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Eternal m-security subdivision numbers in trees

Maryam Atapour

Department of Mathematics, Faculty of basic sciences, University of Bonab
Bonab, I.R. Iran
m.atapour@bonabu.ac.ir

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Abstract: An eternal m-secure set of a graph $G = (V, E)$ is a set $S_0 \subseteq V$ that can defend against any sequence of single-vertex attacks by means of multiple-guard shifts along the edges of G. A suitable placement of the guards is called an eternal m-secure set. The eternal m-security number $\sigma_m(G)$ is the minimum cardinality among all eternal m-secure sets in G. An edge $uv \in E(G)$ is subdivided if we delete the edge uv from G and add a new vertex x and two edges ux and vx. The eternal m-security subdivision number $\mathrm{sd}_{\sigma_m}(G)$ of a graph G is the minimum cardinality of a set of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the eternal m-security number of G. In this paper, we study the eternal m-security subdivision number in trees. In particular, we show that the eternal m-security subdivision number of trees is at most 2 and we characterize all trees attaining this bound.

Keywords: eternal m-secure set, eternal m-security number, eternal m-security subdivision number

AMS Subject classification: 05C69

1. Introduction

Throughout this paper, G is a simple connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The numbers of vertices and edges are called the *order* and *size* of the G, respectively. For every vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* $\deg(v)$ of v is the number of edges incident with v or equivalently $\deg(v) = |N(v)|$. The *minimum* and *maximum* degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A *leaf* of G is a vertex of degree 1 and a *support* *vertex* of G is a vertex adjacent to a leaf. A support vertex is called *strong support vertex* if it is adjacent to at least two leaves. We denote the set of leaves of a graph G and the set of leaves adjacent to $v \in V(G)$ by $L(G)$ and L_v, respectively. For a vertex v in a rooted tree T , let $D(v)$ denote the set of descendants of v. An edge $uv \in E(G)$ is *subdivided* if the edge uv is deleted and a new vertex x and two new edges ux and vx are added in G.

The concept of *domination* in graphs was first defined by Ore in 1962 [7]. A set S of vertices in a graph G is called a *dominating set* if every vertex in V is either an element of S or is adjacent to an element of S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets of G. A $\gamma(G)$*-set* is a dominating set of G of size $\gamma(G)$.

The *domination subdivision number* $\mathrm{sd}_\gamma(G)$ of a graph G is the minimum cardinality of a set of edges of G that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the domination number of G. This concept was first introduced by Velammal in his Ph.D. thesis [8] and since then many results have been obtained on some domination parameters (see for instance [2, 5]).

An *eternal 1-secure set* of a graph G is a set $S_0 \subseteq V$ that can defend against any sequence of single-vertex attacks by means of single-guard shifts along the edges of G. That is, for any k and any sequence $v_1, v_2, \ldots, v_k$ of vertices, there exists a sequence of guards $u_1, u_2 \ldots, u_k$ with $u_i \in S_{i-1}$ and either $u_i = v_i$ or $u_iv_i \in E$, such that each set $S_i = (S_{i-1} \setminus \{u_i\}) \cup \{v_i\}$ is a dominating set. It follows that each S_i can be chosen to be an eternal 1-secure set. The *eternal 1-security number* of G, denoted by $\sigma_1(G)$, is the minimum cardinality among all eternal 1-secure set. The eternal 1-security number was introduced by Burger et al. [3] using the notation γ_∞. In order to reduce the number of guards needed in an eternal secure set, Goddard et al. [4] considered allowing more guards to move. Suppose that in responding to each attack, every guard may shift along an incident edge. The *eternal m-security number* $\sigma_m(G)$ is the minimum number of guards to handle an arbitrary sequence of single attacks using multiple-guard shifts. A suitable placement of the guards is called an *eternal m-secure set* (EmSS) of G. An EmSS of size $\sigma_m(G)$ is called a $\sigma_m(G)$- set. Obviously, any EmSS of G is a dominating set of G. So we have $\gamma(G) \leq \sigma_m(G)$. When an edge $uv \in E(G)$ is subdivided with a vertex x, then the eternal m-security number of G can not decrees. The eternal m-security subdivision number $\mathrm{sd}_{\sigma_m}(G)$ of a graph G is the minimum cardinality of a set of edges of G that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the eternal m-security number of G. Since in the study of eternal m-security subdivision number, the assumption $\sigma_m(G) < n$ is necessary, we always assume that when we discuss $\mathrm{sd}_{\sigma_m}(G)$, all graphs involved satisfy $\sigma_m(G) < n$, i.e., all graphs are nonempty. In this paper, we study of the eternal m-security subdivision number in trees. In particular, we prove that the eternal m-security subdivision number of a tree is at most 2 and we characterize all trees attaining this bound. For a more thorough treatment of domination parameters and for terminology not presented here see [6, 9]. The proof of the following results can be found in [4].

