

A characterization of trees with equal Roman $\{2\}$ -domination and Roman domination numbers

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Dedicated to Prof. Dr. Lutz Volkmann, on the occasion of his 75th birthday.

Abstract: Given a graph $G = (V, E)$ and a vertex $v \in V$, by $N(v)$ we represent the open neighbourhood of v . Let $f : V \rightarrow \{0, 1, 2\}$ be a function on G . The weight of f is $\omega(f) = \sum_{v \in V} f(v)$ and let $V_i = \{v \in V : f(v) = i\}$, for $i = 0, 1, 2$. The function f is said to be

- a Roman $\{2\}$ -dominating function, if for every vertex $v \in V_0$, $\sum_{u \in N(v)} f(u) \geq 2$. The Roman $\{2\}$ -domination number, denoted by $\gamma_{\{R2\}}(G)$, is the minimum weight among all Roman $\{2\}$ -dominating functions on G ;
- a Roman dominating function, if for every vertex $v \in V_0$ there exists $u \in N(v) \cap V_2$. The Roman domination number, denoted by $\gamma_R(G)$, is the minimum weight among all Roman dominating functions on G .

It is known that for any graph G , $\gamma_{\{R2\}}(G) \leq \gamma_R(G)$. In this paper, we characterize the trees T that satisfy the equality above.

Keywords: Roman $\{2\}$ -domination; 2-rainbow domination; Roman domination; tree

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1. Introduction

Throughout this paper we consider $G = (V, E)$ as a simple graph of order $n = |V|$. That is, a graph that is finite, undirected, and without loops or multiple edges.

Given a vertex v of G , $N(v)$ and $N[v]$ represent the *open neighborhood* and the *closed neighborhood* of v , respectively.

Let $f : V \rightarrow \{0, 1, 2\}$ be a function on a graph G . Notice that f generates three sets V_0 , V_1 and V_2 such that $V_i = \{v \in V : f(v) = i\}$ for $i = 0, 1, 2$. In this sense, from now on, we will write $f = (V_0, V_1, V_2)$ so as to refer to the function f . Given a set $S \subseteq V$, $f(S) = \sum_{v \in S} f(v)$. The weight of f is $\omega(f) = f(V) = |V_1| + 2|V_2|$. In this sense, by an $f(V)$ -function, we mean a function of weight $f(V)$. Also $V_{0,2} = \{v \in V_0 : N(v) \cap V_2 \neq \emptyset\}$ and $V_{0,1} = V_0 \setminus V_{0,2}$.

Roman domination in graphs was formally defined by Cockayne, Dreyer, Hedetniemi, and Hedetniemi [4] motivated, in part, by an article in *Scientific American* of Ian Stewart entitled “Defend the Roman Empire” [11]. A *Roman dominating function* (RDF) on a graph G is a function $f = (V_0, V_1, V_2)$ satisfying that every vertex $u \in V_0$ is adjacent to at least one vertex $v \in V_2$. The *Roman domination number*, denoted by $\gamma_R(G)$, is the minimum weight among all Roman dominating functions on G . Further results on Roman domination can be found for example, in [5, 10, 12].

Another kind of functions defined on graphs are the 2-rainbow dominating functions, which were introduced in [2]. Let f be a function on a graph G that assigns a set of colors (possibly empty), chosen from the set $\{1, 2\}$, to each vertex of G . That is, $f : V \rightarrow P(\{1, 2\})$. If for each vertex $v \in V$ such that $f(v) = \emptyset$, we have $\cup_{u \in N(v)} f(u) = \{1, 2\}$, then f is called a *2-rainbow dominating function* (2RDF) on G . The weight of a 2RDF f is defined as $\omega(f) = \sum_{v \in V} |f(v)|$. The *2-rainbow domination number* of G , denoted by $\gamma_{r2}(G)$, is the minimum weight among all 2-rainbow dominating functions.

A generalization of a Roman dominating function, called a *Roman $\{2\}$ -dominating function* (R2DF), was introduced by Chellali et al. in [3] as follows. For a graph G , a *Roman $\{2\}$ -dominating function* $f = (V_0, V_1, V_2)$ is a function having the property that for each vertex $v \in V_0$, it follows $f(N(v)) \geq 2$. That is, either there exists a vertex $u \in N(v) \cap V_2$, or at least two vertices $x, y \in N(v) \cap V_1$. The *Roman $\{2\}$ -domination number*, denoted by $\gamma_{\{R2\}}(G)$, is the minimum weight among all Roman $\{2\}$ -dominating functions on G . This concept was also introduced and barely studied by Brešar et al. in [2], as a monochromatic version of the 2-rainbow domination number, and it was called *weak $\{2\}$ -domination number*. It was also further studied in [8, 9], where it was called *Italian domination number*. In [3], Chellali et al. established the next relationship between Roman $\{2\}$ -domination number, 2-rainbow domination number and Roman domination number.

