AMS Subject classification: 05C50

1. Introduction

Let G be a simple undirected graph with an orientation σ, which assigns to each edge a direction such that G^σ becomes an oriented graph. Then G is usually called the underlying graph of G^σ. The skew-adjacency matrix of G^σ with vertex set $V(G) = \{1, 2, ..., n\}$ is the $n \times n$ matrix $S(G^\sigma) = [s_{ij}]$, where $s_{ij} = 1$ and $s_{ji} = -1$ if (i, j) is an arc of G^σ, and $s_{ij} = s_{ji} = 0$ otherwise. Let u be a vertex of G^σ. The indegree of u, denoted by $id_{G^\sigma}(u)$, is the number of arcs coming to u. The outdegree of u, denoted by $od_{G^\sigma}(u)$, is the number of arcs going out from u.

The skew characteristic polynomial of G^σ is defined as $\phi(G^\sigma; x) = det(xI_n - S(G^\sigma)) = \sum_{i=0}^{n} a_i(G^\sigma)x^{n-i}$, where I_n is the unit matrix of order n. Let λ_i be an eigenvalue of

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[*] *Corresponding author*

G^σ with multiplicity n_i $(i = 1, \dots, k)$, where $\sum_{i=1}^{k} n_i = n$. The skew spectrum of G^σ is

$$Spec(G^\sigma) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ n_1 & n_2 & \cdots & n_k \end{pmatrix}.$$

The skew energy of an oriented graph G^σ, denoted by $E_s(G^\sigma)$, is defined as the sum of absolute values of all eigenvalues of $S(G^\sigma)$ (see [2]), that is

$$E_s(G^\sigma) = \sum_{i=1}^{n} |\lambda_i|.$$

The concept of energy of a graph was introduced by Gutman [10] with an application to chemistry, which is related to the total π-electron energy of the molecule represented by that graph [13]. The energy of a graph G has been extensively studied by many mathematicians and their works can be found in [11, 12, 20] and therein references. Recently, other graph energies were considered, such as the Laplacian energy [14], signless Laplacian energy [1] and distance energy [16]. For other results, one can refer to [11].

Adiga et al. introduced the skew energy of an oriented graph [2]. They showed that the skew energy of an oriented tree is independent of its orientation and obtained bounds for skew energy. Works on skew energy of an oriented graph can be found in [4, 6–9, 15, 17, 19, 22–24]. For a survey on skew energy of oriented graphs, one can refer to [11, 18].

Two oriented graphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ are said to be skew equienergetic if $E_s(G_1^{\sigma_1}) = E_s(G_2^{\sigma_2})$. If two oriented graphs are cospectral, then in a trivial manner, they are skew equienergetic. Therefore, in what follows, we are interested in finding non-cospectral skew equienergetic oriented graphs. Recently in [18] Li and Lian proposed the following problem.

Problem 1. How to construct families of oriented graphs such that they have equal skew energy, but they do not have the same spectra?

The above problem was addressed by Ramane et al. [21] and Adiga et al. [3]. They gave some methods to construct skew-equienergetic digraphs.

This paper is organized as follows: In Section 2, we give some definitions and lemmas, which will be used in the following discussion. In Section 3, we firstly obtain the skew spectra of some new oriented graphs. Further, we construct some new non-cospectral skew equienergetic oriented graphs.

2. Preliminary

In this section, we give some definitions and lemmas which are useful to prove our main results.

Let G_1 and G_2 be two simple graphs on n_1 and n_2 vertices, respectively.

Definition 1. ([21]) The join of oriented graphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$, denoted by $G_1^{\sigma_1} \vee G_2^{\sigma_2}$, is an oriented graph obtained from $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ by adding an arc from each vertex of $G_1^{\sigma_1}$ to all vertices of $G_2^{\sigma_2}$. An example is depicted in Fig. 1.