Theorem A. For any graph G, $\gamma(G) \le \sigma_m(G)$.

Theorem B. *For any graph G, $\sigma_m(G) \le \alpha(G)$.*

Theorem C. *1.* $\sigma_m(K_n) = 1$.

2. $\sigma_m(P_n) = \lceil \frac{n}{2} \rceil$.

3. $\sigma_m(C_n) = \lceil \frac{n}{3} \rceil$.

Theorem D. *For any graph G, $\sigma_m(G) \ge (\mathrm{diam}(G) + 1)/2$.*

Next results are immediate consequence of Propositions C and D.

Corollary 1. For any graph G, $\sigma_m(G) = 1$ if and only if $G \simeq K_n$.

Corollary 2. For $n \ge 2$, $\mathrm{sd}_{\sigma_m}(K_n) = 1$.

Corollary 3. For $n \ge 2$, $\mathrm{sd}_{\sigma_m}(P_n) = \begin{cases} 1, & \text{if } n \text{ is even} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$

2. Main Results

In this section, we show that for any tree T, $\mathrm{sd}_{\sigma_m}(T) \le 2$ and we characterize all trees attaining this bound. We start with two propositions.

Proposition 1. Let G be a connected graph. If G has a vertex u with $|L_u| \ge 3$, then $\mathrm{sd}_{\sigma_m}(G) = 1$.

Proof. Let $w_1, w_2, w_3 \in L_u$ and let G' be the graph obtained from G by subdividing the edge uw_1 by subdivision vertex x. Let S be a $\sigma_m(G')-$ set containing w_2 (we may assume that S is a response to an attack on w_2). To dominate w_3, we may assume that $u \in S$. On the other hand, to dominate w_1, we must have $|S \cap \{w_1, x\}| \ge 1$. It is easy to see that $S \setminus \{w_1, x\}$ is an EmSS of G. This implies that $\mathrm{sd}_{\sigma_m}(T) = 1$. □

Proposition 2. Let G be a connected graph. If G has a vertex u with $|L_u| = 2$, then $\mathrm{sd}_{\sigma_m}(G) \le 2$.

Proof. Let $w_1, w_2 \in L_u$ and let G' is the graph obtained from G by subdividing the edges uw_1 and uw_2 by subdivision vertices x and y respectively. Let S be a $\sigma_m(G')-$ set containing u (we may assume that S is a response to an attack on u). To dominate w_1 and w_2, we must have $|S \cap \{w_1, x\}| \ge 1$ and $|S \cap \{w_1, y\}| \ge 1$, respectively. It is easy to see that $(S \setminus \{w_1, x, y\}) \cup \{w_2\}$ is an EmSS of G. This implies that $\mathrm{sd}_{\sigma_m}(T) \le 2$. □

Theorem 1. For any tree T, $\mathrm{sd}_{\sigma_m}(T) \leq 2$.