Proposition 1. [3] *For every graph G , $\gamma_{\{R2\}}(G) \leq \gamma_{r2}(G) \leq \gamma_R(G)$.*

In concordance with such inequalities above, it is then an interesting problem to investigate the possible equalities that could occur. That is, finding the families of graphs G for which $\gamma_{\{R2\}}(G) = \gamma_{r2}(G)$, $\gamma_{r2}(G) = \gamma_R(G)$ or $\gamma_{\{R2\}}(G) = \gamma_R(G)$. In connection with the second equality, it was shown in [1], that for every fixed non-negative integer k , the recognition of the connected K_4 -free graphs G with $\gamma_R(G) -$

$\gamma_{r2}(G) = k$ is NP-hard. Accordingly, finding a useful characterization of graphs G for which $\gamma_{r2}(G) = \gamma_R(G)$ is quite unlikely. In this sense, our goal in this work is centered into solving the following two problems while we only consider families of trees.

- Characterize the graphs G such that $\gamma_{\{R2\}}(G) = \gamma_R(G)$.
- Characterize the graphs G that satisfy $\gamma_{r2}(G) = \gamma_R(G)$.

As mentioned above, we next settle these two problems for the case of trees, by making some constructive characterizations of the trees achieving such stated equalities.

1.1. Terminology and Notation

We first present some necessary terminology and notation. Given a graph $G = (V, E)$ and a set of vertices S , the *open neighborhood* and *closed neighborhood* of S are $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$, respectively. The *private neighborhood* $pn(v, S)$ of $v \in S \subseteq V$ is defined by $pn(v, S) = \{u \in V : N(u) \cap S = \{v\}\}$. Each vertex in $pn(v, S)$ is called a private neighbor of v . The *external private neighborhood* $epn(v, S)$ of v consists of the private neighbors of v in $V \setminus S$. Thus, $epn(v, S) = pn(v, S) \cap (V \setminus S)$. Also, by $G - S$ we denote the graph obtained from G when removing all the vertices in S , and all the edges incident with a vertex in S (if $S = \{v\}$, for some vertex v , then we simply write $G - v$).

Moreover, we denote the *degree* of a vertex v by $\delta_G(v)$, or simply by $\delta(v)$, if the graph G is clear from the context. The *minimum* and *maximum degrees* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For any two vertices u and v , the *distance* $d(u, v)$ between u and v is the minimum length of a $u-v$ path. The *diameter* of G , $diam(G)$, is the maximum distance among pairs of vertices in G . A *diametral path* in G is a shortest path whose length equals the diameter of the graph. Thus, a diametral path in G is a shortest path joining two vertices that are at distance $diam(G)$ from each other (such vertices are called *diametral vertices*).

A *tree* T is an acyclic connected graph. A *leaf vertex* of T is a vertex of degree one. A *support vertex* of T is a vertex adjacent to a leaf that is no leaf; a *weak support vertex* is a support vertex adjacent to exactly one leaf; a *strong support vertex* is a support vertex that is not a weak support; and a *semi-support vertex* is a vertex adjacent to a support vertex that is neither a leaf nor a support. The set of leaves is denoted by $L(T)$; the set of support vertices is denoted by $S(T)$; the set of weak support vertices is denoted by $S_w(T)$; the set of strong support vertices is denoted by $S_s(T)$; and the set of semi-support vertices is denoted by $SS(T)$.

We will use the following notation for two special families of trees. The *star* $K_{1,k}$ with $k \geq 2$, is a tree with a central vertex of degree k and the remaining vertices are leaves. A *double star* $S_{x,y}$ with $x, y \geq 1$, is a tree with exactly two adjacent vertices of degree $x + 1$ and $y + 1$ respectively, and the remaining vertices are leaves.

A *rooted tree* T is a tree with a distinguished special vertex r , called the root. For each vertex $v \neq r$ of T , the *parent* of v is the neighbor of v on the unique $r-v$ path,

while a *child* of v is any other neighbor of v . A *descendant* of v is a vertex $u \neq v$ such that the unique $r - u$ path contains v . Thus, every child of v is a descendant of v . The set of descendants of v is denoted by $D(v)$, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v . For the remainder of the paper, any necessary definition will be introduced whenever the concept is needed. Moreover, for any other very basic terminology and notation on graph theory, we follow the book [6].

2. The characterizations

We begin this section with a theoretical characterization of the graphs G satisfying the equality $\gamma_{\{R2\}}(G) = \gamma_R(G)$. However, such characterization lacks of usefulness, since it is precisely based on finding a $\gamma_{\{R2\}}(G)$ -function which satisfies a specific condition.