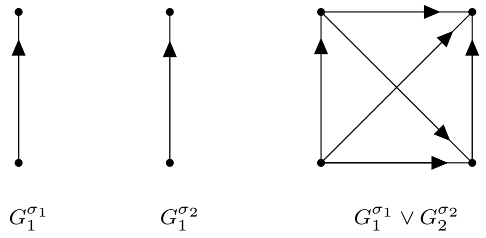


Figure 1. **Oriented graph** $G_1^{\sigma_1} \vee G_2^{\sigma_2}$

Definition 2. The corona of oriented graphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$, denoted by $G_1^{\sigma_1} \circ G_2^{\sigma_2}$, is an oriented graph obtained by taking one copy of $G_1^{\sigma_1}$ and n_1 copies of $G_2^{\sigma_2}$, and then adding an arc from i-th vertex of $G_1^{\sigma_1}$ to every vertex in the i-th copy of $G_2^{\sigma_2}$. An example is depicted in Fig. 2.

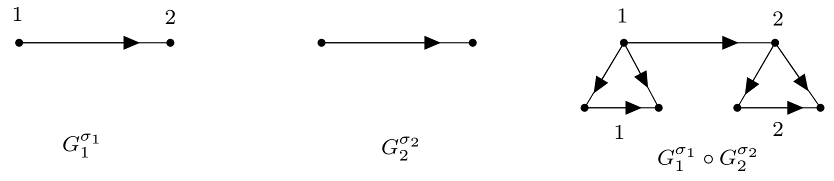
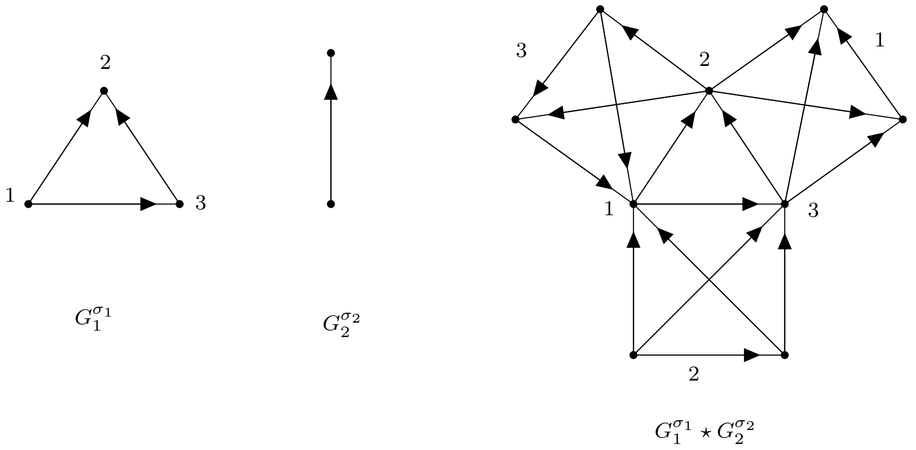


Figure 2. **Oriented graph** $G_1^{\sigma_1} \circ G_2^{\sigma_2}$

Definition 3. The neighborhood corona of oriented graphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$, denoted by $G_1^{\sigma_1} \star G_2^{\sigma_2}$, is an oriented graph obtained by taking one copy of $G_1^{\sigma_1}$ and n_1 copies of $G_2^{\sigma_2}$, and then adding an arc between each neighbor of i-th vertex of $G_1^{\sigma_1}$ and every vertex in the i-th copy of $G_2^{\sigma_2}$. Let j be the neighbor of i. If (i, j) is an arc of $G_1^{\sigma_1}$, then the new arc is from vertex j of $G_1^{\sigma_1}$ to every vertex in the i-th copy of $G_2^{\sigma_2}$. If (j, i) is an arc of $G_1^{\sigma_1}$, then the new arc is from every vertex in the i-th copy of $G_2^{\sigma_2}$ to vertex j of $G_1^{\sigma_1}$. An example is depicted in Fig. 3.

Definition 4. ([5]) Let $A = (a_{ij})$ be an $n \times m$ matrix, $B = (b_{ij})$ be a $p \times q$ matrix.



$G_1^{\sigma_1}$ $G_2^{\sigma_2}$ $G_1^{\sigma_1} \star G_2^{\sigma_2}$

Figure 3. **Oriented graph** $G_1^{\sigma_1} \star G_2^{\sigma_2}$

Then the Kronecker product $A \otimes B$ of A and B is the np by mq matrix obtained by replacing each entry a_{ij} of A by $a_{ij}B$.

