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On independent domination numbers of grid and toroidal grid directed graphs

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Abstract: A subset S of vertex set $V(D)$ is an independent dominating set of a digraph D if S is both an independent and a dominating set of D. The independent domination number $i(D)$ is the minimum cardinality of an independent dominating set of D. In this paper we calculate the independent domination number of the Cartesian product of two directed paths P_m and P_n for arbitraries m and n. Also, we determine the independent domination number of the Cartesian product of two directed cycles C_m and C_n for $m, n \equiv 0 \pmod 3$ and $n \equiv 0 \pmod m$. We note that, there are many values of m and n such that $C_m \square C_n$ does not have an independent dominating set.

Keywords: directed path, directed cycle, cartesian product, independent domination number

AMS Subject classification: 05C69, 05C20

1. Introduction

All digraphs are assumed to be loopless and without duplicate arcs. Let $D = (V, A)$ be a digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. For every vertex $u \in V(D)$, the sets $O(u) = \{v : (u, v) \in A(D)\}$ and $I(u) = \{v : (v, u) \in A(D)\}$ are called the *outset* and *inset* of u, respectively. The *outdegree* of u is $\mathrm{od}(u) = |O(u)|$ and the *indegree* of u is $\mathrm{id}(u) = |I(u)|$. The *maximum outdegree* and *maximum indegree* of all vertices in D are denoted by $\Delta^+(D)$ and $\Delta^-(D)$, respectively. The *minimum outdegree* and minimum indegree of all vertices in D are denoted by $\delta^+(D)$ and $\delta^-(D)$, respectively.
A set S of vertices is an *independent set* if for every two vertices u, v in S, $(u, v) \notin A(D)$

and $(v,u) \notin A(D)$. The *independence number* $\beta(D)$ is the maximum cardinality of an independent set of D. A set S of vertices is a *dominating set* of D if for each vertex $v \in D - S$ there exists a vertex $u \in S$ such that (u,v) is an arc of D. The *domination number* of D $\gamma(D)$ is the minimum cardinality of a *dominating set* of D. A set S of vertices is called an *independent dominating set* of a digraph D if S is both an independent and a dominating set of D. The *independent domination number* $i(D)$ is the minimum cardinality of an independent dominating set. An $i(D)$-*set* is an independent dominating set of D of size $i(D)$.

The *Cartesian product* $D_1 \Box D_2$ of two digraphs D_1 and D_2 is the digraph with vertex set $V(D_1 \Box D_2) = V(D_1) \times V(D_2)$ and $((u_1,u_2),(v_1,v_2)) \in A(D_1 \Box D_2)$ if and only if either $u_1 = v_1$ and $(u_2,v_2) \in A(D_2)$ or $u_2 = v_2$ and $(u_1,v_1) \in A(D_1)$.

Let P_n denote a directed path on n vertices with vertex set $V(P_n) = \{1,2,\ldots,n\}$ and arc set $A(P_n) = \{(i,i+1) : 1 \leq i \leq n-1\}$. Then for two paths P_m and P_n, $V(P_m \Box P_n) = \{(i,j) : 1 \leq i \leq m$ and $1 \leq j \leq n\}$ and there is an arc from (i,j) to (p,q) if and only if $i = p$ and $q = j+1$ or $j = q$ and $p = i+1$. The vertices of the jth column for $2 \leq j \leq n$, can only be dominated by themselves and vertices in $(j-1)$th column.

Let C_n denote a directed cycle on n vertices with vertex set $V(C_n) = \{1,2,...,n\}$ and arc set $A(C_n) = \{(i,i+1) : 1 \leq i \leq n-1\} \cup \{(n,1)\}$. Then we have $V(C_m \Box C_n) = \{(i,j) : 1 \leq i \leq m$ and $1 \leq j \leq n\}$ and there is an arc from (i,j) to (p,q) if and only if either ($i = p$ and $q = j+1$ or $q = j+1-n$) or ($j = q$ and $p = i+1$ or $p = i+1-m$). Hence, the vertices of the jth column for $2 \leq j \leq n$, can only be dominated by themselves and vertices of $(j-1)$th column and the vertices in the first column can be dominated by themselves and vertices of nth column.