Theorem 1. *Let G be a graph. Then $\gamma_{\{R2\}}(G) = \gamma_R(G)$ if and only if there exists a $\gamma_{\{R2\}}(G)$ -function $f = (V_0, V_1, V_2)$ such that $V_{0,1} = \emptyset$.*

Proof. Suppose that $\gamma_{\{R2\}}(G) = \gamma_R(G)$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$ -function. Since every RDF is a R2DF on G , f is a $\gamma_{\{R2\}}(G)$ -function as well, and satisfies that $V_{0,1} = \emptyset$. Conversely, suppose there exists a $\gamma_{\{R2\}}(G)$ -function $f' = (V'_0, V'_1, V'_2)$ such that $V'_{0,1} = \emptyset$. So, $V'_0 = V'_{0,2}$, which implies that f' is a RDF on G . Thus, $\gamma_R(G) \leq \omega(f') = \gamma_{\{R2\}}(G)$. Since $\gamma_{\{R2\}}(G) \leq \gamma_R(G)$, by Proposition 1, we consequently deduce $\gamma_{\{R2\}}(G) = \gamma_R(G)$. \square

We now continue with some results (some of them are already known) which are useful for our purposes.

Proposition 2. [4] *Let $f = (V_0, V_1, V_2)$ be any $\gamma_R(G)$ -function. Then*

- (i) *The subgraph induced by the vertices of V_1 has maximum degree at most one.*
- (ii) *No edge of G joins V_1 to V_2 .*

Observation 2. Let G be a graph. If $v \in S_s(G)$, then there exists a $\gamma_R(G)$ -function f such that $f(v) = 2$ and $f(h) = 0$ for every $h \in N(v) \cap L(G)$.

Observation 3. Let G be a graph. If $v \in S_s(G)$, then there exists a $\gamma_{\{R2\}}(G)$ -function f such that $f(v) = 2$ and $f(h) = 0$ for every $h \in N(v) \cap L(G)$.

Observation 4. If T' is a subtree of a tree T , then $\gamma_{\{R2\}}(T') \leq \gamma_{\{R2\}}(T)$ and $\gamma_R(T') \leq \gamma_R(T)$.

In [8], the trees T for which $\gamma_{\{R2\}}(T) = 2\gamma(T)$ were characterized. On the other hand, the trees T for which $\gamma_R(T) = 2\gamma(T)$ (known as Roman trees) were characterized in [7]. Since $\gamma_{\{R2\}}(G) \leq \gamma_R(G)$ and $\gamma_R(G) \leq 2\gamma(G)$ are satisfied for any graph G , we can deduce that $\gamma_{\{R2\}}(T) = 2\gamma(T)$ if and only if $\gamma_{\{R2\}}(T) = \gamma_R(T)$ and $\gamma_R(T) = 2\gamma(T)$. Consequently, we observe that the trees belonging to the family given in [8] form a subfamily of the family of trees which we construct in our work. In this sense, we need to introduce some terminology previously used in [8]. A *near Roman $\{2\}$ -dominating function relative to a vertex v* , abbreviated near-R2DF relative to v , on a graph $G = (V, E)$, is a function $f = (V_0, V_1, V_2)$ satisfying the following. For each vertex u in V such that $f(u) = 0$, if $u = v$, then $\sum_{u \in N(v)} f(u) \geq 1$, while if $u \neq v$, then $\sum_{u \in N(v)} f(u) \geq 2$. The weight of a near-R2DF relative to v on G is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a near-R2DF relative to v on G is called the *near Roman $\{2\}$ -domination number relative to v of G* , which we denote as $\gamma_{\{R2\}}^n(G; v)$. Since every R2DF is a near-R2DF, we note that $\gamma_{\{R2\}}^n(G; v) \leq \gamma_{\{R2\}}(G)$ for any vertex v of G . The authors of [8] defined a vertex $v \in V(G)$ to be a *stable vertex* in G , if $\gamma_{\{R2\}}(G - v) \geq \gamma_{\{R2\}}(G)$; while v is a *near stable vertex* in G if $\gamma_{\{R2\}}^n(G; v) = \gamma_{\{R2\}}(G)$. Moreover, we denote by $S_{2,R}(G)$ the set of support vertices labelled with two by some $\gamma_R(G)$ -function.

Now on, in order to provide a constructive characterization of the trees having equal Roman $\{2\}$ -domination and Roman domination numbers, we need to introduce the next family of trees. Before, we assume that every tree T has order $n \geq 3$, since the case $n \in \{1, 2\}$ means that T is either a path P_1 or a path P_2 and it is straightforward to see that in both cases $\gamma_{\{R2\}}(T) = \gamma_R(T)$. Let \mathcal{F} be the family of trees T that can be obtained from a sequence of trees T_0, \dots, T_k , where $k \geq 0$, $T_0 \cong P_3$ and $T \cong T_k$. Furthermore, if $k \geq 1$, then for each $i \in \{1, \dots, k\}$, the tree T_i can be obtained from the tree $T' \cong T_{i-1}$ by one of the following operations F_1, F_2, F_3, F_4, F_5 or F_6 . In such operations, by a join of two vertices we mean adding an edge between these two vertices.