Proof. The result is obvious for $n(T) \leq 3$. Let $n(T) \geq 4$. If T is a star, then the result follows from Proposition 1. Assume that T is not a star and $v_1v_2 \dots v_k$ be a diametrical path in T. Root T at v_k. If $\deg(v_2) \geq 3$, then the result follows from Proposition 2. Suppose that $\deg(v_2) = 2$ and T' is the tree obtained from T by subdividing the edges v_1v_2 and v_2v_3 by subdivision vertices x and y, respectively. Let S be a $\sigma_m(T')$-set containing v_2. To dominate v_1, we may assume that $x \in S$. Let $S' = S \setminus \{x\}$ if $y \notin S$, $S' = (S \setminus \{x, y\}) \cup \{v_3\}$ if $v_3 \notin S$ and $y \in S$ and $S' = (S \setminus \{x, y\}) \cup \{w\}$ if $v_3, y \in S$, where $w \in N_T(v_3) \setminus \{v_2\}$. Clearly, S' is an EmSS of T of size $|S| - 1$ and this completes the proof. $\square$

Now we give a constructive characterization of trees T for which $\mathrm{sd}_{\sigma_m}(T) = 2$. For this purpose, we describe a procedure to build a family $\mathfrak{T}$ of trees as follows. Let $\mathfrak{T}$ be the family of trees that: A path P_3 is a tree in $\mathfrak{T}$ and if T is a tree in $\mathfrak{T}$, then the tree T' obtained from T by the following four operations which extend the tree T by attaching a tree to a vertex $v \in V(T)$, called an attacher, is also a tree in $\mathfrak{T}$.

Operation $\mathfrak{T}_1$. If $v \in V(T)$, then $\mathfrak{T}_1$ adds a path vxy to T.
Operation $\mathfrak{T}_2$. If $v \in V(T)$, then $\mathfrak{T}_2$ adds a star $K_{1,3}$ with center y and leaves x, w, z and joins x to v.
Operation $\mathfrak{T}_3$. If $v \in V(T)$ is a leaf of T, then $\mathfrak{T}_3$ adds a pendant edge vw and a star $K_{1,2}$ with center x and leaves y, z and joins x to v.
Operation $\mathfrak{T}_4$. If v is a leaf of T, then $\mathfrak{T}_4$ adds two new stars $K_{1,2}$ with centers x_1 and x_2 and joins v to x_1 and x_2 (see Fig. 1).

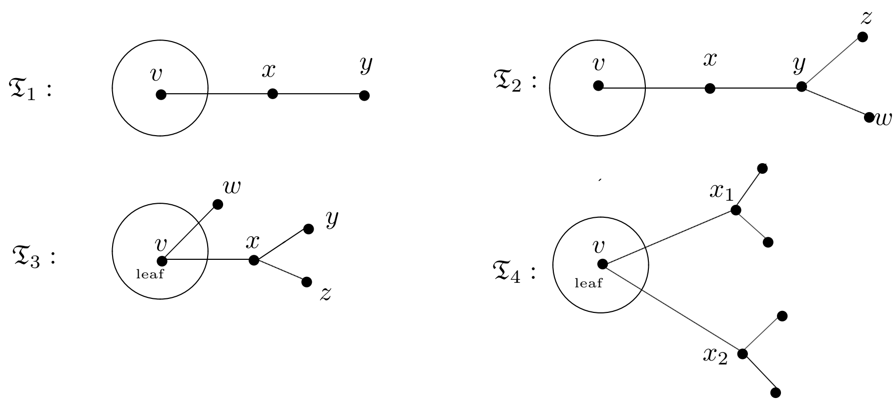

Fig. 1. The four operations

The proof of the following Lemmas can be found in [1].

Lemma 1. Let T' be a tree, $v \in V(T')$ and T be obtained from T' by Operation $\mathfrak{T}_1$. Then $\sigma_m(T) = \sigma_m(T') + 1$.

Lemma 2. Let T' be a tree and $v \in V(T')$. If T is the tree obtained from T' by Operation $\mathfrak{T}_2$, then $\sigma_m(T) = \sigma_m(T') + 2$.

Lemma 3. Let T' be a tree and $v \in L(T')$. If T is the tree obtained from T' by Operation $\mathfrak{T}_3$, then $\sigma_m(T) = \sigma_m(T') + 2$.

Lemma 4. Let T' be a tree and let $v \in L(T')$. If T is the tree obtained from T' by Operation $\mathfrak{T}_4$, then $\sigma_m(T) = \sigma_m(T') + 3$.