Lemma 1. *([5]) If M, N, P, Q are matrices with M being a non-singular matrix, then*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M||Q - PM^{-1}N|.$$

Lemma 2. *([21]) Let D_a and D_b be two oriented graphs with $id_{D_a}(u) = od_{D_a}(u)$ and $id_{D_b}(v) = od_{D_b}(v)$ for all $u \in D_a$ and $v \in D_b$. Then D_a and D_b have different skew spectra but $E_s(D_a) = E_s(D_b)$, where D_a and D_b are depicted in Fig. 4.*

3. Main Results

In this section, we firstly compute the skew spectrum of $(G_1^{\sigma_1} \vee G_2^{\sigma_2}) \uplus (G_1^{\sigma_1} \circ G_3^{\sigma_3})$, which is obtained by join of oriented graphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$, at the same time corona of oriented graphs $G_1^{\sigma_1}$ and $G_3^{\sigma_3}$.

Theorem 1. *Let $G_1^{\sigma_1}$, $G_2^{\sigma_2}$ and $G_3^{\sigma_3}$ be oriented graphs with n, m and l vertices, respectively. And $id_{G_s^{\sigma_s}}(v_s) = od_{G_s^{\sigma_s}}(v_s)$ for all $v_s \in G_s^{\sigma_s}$, $s = 1, 2, 3$. Suppose $Spec(G_1^{\sigma_1}) = \{\lambda_1 =$*

Figure 4. **Oriented graphs D_a and D_b**

$0, \lambda_2, \ldots, \lambda_n\}$, *Spec*$(G_2^{\sigma_2}) = \{\mu_1 = 0, \mu_2, \ldots, \mu_m\}$, *Spec*$(G_3^{\sigma_3}) = \{\gamma_1 = 0, \gamma_2, \ldots, \gamma_l\}$. *Then the skew spectrum of* $G^\sigma = (G_1^{\sigma_1} \vee G_2^{\sigma_2}) \uplus (G_1^{\sigma_1} \circ G_3^{\sigma_3})$ *is*

$$Spec(G^\sigma) = \begin{pmatrix} \gamma_t & \mu_j & (\lambda_k \pm \sqrt{\lambda_k^2 - 4l})/2 & i\sqrt{mn+l} & -i\sqrt{mn+l} & 0 \\ n & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

where $t = 2, \ldots, l$, $j = 2, \ldots, m$, $k = 2, \ldots, n$, i *is imaginary unit.*

Proof. With suitable labelling of the vertices of G^σ, the skew-adjacency matrix of G^σ can be formulated as follows:

$$S(G^\sigma) = \begin{pmatrix} I_n \otimes S(G_3^{\sigma_3}) & 0 & -I_n \otimes e \\ 0 & S(G_2^{\sigma_2}) & -J \\ I_n \otimes e^T & J^T & S(G_1^{\sigma_1}) \end{pmatrix},$$

where e is the l dimensional column vector with all its entries are 1, I_n is the identity matrix of order n, and J is the $m \times n$ matrix with all its entries are 1.

Since $S(G_1^{\sigma_1})$, $S(G_2^{\sigma_2})$ and $S(G_3^{\sigma_3})$ are normal matrices, they are unitarily diagonalizable. And $id_{G_s^{\sigma_s}}(v_s) = od_{G_s^{\sigma_s}}(v_s)$, for all $v_s \in G_s^{\sigma_s}$, $s = 1, 2, 3$. So we have $S(G_1^{\sigma_1}) = U_1 D_1 U_1^H$, $S(G_2^{\sigma_2}) = U_2 D_2 U_2^H$ and $S(G_3^{\sigma_3}) = U_3 D_3 U_3^H$, where U_1, U_2 and U_3 are unitary matrix having its first column vector as $\frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$, $\frac{1}{\sqrt{m}}(1, 1, \ldots, 1)^T$ and $\frac{1}{\sqrt{l}}(1, 1, \ldots, 1)^T$, $D_1 = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $D_2 = diag(\mu_1, \mu_2, \ldots, \mu_m)$, $D_3 =$