Operation F_1 : Add a star $K_{1,3}$, and join a leaf u of the star to an arbitrary vertex v of T' .

Operation F_2 : Add a double star $S_{1,2}$, and join the weak support u of the double star to an arbitrary vertex v of T' .

Operation F_3 : Add a path P_3 with support vertex u , and join u to a stable vertex v of T' .

Operation F_4 : Add a path P_3 , and join a leaf to a near stable vertex v of T' .

Operation F_5 : Add a new vertex u to T' and join u to a vertex $v \in S_{2,R}(T')$.

Operation F_6 : Add a new vertex u to T' and join u to a near stable vertex $v \in L(T')$.

We next show that every tree T in the family \mathcal{F} satisfies that $\gamma_{\{R2\}}(T) = \gamma_R(T)$.

Lemma 1. *If $T \in \mathcal{F}$, then $\gamma_{\{R2\}}(T) = \gamma_R(T)$.*

Proof. We proceed by induction on the number $r(T)$ of operations required to construct the tree T . If $r(T) = 0$, then $T \cong P_3$ that satisfies $\gamma_{\{R2\}}(T) = 2 = \gamma_R(T)$. This establishes the base case. Hence, we now assume that $k \geq 1$ is an integer and that each tree $T' \in \mathcal{F}$ with $r(T') < k$ satisfies that $\gamma_{\{R2\}}(T') = \gamma_R(T')$. Let $T \in \mathcal{F}$ be a tree with $r(T) = k$. Then, T can be obtained from a tree $T' \in \mathcal{F}$ with $r(T') = k - 1$ by one of the six operations above. We shall prove that T satisfies that $\gamma_{\{R2\}}(T) = \gamma_R(T)$. We consider six cases, depending on which operation is used to construct the tree T from T' .

Case 1. T is obtained from T' by operation F_1 . Assume T is obtained from T' by adding a star $K_{1,3}$, with central vertex u and leaves h_1, h_2, h_3 , and adding the edge h_1v , where v is an arbitrary vertex of T' . Notice that every RDF on T' can be extended to a RDF on T by assigning the weight 2 to u and the weight 0 to the three neighbors of u . Hence, by the statement above, Proposition 1 and inductive hypothesis, we obtain

$$\gamma_{\{R2\}}(T) \leq \gamma_R(T) \leq \gamma_R(T') + 2 = \gamma_{\{R2\}}(T') + 2. \quad (1)$$

Conversely, the vertex u is a strong support of T and so, by Observation 3, there exists a $\gamma_{\{R2\}}(T)$ -function f satisfying that $f(u) = 2$ and $f(h_2) = f(h_3) = 0$. Also, we may assume that $f(h_1) = 0$. Otherwise, if $f(h_1) \geq 1$, then the function g defined by $g(v) = \max\{f(v), f(h_1)\}$, $g(h_1) = 0$, and $g(x) = f(x)$ if $x \in V(T) \setminus \{v, h_1\}$, is a R2DF on T and $\omega(g) \leq \omega(f) = \gamma_{\{R2\}}(T)$, which implies that either $f(h_1) = 0$, or that g is a $\gamma_{\{R2\}}(T)$ -function with $g(h_1) = 0$. Thus, $f(h_1) = 0$ implies f restricted to $V(T')$ is a R2DF on T' , from which we deduce that $\gamma_{\{R2\}}(T') \leq f(V(T')) = \omega(f) - f(N[u]) = \gamma_{\{R2\}}(T) - 2$. In consequence, we must have equalities throughout the inequality chain (1). In particular, $\gamma_{\{R2\}}(T) = \gamma_R(T)$.