The ith *row* of $V(P_m \Box P_n)$ or $V(C_m \Box C_n)$ is $R_i = \{(i,j) : j = 1,2,\ldots,n\}$ and its jth *column* is $K_j = \{(i,j) : i = 1,2,\ldots,m\}$. If S is an independent dominating set for $P_m \Box P_n$ or $C_m \Box C_n$, then we denote $W_j^S = S \cap K_j$ and $s_j = |W_j^S|$. For any independent dominating set S, the *independent dominating sequence corresponding* to S is $(s_1,s_2,\ldots,s_n)$. For the vertex (i,j), i is always modulo m, and j is always modulo n.

The theory of independent domination was formalized by Berge [2] and Ore [13] in 1962. The independent domination number and the notation $i(G)$ were introduced by Cockayne and Hedetniemi in [3, 4]. Independent dominating sets and variations of independent dominating sets are now extensively studied in the literature (see [1, 7, 17]). Independent dominating sets in regular graphs, in particular in cubic graphs, are also well studied (see [5, 6, 9, 18]). Also, independent dominating sets were introduced into the theory of games by Morgenstern [12]. In [8], Klobucar, established the independent domination number of the strong product of undirected paths, undirected cycles and undirected path with cycle. In [10, 11], Lee established an upper bound on the domination number of a digraph D and gave an upper bound for the domination number of a tournament. In [14, 15], the author established the domination number of grid, torodidal grid digraphs. Also in [16], the total domination number of products of two directed cycles has determined. Anyway, there are very few papers that studied the subject of independent domination in directed graphs.

In this paper we continue the study of study independent domination in directed graphs and we determine the independent domination number of $P_m \Box P_n$ for arbitraries m and n, and $C_m \Box C_n$ when $m, n \equiv 0 \pmod 3$ and $n \equiv 0 \pmod m$.
The proof of the following Observations are easy to check and so omitted.

Observation 1. *For any path P_n, $i(P_n) = \lceil \frac{n}{2} \rceil$.*

Observation 2. *For any cycle C_n with $n \equiv 0 (\text{mod } 2)$, $i(C_n) = \frac{n}{2}$.*

2. Independent domination number of Cartesian product of paths

We use a $(0,1)$-matrix pattern $\mathcal{A}$ with m rows and n columns to represent an IDS S of $P_m \Box P_n$ (resp. $C_m \Box C_n$), where the value at the entry (i,j) of $\mathcal{A}$ is 0 if $(i,j) \notin S$ and 1 if $(i,j) \in S$.

Lemma 1. For any $i(P_m \Box P_n)$-set S with independent dominating sequence $(s_1, s_2, \ldots, s_n)$, $\lceil \frac{m-1}{3} \rceil \leq s_j \leq \lceil \frac{m}{2} \rceil$ for each $j \in \{1, 2, \ldots, n\}$.

Proof. Let S be an $i(P_m \Box P_n)$-set . Since S is independent we have $(i,j) \notin S$ or $(i+1, j) \notin S$ and so $s_j \leq \lceil \frac{m}{2} \rceil$ for each j.
To prove the lower bound, suppose that $(i,j), (q,j) \in K_j \cap S$ and $(i+1, j), (i+2, j), \ldots, (q-1, j) \notin K_j \cap S$. If $|q - i| \geq 4$, then to dominate the vertices $(i+2, j)$ and $(i+3, j)$, we must have $\{(i+2, j-1), (i+3, j-1)\} \subseteq K_{j-1} \cap S$ which is a contradiction. Thus, $\{(i+1, j), (i+2, j), (i+2, j)\} \cap S \neq \emptyset$ for each $i \geq 2$. Since either $(1, j) \in S$ or $(2, j) \in S$, we can start from the 2nd row. This implies that $s_j \geq \lceil \frac{m-1}{3} \rceil$ and the proof is complete. □

Theorem 3. For two paths P_n and P_m, $i(P_m \Box P_n) \leq \lceil \frac{mn}{2} \rceil$.

Proof. Let $D = \{(2i-1, 2j-1) \mid 1 \leq j \leq \lceil \frac{n}{2} \rceil \text{ and } 1 \leq i \leq \lceil \frac{m}{2} \rceil\} \cup \{(2i, 2j) \mid 1 \leq j \leq \lfloor \frac{n}{2} \rfloor \text{ and } 1 \leq i \leq \lfloor \frac{m}{2} \rfloor\}$ (see pattern of D for $P_5 \Box P_8$). Clearly $|D| = \lceil \frac{mn}{2} \rceil$ and we can check that D is an independent dominating set for $P_m \Box P_n$. This implies that $i(P_m \Box P_n) \leq \lceil \frac{mn}{2} \rceil$ and the proof is complete. □

$$Fig\ 1: \ An\ IDS\ of\ P_5 \Box P_8 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Proposition 1. For any independent dominating set D of $P_m \Box P_n$,

$$\{(2i-1,1) : 1 \le i \le \lceil \frac{m}{2} \rceil\} \cup \{(1,2j-1) : 1 \le j \le \lceil \frac{n}{2} \rceil\} \subseteq D.$$