$diag(\gamma_1, \gamma_2, \ldots, \gamma_l)$. Hence,

$$S(G^\sigma) = \begin{pmatrix} I_n \otimes U_3 D_3 U_3^H & 0 & -I_n \otimes e \\ 0 & S(G_2^{\sigma_2}) & -J \\ I_n \otimes e^T & J^T & S(G_1^{\sigma_1}) \end{pmatrix}$$

$$= \begin{pmatrix} I_n \otimes U_3 & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_n \end{pmatrix} \begin{pmatrix} I_n \otimes D_3 & 0 & -I_n \otimes U_3^H e \\ 0 & S(G_2^{\sigma_2}) & -J \\ I_n \otimes e^T U_3 & J^T & S(G_1^{\sigma_1}) \end{pmatrix}$$

$$\begin{pmatrix} I_n \otimes U_3^H & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_n \end{pmatrix}$$

$$= \begin{pmatrix} I_n \otimes U_3 & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_n \end{pmatrix} \begin{pmatrix} I_n \otimes D_3 & 0 & -I_n \otimes \sqrt{l} e_1 \\ 0 & S(G_2^{\sigma_2}) & -J \\ I_n \otimes \sqrt{l} e_1^T & J^T & S(G_1^{\sigma_1}) \end{pmatrix}$$

$$\begin{pmatrix} I_n \otimes U_3^H & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_n \end{pmatrix},$$

where $e_1 = (1, 0, \ldots, 0)^T$.
For convenience, let

$$B = \begin{pmatrix} I_n \otimes D_3 & 0 & -I_n \otimes \sqrt{l} e_1 \\ 0 & S(G_2^{\sigma_2}) & -J \\ I_n \otimes \sqrt{l} e_1^T & J^T & S(G_1^{\sigma_1}) \end{pmatrix}.$$

Then by the above equation, we have $det(xI - S(G^\sigma)) = det(xI - B)$. Expanding $|xI - B|$ by Laplace's method along $lt+2, lt+3, \ldots, lt+l$ $(t = 0, 1, \ldots, n-1)$ columns, we see that the only non zero $(l-1)n \times (l-1)n$ minor is

$$M = |I_n \otimes diag(x - \gamma_2, x - \gamma_3, \ldots, x - \gamma_l)|.$$

The complementary minor of M is

$$M_1 = \begin{vmatrix} (x - \gamma_1) I_n & 0 & \sqrt{l} I_n \\ 0 & xI_m - S(G_2^{\sigma_2}) & J \\ -\sqrt{l} I_n & -J^T & xI_n - S(G_1^{\sigma_1}) \end{vmatrix}.$$

To calculate M_1, let

$$C = \begin{pmatrix} (x - \gamma_1) I_n & 0 & \sqrt{l} I_n \\ 0 & xI_m - S(G_2^{\sigma_2}) & J \\ -\sqrt{l} I_n & -J^T & xI_n - S(G_1^{\sigma_1}) \end{pmatrix}.$$

Similarly, we have

$$\begin{aligned} C \;&=\; \begin{pmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & U_1 \end{pmatrix} \begin{pmatrix} (x-\gamma_1)I_n & 0 & \sqrt{l}I_n \\ 0 & xI_m - D_2 & U_2^H J U_1 \\ -\sqrt{l}I_n & -U_1^H J^T U_2 & xI_n - D_1 \end{pmatrix} \\ &\quad \begin{pmatrix} U_1^H & 0 & 0 \\ 0 & U_2^H & 0 \\ 0 & 0 & U_1^H \end{pmatrix} \\ &= \begin{pmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & U_1 \end{pmatrix} \begin{pmatrix} (x-\gamma_1)I_n & 0 & \sqrt{l}I_n \\ 0 & xI_m - D_2 & \sqrt{mn}J_1 \\ -\sqrt{l}I_n & -\sqrt{mn}J_1^T & xI_n - D_1 \end{pmatrix} \\ &\quad \begin{pmatrix} U_1^H & 0 & 0 \\ 0 & U_2^H & 0 \\ 0 & 0 & U_1^H \end{pmatrix}, \end{aligned}$$

where J_1 is the matrix obtained by replacing every entries of J except the first diagonal entry by 0. Let

$$D = \begin{pmatrix} (x-\gamma_1)I_n & 0 & \sqrt{l}I_n \\ 0 & xI_m - D_2 & \sqrt{mn}J_1 \\ -\sqrt{l}I_n & -\sqrt{mn}J_1^T & xI_n - D_1 \end{pmatrix}.$$