Lemma 5. Let $T' \in \mathfrak{T}$ and $u \in V(T')$. If T is a tree obtained from T' by adding a pendant edge uu', then $\sigma_m(T) = \sigma_m(T')$.

Observation 2. *Let T' be a tree and T be obtained from T' by an operation from the set $\{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4\}$. Then $\mathrm{sd}_{\sigma_m}(T) \leq \mathrm{sd}_{\sigma_m}(T')$.*

Proof. Let F be a set of edges in T' where subdividing the edges in F increases the eternal $m-$ security number of T'. Let T_1 and T_2 be the trees obtained from T' and T, by subdividing the edges in F, respectively. Then T_2 is obtained from T_1 by one of the Operations $\mathfrak{T}_1, \ldots, \mathfrak{T}_4$ and the result follows from Lemmas 1, 2, 3 and 4. □

Theorem 3. Let $T \in \mathfrak{T}$ and let T' be a tree obtained from T by subdividing an edge of T. Then $\sigma_m(T') = \sigma_m(T)$.

Proof. Let $T \in \mathfrak{T}$, $e \in E(T)$ and let T' be the tree obtained from T by subdividing the edge e by subdivision vertex u. First note that $\sigma_m(T') \geq \sigma_m(T)$. Let T be obtained from a path P_3 by successive operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$, respectively, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4\}$ for $1 \leq i \leq m$ if $m \geq 1$ and $T = P_3$ if $m = 0$. We proceed by induction on m. If $m = 0$, then clearly the statement is true by Corollary 3. Assume $m \geq 1$ and that the statement holds for all trees which are obtained from P_3 by applying at most $m - 1$ operations. Suppose T_{m-1} is a tree obtained by applying the first $m - 1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{m-1}$. When $e \in E(T_{m-1})$, let T'_{m-1} be obtained from T_{m-1} by subdividing the edge e. We consider the following cases:

Case 1. $\mathfrak{T}^m = \mathfrak{T}_1$. Then T is obtained from T_{m-1} by attaching a path vxy to $v \in V(T_{m-1})$. If $e \in E(T_{m-1})$, then by the inductive hypothesis, $\sigma_m(T'_{m-1}) = \sigma_m(T_{m-1})$ and by Lemma 1,

$$\sigma_m(T') = \sigma_m(T'_{m-1}) + 1 = \sigma_m(T_{m-1}) + 1 = \sigma_m(T).$$

Assume that $e = xy$ (the case $e = vx$ is similar). Let $T^* = T' - \{u, y\}$. Then T^* is obtained from T_{m-1} by attaching a pendant edge vx. By Lemma 5, $\sigma_m(T^*) = \sigma_m(T_{m-1})$ and by Lemma 1, we have

$$\sigma_m(T') = \sigma_m(T^*) + 1 = \sigma_m(T_{m-1}) + 1 = \sigma_m(T).$$

Case 2. $\mathfrak{T}^m = \mathfrak{T}_2$. Then T is obtained from T_{m-1} by adding a star $K_{1,3}$ centered at y and leaves x, w, z and joining x to v. If $e \in E(T_{m-1})$, then by the inductive hypothesis, $\sigma_m(T'_{m-1}) = \sigma_m(T_{m-1})$ and by Lemma 2,

$$\sigma_m(T') = \sigma_m(T'_{m-1}) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

If $e = xy$ (the case $e = vx$ is similar), then let $T^* = T' - \{u, y, z, w\}$. Then T^* is obtained from T_{m-1} by attaching a pendant edge vx. By Lemma 5, $\sigma_m(T^*) = \sigma_m(T_{m-1})$ and by Lemma 2, we have

$$\sigma_m(T') = \sigma_m(T^*) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

Assume that $e = yz$ (the case $e = yw$ is similar). Let $T^* = T' - z$. Then $T^* \simeq T$ and T' is obtained from T^* by attaching a pendant edge uz. It follows from Lemma 5 that $\sigma_m(T') = \sigma_m(T^*) = \sigma_m(T)$.