Case 2. T is obtained from T' by operation F_2 . Assume T is obtained from T' by adding a double star $S_{1,2}$, where u is the weak support, h_1 is the leaf-neighbour of u , w is the strong support; and we add the edge uv , such that v is an arbitrary vertex of T' . Notice that every RDF on T' can be extended to a RDF on T by assigning the weight 2 to w , the weight 0 to the three neighbors of w , and the weight 1 to h_1 . Hence, by the statement above, Proposition 1 and inductive hypothesis, we obtain

$$\gamma_{\{R2\}}(T) \leq \gamma_R(T) \leq \gamma_R(T') + 3 = \gamma_{\{R2\}}(T') + 3. \quad (2)$$

Conversely, since the vertex w is a strong support of T , by Observation 3, there exists a $\gamma_{\{R2\}}(T)$ -function f satisfying that $f(w) = 2$ and $f(h) = 0$ for every $h \in N(w) \cap L(T)$. If $f(u) = 0$, then $f(h_1) = 1$ and also f restricted to $V(T')$ is a R2DF on T' , implying that $\gamma_{\{R2\}}(T') \leq f(V(T')) = \omega(f) - f(N[u]) = \gamma_{\{R2\}}(T) - 3$, and by the inequality chain (2) it follows that $\gamma_{\{R2\}}(T) = \gamma_R(T)$. Conversely, if $f(u) > 0$, then the function g , defined by $g(v) = g(h_1) = 1$, $g(u) = 0$ and $g(x) = f(x)$

if $x \in V(T) \setminus \{v, h_1, u\}$, is a R2DF on T with weight $\omega(g) \leq \omega(f) = \gamma_{\{R2\}}(T)$. So, g is a $\gamma_{\{R2\}}(T)$ -function as well, and consequently, g restricted to $V(T')$ is a R2DF on T' . Thus, $\gamma_{\{R2\}}(T') \leq g(V(T')) = \omega(g) - (g(N[w]) + g(h_1)) = \gamma_{\{R2\}}(T) - 3$. Therefore, we must have equality throughout the inequality chain (2). In particular $\gamma_{\{R2\}}(T) = \gamma_R(T)$.

Case 3. T is obtained from T' by operation F_3 . Assume T is obtained from T' by adding a path u_1uu_2 and the edge uv , where v is a stable vertex of T' . Again, notice that every RDF on T' can be extended to a RDF on T by assigning the weight 2 to u and the weight 0 to u_1 and u_2 . Hence, by the statement above, Proposition 1 and inductive hypothesis, we obtain the inequality chain (1).

We now show that $\gamma_{\{R2\}}(T') \leq \gamma_{\{R2\}}(T) - 2$. As u is a strong support of T , by Observation 3, there exists a $\gamma_{\{R2\}}(T)$ -function f satisfying that $f(u) = 2$ and $f(u_1) = f(u_2) = 0$. If f restricted to $V(T')$ is a R2DF on T' , then $\gamma_{\{R2\}}(T') \leq f(V(T')) = \gamma_{\{R2\}}(T) - 2$. Conversely, if f restricted to $V(T')$ is not a R2DF on T' , then, among other facts, $f(v) = 0$. This implies that f restricted to $V(T' - v)$ is a R2DF on $T' - v$. Also, as v is a stable vertex of T' , it follows that $\gamma_{\{R2\}}(T') \leq \gamma_{\{R2\}}(T' - v) \leq f(V(T' - v)) = \gamma_{\{R2\}}(T) - 2$. In consequence, we must have equality throughout the inequality chain (1). In particular, $\gamma_{\{R2\}}(T) = \gamma_R(T)$.

Case 4. T is obtained from T' by operation F_4 . Assume T is obtained from T' by adding a path uu_1u_2 and the edge uv , where v is a near stable vertex of T' . Again, notice that every RDF on T' can be extended to a RDF on T by assigning the weight 2 to u_1 and the weight 0 to u and u_2 . Hence, by the statement above, Proposition 1 and the inductive hypothesis, we again obtain the inequality chain (1).

On the other hand, let f be a $\gamma_{\{R2\}}(T)$ -function. If f restricted to $V(T')$ is a R2DF on T' , then $\gamma_{\{R2\}}(T') \leq f(V(T')) \leq \gamma_{\{R2\}}(T) - 2$. Conversely, if f restricted to $V(T')$ is not a R2DF on T' , then $f(v) = 0$ and $f(u) = 1$. These imply that f restricted to $V(T')$ is a near-R2DF relative to v on T' . Also, as v is a near stable vertex of T' , it follows that $\gamma_{\{R2\}}(T') = \gamma_{\{R2\}}^n(T', v) \leq f(V(T')) = \gamma_{\{R2\}}(T) - 2$. Therefore, $\gamma_{\{R2\}}(T') + 2 \leq \gamma_{\{R2\}}(T)$ and so, we must have equality throughout the inequality chain (1). Particularly, $\gamma_{\{R2\}}(T) = \gamma_R(T)$.

Case 5. T is obtained from T' by operation F_5 . Assume T is obtained from T' by adding a new vertex u and the edge uv , where $v \in S_{2,R}(T')$. Thus, there exists a $\gamma_R(T')$ -function f satisfying that $f(v) = 2$. Also, f can be extended to a RDF on T by assigning the weight 0 to u . Hence, $\gamma_R(T) \leq \gamma_R(T')$. By Proposition 1, previous inequality, inductive hypothesis and Observation 4 we deduce $\gamma_{\{R2\}}(T) \leq \gamma_R(T) \leq \gamma_R(T') = \gamma_{\{R2\}}(T') \leq \gamma_{\{R2\}}(T)$. Therefore, we must have equality throughout the inequality chain above. In particular, $\gamma_{\{R2\}}(T) = \gamma_R(T)$.