So M_1 and $det(D)$ have same value.
By lemma 1, we have

$$M_1 = x^n \times \begin{vmatrix} diag(x-\mu_1,\ldots,x-\mu_m) & \sqrt{mn}J_1 \\ -\sqrt{mn}J_1^T & diag(x-\lambda_1+l/x,\ldots,x-\lambda_n+l/x) \end{vmatrix},$$

Applying Laplace's method along $2,\ldots,m,m+2,\ldots,m+n$ columns in the above determinant, we see that the only non zero $(m+n-2)\times(m+n-2)$ minor is

$$M_2 = \left| diag(x-\mu_2,\ldots,x-\mu_m,x-\lambda_2+l/x,\ldots,x-\lambda_n+l/x) \right|,$$

Its complementary minor is

$$M_3 = \begin{vmatrix} x-\mu_1 & \sqrt{mn} \\ -\sqrt{mn} & x-\lambda_1+l/x \end{vmatrix},$$

So by the value of M, M_1, M_2, M_3, we have the skew spectrum of G^σ. $\square$

As an immediate consequence of the above theorem, we have the following result.

Theorem 2. *Let $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ be oriented graphs with m and l vertices, respectively. And $id_{G_k^{\sigma_k}}(v_k) = od_{G_k^{\sigma_k}}(v_k)$ for all $v_k \in G_k^{\sigma_k}$, $k = 1, 2$. Then the skew energy of $G^\sigma = (nK_1 \vee G_1^{\sigma_1}) \uplus (nK_1 \circ G_2^{\sigma_2})$ is $E_s(G^\sigma) = E_s(G_1^{\sigma_1}) + nE_s(G_2^{\sigma_2}) + 2\sqrt{mn+l} + 2(n-1)\sqrt{l}$.*

Corollary 1. *Let $G_k^{\sigma_k}$ be skew equienergetic non-cospectral oriented graphs, $id_{G_k^{\sigma_k}}(v_k) = od_{G_k^{\sigma_k}}(v_k)$ for all $v_k \in G_k^{\sigma_k}$, $k = 1, 2$. Let $H_j^{\gamma_j}$ be skew equienergetic non-cospectral oriented graphs, $id_{H_j^{\sigma_j}}(v_j) = od_{H_j^{\sigma_j}}(v_j)$ for all $v_j \in H_j^{\sigma_j}$, $j = 1, 2$. Then $E_s((nK_1 \vee G_1^{\sigma_1}) \uplus (nK_1 \circ H_1^{\gamma_1})) = E_s((nK_1 \vee G_2^{\sigma_2}) \uplus (nK_1 \circ H_2^{\gamma_2}))$.*

Theorem 3. *There exists a pair of skew equienergetic non-cospectral oriented graphs on n vertices for all $n \geq 6$.*

Proof. By Lemma 2, we have D_a and D_b with 6 vertices have different skew spectra but $E_s(D_a) = E_s(D_b)$.
Let G^σ be an oriented graph with $n(\geq 1)$ vertices. Then $E_s((K_1 \vee D_a) \uplus (K_1 \circ G^\sigma)) = E_s((K_1 \vee D_b) \uplus (K_1 \circ G^\sigma))$, but they have different skew spectra. ☐

Next we will compute the skew spectrum of $(G_1^{\sigma_1} \vee G_2^{\sigma_2}) \uplus (G_1^{\sigma_1} \star G_3^{\sigma_3})$, which is obtained by join of oriented graphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$, at the same time neighborhood corona of oriented graphs $G_1^{\sigma_1}$ and $G_3^{\sigma_3}$.

Theorem 4. *Let $G_1^{\sigma_1}$, $G_2^{\sigma_2}$ and $G_3^{\sigma_3}$ be oriented graphs with n, m and l vertices, respectively. And $id_{G_s^{\sigma_s}}(v_s) = od_{G_s^{\sigma_s}}(v_s)$ for all $v_s \in G_s^{\sigma_s}$, $s = 1, 2, 3$. Suppose $Spec(G_1^{\sigma_1}) = \{\lambda_1 = 0, \lambda_2, \ldots, \lambda_n\}$, $Spec(G_2^{\sigma_2}) = \{\mu_1 = 0, \mu_2, \ldots, \mu_m\}$, $Spec(G_3^{\sigma_3}) = \{\gamma_1 = 0, \gamma_2, \ldots, \gamma_l\}$. Then the skew spectrum of $G^\sigma = (G_1^{\sigma_1} \vee G_2^{\sigma_2}) \uplus (G_1^{\sigma_1} \star G_3^{\sigma_3})$ is*