Case 3. $\mathfrak{T}^m = \mathfrak{T}_3$. Then T is obtained from T_{m-1} by attaching a pendant edge vw to the leaf $v \in V(T_{m-1})$ and adding a star $K_{1,2}$ with center x and leaves y, z and joining x to v. If $e \in E(T_{m-1})$, then by the inductive hypothesis, $\sigma_m(T'_{m-1}) = \sigma_m(T_{m-1})$ and by Lemma 3,

$$\sigma_m(T') = \sigma_m(T'_{m-1}) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

If $e = xv$, then let $T^* = T' - \{u, x, y, z\}$. Then T' is obtained from T^* by adding a star with center x and leaves u, y, z and joining u to v. On the other hand, T^* is obtained from T_{m-1} by attaching a pendant edge vw. By Lemmas 5 and 2,

$$\sigma_m(T') = \sigma_m(T^*) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

If $e = vw$, then let $T^* = T' - w$. Then $T^* \simeq T$ and T' is obtained from T^* by attaching a pendant edge $u'w$. By Lemma 5, $\sigma_m(T') = \sigma_m(T^*) = \sigma_m(T)$. If $e = xy$ (the case $e = xz$ is similar), then let $T^* = T' - y$. Then $T^* \simeq T$ and T' is obtained from T^* by attaching a pendant edge uy. By Lemma 5, $\sigma_m(T') = \sigma_m(T^*) = \sigma_m(T)$.

Case 4. $\mathfrak{T}^m = \mathfrak{T}_4$. Then T is obtained from T_{m-1} by adding two stars $K_{1,2}$ with centers x_1 and x_2 and joining x_1, x_2 to v. Let y_i, z_i be the leaves adjacent to x_i, for $i = 1, 2$. If $e \in E(T_{m-1})$, then by the inductive hypothesis, $\sigma_m(T'_{m-1}) = \sigma_m(T_{m-1})$ and by Lemma 4,

$$\sigma_m(T') = \sigma_m(T'_{m-1}) + 3 = \sigma_m(T_{m-1}) + 3 = \sigma_m(T).$$

If $e = x_1v$ (the case $e = x_2v$ is similar), then let $T^* = T' - \{x_1, y_1, z_1\}$. Then T^* is obtained from T_{m-1} by an Operation $\mathfrak{T}_3$. By Lemma 3, $\sigma_m(T^*) = \sigma_m(T_{m-1}) + 2$. Let S be a $\sigma_m(T^*)-$ set which contains u. Then obviously, $S \cup \{x_1\}$ is an EmSS of T' and so

$$\sigma_m(T') \leq |S| + 1 = \sigma_m(T^*) + 1 = \sigma_m(T_{m-1}) + 3 = \sigma_m(T).$$

Assume that $e = x_1y_1$ (the cases $e = x_1z_1, x_2y_2, x_2z_2$ are similar). Let $T^* = T' - y_1$. Then $T^* \simeq T$ and T' is obtained from T^* by attaching a pendant edge uy_1. By Lemma 5, $\sigma_m(T') = \sigma_m(T^*) = \sigma_m(T)$. This completes the proof. □

An immediate consequence of Theorem 3 and Corollary 1 now follows.

Corollary 4. For any tree $T \in \mathfrak{T}$, $\mathrm{sd}_{\sigma_m}(T) = 2$.

Proposition 3. Let T be a tree and v be a strong support vertex in T such that v is adjacent to the center of a star $K_{1,2}$. Then $\mathrm{sd}_{\sigma_m}(T) = 1$.

Proof. Let v be adjacent to the vertex u which is a center of a star $K_{1,2}$ and let T' be the tree obtained from T by subdividing the edge vu with the subdivision vertex z. Assume that S is a $\sigma_m(T')-$set containing z (we may assume S as a response to an attack on z). To dominate the leaves in L_u and L_v, we have $|(N_{T'}[u] \cup N_{T'}[v]) \cap S| \geq 4$ and we may assume that $v, u \in S$. It is easy to see that $S \setminus \{z\}$ is an EmSS of T of size $|S| - 1$. This completes the proof. □

Proposition 4. Let T be a tree and $v \in V(T)$. If T' is a tree obtained from T by adding three copies $x_iy_iz_i$ $(1 \leq i \leq 3)$ of P_3 and joining v to y_1, y_2, y_3, then $\mathrm{sd}_{\sigma_m}(T') = 1$.