Case 6. T is obtained from T' by operation F_6 . Assume T is obtained from T' by adding a new vertex u and the edge uv , where v is both a near stable vertex and a leaf of T' . Notice that every RDF on T' can be extended to a RDF on T by assigning

the weight 1 to u . Hence, by the statement above, Proposition 1 and the inductive hypothesis, we obtain

$$\gamma_{\{R2\}}(T) \leq \gamma_R(T) \leq \gamma_R(T') + 1 = \gamma_{\{R2\}}(T') + 1. \quad (3)$$

On the other hand, as v is a leaf of T' , we consider the vertex $s \in N(v) \cap S(T')$ and notice that there exists a $\gamma_{\{R2\}}(T)$ -function f satisfying that $f(v) = 0$, $f(u) = 1$ and $f(s) > 0$. Thus, f restricted to $V(T')$ is a near-R2DF relative to v on T' , and as v is a near stable vertex of T' , it follows that $\gamma_{\{R2\}}(T') = \gamma_{\{R2\}}^n(T', v) \leq f(V(T')) = f(V(T)) - f(u) = \gamma_{\{R2\}}(T) - 1$. Hence, $\gamma_{\{R2\}}(T') + 1 \leq \gamma_{\{R2\}}(T)$. Therefore, we must have equality throughout the inequality chain above. In particular, $\gamma_{\{R2\}}(T) = \gamma_R(T)$. \square

We now turn our attention to the opposite direction concerning the lemma above. That is, we show that if a tree T satisfies $\gamma_{\{R2\}}(T) = \gamma_R(T)$, then it belongs to the family \mathcal{F} .

Lemma 2. *Let T be a tree. If $\gamma_{\{R2\}}(T) = \gamma_R(T)$, then $T \in \mathcal{F}$.*

Proof. In order to easily proceed with the proof, from now on we say that a tree T belongs to the family $T_{\{R2\},R}$ if $\gamma_{\{R2\}}(T) = \gamma_R(T)$. We proceed by induction on the order $n \geq 3$ of the trees $T \in T_{\{R2\},R}$. If T is a star, then $\gamma_{\{R2\}}(T) = 2 = \gamma_R(T)$ by assigning two to the central vertex and zero to the other vertices. Thus, T can be obtained from P_3 by repeatedly applying operation F_5 . Therefore, $T \in \mathcal{F}$. This establishes the base case. We assume now that $k > 3$ is an integer and that each tree $T' \in T_{\{R2\},R}$ with $|V(T')| < k$ satisfies that $T' \in \mathcal{F}$. Let T be a tree with $|V(T)| = k$ such that $T \in T_{\{R2\},R}$ and we may assume that $\text{diam}(T) \geq 3$.

First, suppose that $\text{diam}(T) = 3$. Therefore, T is a double star $S_{x,y}$ for some integers $x \geq y \geq 1$. If $T \cong P_4$ then T can be obtained from a path P_3 by applying operation F_6 . If $T \cong S_{x,1}$ with $x \geq 2$, then T can be obtained from a path P_3 by first applying operation F_6 , thereby producing a path P_4 , and then repeating operation F_5 as required. Now, if $T \cong S_{x,y}$ with $x \geq y \geq 2$, then T can be obtained from a path P_3 by first applying operation F_3 , thereby producing a double star $S_{2,2}$ and then doing repeated applications of operation F_5 in both strong support vertices of the double star. Therefore, $T \in \mathcal{F}$.

We may now assume that $\text{diam}(T) \geq 4$, and we root the tree T at a vertex r located at the end of a longest path in T . Let h be a vertex at maximum distance from r . Notice that, necessarily, r and h are leaves (and diametral vertices). Let s be the parent of h ; let v be the parent of s ; let w be the parent of v ; and let z be the parent of w . Notice that all these vertices exist since $\text{diam}(T) \geq 4$, and it could happen $z = r$. Since h is a vertex at maximum distance from the root r , every child of s is a leaf. We proceed further with the following claims.

Claim I. If $\delta_T(s) \geq 4$, then $T \in \mathcal{F}$.