Proof. First we note that the vertices in K_1 can be dominated only by the vertices of K_1 and so for R_1. Since $\mathrm{id}((1,1)) = 0$, we have $(1,1) \in D$ and this implies that $(1,2), (2,1) \notin D$ because D is an independent set. Now to dominate the vertices $(1,3)$ and $(3,1)$, we must have $(1,3), (3,1) \in D$ implying that $(1,4), (4,1) \notin D$. By repeating the process we obtain the result. □

Proposition 2. Let D be an independent dominating set of $P_m \Box P_n$. If there is a vertex $(i,j) \in K_j \cap D$ where $i \ne m$ and $j \ne n$, then $(i+1, j+1) \in K_{j+1} \cap D$.

Proof. Since D is an independent dominating set and since $(i,j) \in K_j \bigcap D$, we have $(i, j+1), (i+1, j) \notin D$. Now to dominate the vertex $(i+1, j+1)$, we must have $(i+1, j+1) \in K_{j+1} \cap D$. □

Proposition 3. Let D be an independent dominating set of $P_m \Box P_n$. If there is a vertex $(i,j) \in K_j \cap D$ where $1 \notin \{i,j\}$, then $(i-1, j-1) \in K_{j-1} \cap D$.

Proof. Suppose, to the contrary, $(i-1, j-1) \notin K_{j-1} \cap D$. Since D is an independent dominating set and since $(i,j) \in K_j \cap D$, we have $(i-1, j), (i, j-1) \notin D$. Hence to dominate $(i-1, j)$, we must have $(i-2, j) \in D$. It follows that $(i-2, j-1) \notin D$ because D is independent. Now to dominate $(i-1, j-1)$, we must have $(i-1, j-2) \in D$ implying that $(i, j-2) \in D$. But then $(i, j-1)$ is not dominated by D which is a contradiction. Thus $(i-1, j-1) \in K_{j-1} \cap D$ and the proof is complete. □

Proposition 4. Let D be an independent dominating set of $P_m \Box P_n$. If there is a vertex $(i,j) \in K_j \cap D$ such that $i \le m-2$ and $i, j > 1$, then $(i+2, j) \in K_j \cap D$.

Proof. Since D is an independent dominating set and since $(i,j) \in K_j \cap D$, we have $(i-1, j), (i, j-1) \notin D$. Suppose, to the contrary, that $(i+2, j) \notin K_j \cap D$. To dominate $(i+2, j)$, we must have $(i+2, j-1) \in K_{j-1} \cap D$. We deduce from Proposition 3 that $(i-1, j-1) \in K_{j-1} \cap D$ and $(i+1, j-2) \in K_{j-2} \cap D$. By repeating this argument, we have $(i-1-q_1, j-1-q_1) \in K_{j-1-q_1} \cap D$ for $0 \le q_1 \le \min\{i-2, j-2\}$ and $(i+1-q_2, j-2-q_2) \in K_{j-2-q_2} \cap D$ for $0 \le q_2 \le \min\{i, j-3\}$. If $i \ge j$, then Proposition 3 yields to $(i-j+1, 1) \in K_1 \cap D$ and $(i-j+4, 1) \in K_1 \cap D$ and this leads to $|(i-j+1) - (i-j+4)| = 3$, a contradiction with Proposition 1. Assume that $i < j$. If $j = i+1$ (resp. $j = i+2$), then we have $(1,2) \in K_2 \cap D$ (resp. $(2,1) \in K_1 \cap D$), a contradiction with $(1,1) \in D$ (see Proposition 1). Let $i+3 \le j$. Using above argumentation, we have $(1, j-i+1) \in R_1 \cap D$ and $(1, j-i-2) \in R_1 \cap D$. But then $(1, j-i-1), (1, j-i) \notin D$ and so $(1, j-i)$ is not dominated, a contradiction. Thus $(i+2, j) \in K_j \cap D$ and the proof is complete. □

The proof of next result is similar to the proof of Proposition 4 and therefore omitted.