Proof. Let T_1 be the tree obtained from T by subdividing the edge vy_1 by subdivision vertex w. Let S be a $\sigma_m(T_1)$-set containing w (we may consider a response to an attack on w). To dominate x_1 and z_1, we may assume that $y_1 \in S$. If $v \notin S$, then we may assume that $z_i, y_i \in S$ for $i = 2, 3$ and the set $S' = (S \setminus \{w, y_2\}) \cup \{v\}$ is clearly an EmSS of T' of size less than $\sigma_m(T_1)$. If $v \in S$, then it is not hard to see that the set $S' = S \setminus \{w\}$ is an EmSS of T of size $|S| - 1$. This implies that $\mathrm{sd}_{\sigma_m}(T) = 1$. □

Now we are ready to prove the main theorem of this section.

Theorem 4. For any tree T of order $n \geq 3$, $\mathrm{sd}_{\sigma_m}(T) = 2$ if and only if $T \in \mathfrak{T}$.

Proof. According to Corollary 4, we only need to prove the necessity. We proceed by induction n. The result is trivial for $n = 3$. Let $n \geq 4$ and assume that the statement holds for any tree of order less than n. Let T be a tree of order n and $\mathrm{sd}_{\sigma_m}(T) = 2$. By Proposition 1, T is not a star and so $\mathrm{diam}(T) \geq 3$. Assume $P := v_1 \ldots v_k$ is the diametrical path in T such that $\deg(v_2)$ is as small as possible. Suppose that T is rooted at v_k. It follows from Proposition 1 and the assumption $\mathrm{sd}_{\sigma_m}(T) = 2$ that $\deg(v_2) \leq 3$. If $\deg(v_2) = 2$, then let $T' = T - \{v_1, v_2\}$. It follows from Lemma 1 and Observation 2, that $\mathrm{sd}_{\sigma_m}(T') = 2$. By the inductive hypothesis, $T' \in \mathfrak{T}$. Now T can be obtained from T' by Operation $\mathfrak{T}_1$ and so $T \in \mathfrak{T}$. Let $\deg(v_2) = 3$. Suppose that $w \neq v_1$ is a leaf adjacent to v_2. If $\deg(v_3) = 2$, then let $T' = T - \{v_1, v_2, v_3, w\}$. It follows from Lemma 2 and Observation 2 that $\mathrm{sd}_{\sigma_m}(T') = 2$. By the inductive hypothesis, $T' \in \mathfrak{T}$. Now T can be obtained from T' by Operation $\mathfrak{T}_2$ and so $T \in \mathfrak{T}$. Let $\deg(v_3) \geq 3$. It follows from the assumption about v_2 and Propositions 3 and 4 that $\deg(v_3) = 3$ and there is only two possible cases.

Case 1. v_3 is adjacent to a leaf x. Let $T' = T - \{v_1, v_2, x, w\}$. It follows from Lemma 3 and Observation 2 that $\mathrm{sd}_{\sigma_m}(T') = 2$. By the inductive hypothesis, $T' \in \mathfrak{T}$. Now T can be obtained from T' by Operation $\mathfrak{T}_3$ and so $T \in \mathfrak{T}$.

Case 2. v_3 is adjacent to the center of a star $K_{1,2}$ other than v_2. Let $T' = T - D(v_3)$. It follows from Lemma 4 and Observation 2 that $\mathrm{sd}_{\sigma_m}(T') = 2$. By the inductive hypothesis, $T' \in \mathfrak{T}$. Now T can be obtained from T' by Operation $\mathfrak{T}_4$ and so $T \in \mathfrak{T}$. This completes the proof. □

We conclude this paper with the following problem.

Problem. Prove or disprove: For any nonempty graph G, $1 \leq \mathrm{sd}_{\sigma_m}(G) \leq 3$.

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