$$Spec(G^\sigma) = \begin{pmatrix} \gamma_t & \mu_j & (\lambda_k \pm \sqrt{\lambda_k^2 + 4l\lambda_k^2})/2 & i\sqrt{mn} & -i\sqrt{mn} & 0 \\ n & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

where $t = 2, \ldots, l$, $j = 2, \ldots, m$, $k = 2, \ldots, n$, i is imaginary unit.

Proof. With suitable labelling of the vertices of G^σ, the skew-adjacency matrix of G^σ can be formulated as follows:

$$S(G^\sigma) = \begin{pmatrix} I_n \otimes S(G_3^{\sigma_3}) & 0 & -S(G_1^{\sigma_1}) \otimes e \\ 0 & S(G_2^{\sigma_2}) & -J \\ -S(G_1^{\sigma_1}) \otimes e^T & J^T & S(G_1^{\sigma_1}) \end{pmatrix},$$

where e is the l dimensional column vector with all its entries are 1, I_n is the identity matrix of order n, and J is the $m \times n$ matrix with all its entries are 1.
The next proof is similar to that of Theorem 1, we can obtain the skew spectrum of oriented graph G^σ. ☐

As an immediate consequence of the Theorem 4, we have the following result.

Theorem 5. *Let $G_1^{\sigma_1}$, $G_2^{\sigma_2}$ and $G_3^{\sigma_3}$ be oriented graphs with n, m and l vertices, respectively. And $id_{G_s^{\sigma_s}}(v_s) = od_{G_s^{\sigma_s}}(v_s)$ for all $v_s \in G_s^{\sigma_s}$, $s = 1, 2, 3$. Then the skew energy of oriented graph $G^{\sigma} = (G_1^{\sigma_1} \vee G_2^{\sigma_2}) \uplus (G_1^{\sigma_1} \star G_3^{\sigma_3})$ is $E_s(G^{\sigma}) = E_s(G_2^{\sigma_2}) + nE_s(G_3^{\sigma_3}) + \sqrt{4l+1}E_s(G_1^{\sigma_1}) + 2\sqrt{mn}$.*

Corollary 2. *Let $D_s^{\sigma_s}$ be skew equienergetic non-cospectral oriented graphs, $id_{D_s^{\sigma_s}}(v_s) = od_{D_s^{\sigma_s}}(v_s)$ for all $v_s \in D_s^{\sigma_s}$, $s = 1, 2$. Let $G_k^{\sigma_k}$ be skew equienergetic non-cospectral oriented graphs, $id_{G_k^{\sigma_k}}(v_k) = od_{G_k^{\sigma_k}}(v_k)$ for all $v_k \in G_k^{\sigma_k}$, $k = 1, 2$. Let $H_j^{\gamma_j}$ be skew equienergetic non-cospectral oriented graphs, $id_{H_j^{\sigma_j}}(v_j) = od_{H_j^{\sigma_j}}(v_j)$ for all $v_j \in H_j^{\sigma_j}$, $j = 1, 2$. Then $E_s((D_1^{\sigma_1} \vee G_1^{\alpha_1}) \uplus (D_1^{\sigma_1} \star H_1^{\gamma_1})) = E_s((D_2^{\sigma_2} \vee G_2^{\alpha_2}) \uplus (D_2^{\sigma_2} \star H_2^{\gamma_2}))$.*

Now we construct some new pairs of skew equienergetic non-cospectral oriented graphs.

Theorem 6. *There exists a pair of skew equienergetic non-cospectral oriented graphs on n vertices for all $n \geq 6$.*

Proof. By Lemma 2, we have D_a and D_b with 6 vertices have different skew spectra but $E_s(D_a) = E_s(D_b)$.
Let G^{σ} be an oriented graph with $n(\geq 1)$ vertices. Then $E_s((K_1 \vee D_a) \uplus (K_1 \star G^{\sigma})) = E_s((K_1 \vee D_b) \uplus (K_1 \star G^{\sigma}))$, but they have different skew spectra. □

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