Proposition 5. Let D be an independent dominating set of $P_m \Box P_n$. If there is a vertex $(i,j) \in K_j \cap S'$ such that $j \leq n-2$, then $(i, j+2) \in K_{j+2} \cap D$.

Theorem 4. For two paths P_n and P_m, $i(P_m \Box P_n) = \lceil \frac{mn}{2} \rceil$.

Proof. By Propositions 1,2,3,4 and 5, $P_m \Box P_n$ has exactly one independent dominating set which presented in Theorem 3 and this implies that $i(P_m \Box P_n) \geq \lceil \frac{mn}{2} \rceil$. Now by Theorem 3 we have $i(P_m \Box P_n) = \lceil \frac{mn}{2} \rceil$ and the proof is complete. □

3. Independent domination number of $C_m \Box C_n$

In this section we determine the independent domination number of $C_m \Box C_n$ for some values of m and n. We start with a definition. Suppose that D is an independent dominating set of $C_m \Box C_n$ and assume that $1 \leq j, h \leq n$. We say that the hth column is an t-shift of the jth column with respect to D, if $(i,j) \in D$ implies that $(i+t, h) \in D$ and vice versa, where the indices $i,\ i+t$ are taken modulo m and $j,\ h$ are taken modulo n.
We observe that Propositions 2 and 3 are true for $C_m \Box C_n$, but here for any i, j. The proof of the next lemma is similar to that of Lemma 1 and therefore omitted.

Lemma 2. For any $i(C_m \Box C_n)$-set S with independent dominating sequence $(s_1, s_2, \ldots, s_n)$, $\lceil \frac{m-1}{3} \rceil \leq s_j \leq \lceil \frac{m}{2} \rceil$ for each $j \in \{1, 2, \ldots, n\}$.

We note that C_n has no independent dominating set when $n \equiv 1 \pmod 2$.

Theorem 5. Let m, n be positive integers. If $m \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 3$, then $i(C_m \Box C_n) = \frac{mn}{3}$.

Proof. Let $S = \{(i,j) \mid 1 \leq i \leq n \text{ and } j \equiv i \pmod 3\}$ (see Figure 2). Obviously, and it is easy to see that S is an independent dominating set of $C_m \Box C_n$. This implies that

$$i(C_m \Box C_n) \leq \frac{mn}{3} \tag{1}$$

$$\textit{Fig 2: An IDS of } C_6 \Box C_9 = \begin{bmatrix} 1&0&0&1&0&0&1&0&0 \\ 0&1&0&0&1&0&0&1&0 \\ 0&0&1&0&0&1&0&0&1 \\ 1&0&0&1&0&0&1&0&0 \\ 0&1&0&0&1&0&0&1&0 \\ 0&0&1&0&0&1&0&0&1 \end{bmatrix}$$

To prove the inverse inequality, let D be an $i(C_m \Box C_n)$-set. Since $C_m \Box C_n$ is a 2-regular digraph, any vertex of S dominates exactly two vertices in $V(C_m \Box C_n) - S$ and so

$$i(C_m \Box C_n) \geq \frac{mn}{3} \tag{2}$$

By (1) and (2), we have $i(C_m \Box C_n) = \frac{mn}{3}$ as desired. □

Next we consider the case $n \equiv 0 \pmod{m}$.

Theorem 6. Let m, n be positive integers. If $n \equiv 0 \pmod{m}$, then $i(C_m \Box C_n) = \lceil \frac{m}{3} \rceil n$.

Proof. Let $S = \{(i,j) \mid 1 \leq j \leq n \text{ and } i \equiv j \pmod{m}\} \cup \{(i,j) \mid 1 \leq j \leq n \text{ and } i \equiv j + 2 + 3q \pmod{m} \text{ for } 0 \leq q \leq \lceil \frac{m}{3} \rceil - 2\}$. Clearly $|S| = \lceil \frac{m}{3} \rceil n$ and it is easy to see that S is an independent dominating set of $C_m \Box C_n$. Hence $i(C_m \Box C_n) \leq \lceil \frac{m}{3} \rceil n$. Now let D be an independent dominating set of $C_m \Box C_n$ with independent dominating sequence $(s_1, s_2, \ldots, s_n)$. By Lemma 2, we have $s_j \geq \lceil \frac{m}{3} \rceil$ for each $j \in \{1, 2, \ldots, n\}$. Thus $i(C_m \Box C_n) \geq \lceil \frac{m}{3} \rceil n$, and hence $i(C_m \Box C_n) = \lceil \frac{m}{3} \rceil n$. □

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