Proof. Suppose that $\delta_T(s) \geq 4$ and let $T' = T - h$. Thus, $\delta_{T'}(s) \geq 3$ and consequently, since every child of s is a leaf, s is a strong support vertex of T' . Therefore, by Observation 3, there exists a $\gamma_{\{R2\}}(T')$ -function, that assigns the weight 2 to s and the weight 0 to every leaf-neighbor of s in T' . The function above can be extended to a R2DF on T by assigning the weight 0 to the leaf h , implying that $\gamma_{\{R2\}}(T) \leq \gamma_{\{R2\}}(T')$. Thus, by Proposition 1, Observation 4, hypothesis and the previous inequality, we obtain $\gamma_{\{R2\}}(T') \leq \gamma_R(T') \leq \gamma_R(T) = \gamma_{\{R2\}}(T) \leq \gamma_{\{R2\}}(T')$. Hence, we must have equality throughout this inequality chain. In particular $\gamma_{\{R2\}}(T') = \gamma_R(T')$. Applying the inductive hypothesis to T' , it follows that $T' \in \mathcal{F}$. Since $s \in S_s(T')$, and by Observation 2, $s \in S_{2,R}(T')$. Therefore, T can be obtained from T' by Operation F_5 , and consequently, $T \in \mathcal{F}$. (\square)

By the proof of Claim I, we may henceforth assume that $|N(x) \cap L(T)| = 2$ for every strong support vertex x of T .

Claim II. If $\delta_T(s) = 3$ and $\delta_T(v) \geq 3$, then $T \in \mathcal{F}$.

Proof. Suppose that $\delta_T(s) = 3$ and $\delta_T(v) \geq 3$. Thus, s is a strong support vertex and has two leaf neighbours, say h, h_1 . Also, observe that v has at least one child, say s' , different from s , and moreover, s' is either a support vertex or a leaf vertex of T , according to the choice of r and h .

By using Theorem 1, there exists a $\gamma_{\{R2\}}(T)$ -function $f = (V_0, V_1, V_2)$ with $V_{0,1} = \emptyset$, and without loss of generality, we assume that $|V_2|$ is maximum. So, f is a $\gamma_R(T)$ -function as well. Let $T' = T - V(T_s) = T - \{h, h_1, s\}$.

Suppose first that f restricted to $V(T')$ is a RDF on T' , which means $\gamma_R(T') \leq \gamma_R(T) - 2$. Now, notice that every R2DF on T' can be extended to a R2DF of T by assigning the weight 2 to s and the weight 0 to every leaf-neighbor of s . Hence $\gamma_{\{R2\}}(T) \leq \gamma_{\{R2\}}(T') + 2$. Consequently, by these previous inequalities, and Proposition 1, we obtain $\gamma_{\{R2\}}(T) \leq \gamma_{\{R2\}}(T') + 2 \leq \gamma_R(T') + 2 \leq \gamma_R(T) = \gamma_{\{R2\}}(T)$. Therefore, we must have equality throughout this inequality chain. In particular, $\gamma_{\{R2\}}(T') = \gamma_R(T')$. Also, note that $\gamma_{\{R2\}}(T) = \gamma_{\{R2\}}(T') + 2$. Applying the inductive hypothesis to T' , it follows that $T' \in \mathcal{F}$. Moreover, every R2DF on $T' - v$ can be extended to T by assigning the weight 2 to s and the weight 0 to each neighbour of s , and so $\gamma_{\{R2\}}(T) \leq \gamma_{\{R2\}}(T' - v) + 2$. In addition, if v is not a stable vertex of T' , then $\gamma_{\{R2\}}(T' - v) < \gamma_{\{R2\}}(T')$, implying that $\gamma_{\{R2\}}(T) \leq \gamma_{\{R2\}}(T' - v) + 2 < \gamma_{\{R2\}}(T') + 2$, which is a contradiction with the related equality noticed above. Therefore, s is a stable vertex of T' , and hence, T can be obtained from the tree T' by applying operation F_3 . Thus, $T \in \mathcal{F}$.

Conversely, suppose that f restricted to $V(T')$ is not a RDF on T' . In this case, $f(v) = 0$. If there exists a vertex not leaf $x \neq w$ adjacent to v , then x is a support (by the choice of r and h), and it must happen $f(x) = 2$, which contradicts the fact that f restricted to $V(T')$ is not a RDF on T' . Thus, each

vertex adjacent to v , different to w , must be a leaf, and all of them must have weight 1. If $|(N(v) \setminus \{w\}) \cap L(T)| \geq 2$, then reassigning the weight 2 to v and the weight 0 to each vertex in $(N(v) \setminus \{w\}) \cap L(T)$, we produce a RDF on T of weight at most that of f , either contradicting our choice of f , or the fact that f is a $\gamma_{\{R2\}}(T)$ -function. Therefore $N(v) = \{w, s, s'\}$ and $f(s') = 1$. Now, let $T'' = T - V(T_v) = T - \{v, s', s, h, h_1\}$. Notice that every R2DF on T'' can be extended to a R2DF on T by assigning the weight 1 to s' ; the weight 2 to s ; and the weight 0 to each neighbor vertex of s . Hence, $\gamma_{\{R2\}}(T) \leq \gamma_{\{R2\}}(T'') + 3$. Moreover, since $f(v) = 0$, the labelling of vertices in $V(T'')$ under f is not influenced by the labels in $V(T_v)$. Thus, we deduce that f restricted to $V(T'')$ is a RDF on T'' , and so $\gamma_R(T'') \leq \gamma_R(T) - 3$. By using Proposition 1, hypothesis and the previous inequalities, we get $\gamma_{\{R2\}}(T'') \leq \gamma_R(T'') \leq \gamma_R(T) - 3 = \gamma_{\{R2\}}(T) - 3 \leq \gamma_{\{R2\}}(T'')$. So, we must have equality throughout this inequality chain. In particular, $\gamma_{\{R2\}}(T'') = \gamma_R(T'')$. Applying the inductive hypothesis to T'' , it follows that $T'' \in \mathcal{F}$. Since T can be obtained from the tree T'' by applying operation F_2 , we have $T \in \mathcal{F}$. (\square)

Claim III. If $\delta_T(s) = 3$ and $\delta_T(v) = 2$, then $T \in \mathcal{F}$.

Proof. Clearly, s is a strong support vertex and has two leaf neighbors, say h, h_1 . We now consider the tree $T' = T - V(T_v) = T - \{h, h_1, s, v\}$. Notice that a R2DF on T' can be extended to a R2DF of T by assigning the weight 2 to s and the weight 0 to every neighbor of s , implying that $\gamma_{\{R2\}}(T) \leq \gamma_{\{R2\}}(T') + 2$. By Observation 2, there exists a $\gamma_R(T)$ -function $f = (V_0, V_1, V_2)$ that assigns the weight 2 to s and the weight 0 to h and h_1 . Suppose $f(v) \geq 1$. First notice that by Proposition 2 (ii), $f(v) \neq 1$. So $f(v) = 2$. In consequence, it must be $f(w) = 0$ and as $epn(v, V_1 \cup V_2) \neq \emptyset$, we get $w \in epn(v, V_1 \cup V_2)$ (in fact $epn(v, V_1 \cup V_2) = \{w\}$ in such case). Now, by reassigning to v the weight 0 and to w the weight 1, and leaving all other weights unchanged, we produce a R2DF function of weight less than that of f , which is not possible. Hence $f(v) = 0$. Consequently, f restricted to $V(T')$ is a RDF on T' , and so $\gamma_R(T') \leq f(V(T')) = \omega(f) - f(N[s]) = \gamma_R(T) - 2$. By the previous inequalities, Proposition 1 and hypothesis, we obtain that $\gamma_{\{R2\}}(T) \leq \gamma_{\{R2\}}(T') + 2 \leq \gamma_R(T') + 2 \leq \gamma_R(T) = \gamma_{\{R2\}}(T)$. Therefore, we must have equality throughout this inequality chain. In particular, $\gamma_{\{R2\}}(T') = \gamma_R(T')$. Moreover, observe that the vertices $w, z \in V(T')$ and $V(T') \neq \{w, z\}$ (for otherwise, this implies that $T \notin T_{\{R2\}, R}$, which is a contradiction). Hence, we may assume that T' has order at least 3. Applying the inductive hypothesis to T' , it follows that $T' \in \mathcal{F}$. Since the tree T can be obtained from the tree T' by applying operation F_1 , we obtain $T \in \mathcal{F}$. (\square)

Claim IV. If $\delta_T(s) = 2$ and $\delta_T(v) \geq 3$, then $T \in \mathcal{F}$.

Proof. Clearly, v has at least one child, say s' , different from s , implying that s' is either a support vertex or a leaf vertex of T . We now consider the tree $T' = T - h$. Notice that a R2DF on T' can be extended to a R2DF of T by assigning the weight

1 to h , implying that $\gamma_{\{R2\}}(T) \leq \gamma_{\{R2\}}(T') + 1$.

By using Theorem 1, there exists a $\gamma_{\{R2\}}(T)$ -function $f = (V_0, V_1, V_2)$ with $V_{0,1} = \emptyset$, and without loss of generality, we assume that $|V_2|$ is maximum. Observe that f is a $\gamma_R(T)$ -function as well. If $f(v) = 1$, then $f(s) = 0$ and $f(h) = 1$, implying that $s \in V_{0,1}$, which is a contradiction. Thus $f(v) \in \{0, 2\}$. If $f(v) = 2$, then $f(s) = 0$ and $f(h) = 1$. Hence, f restricted to $V(T')$ is a RDF on T' and so, $\gamma_R(T') \leq \gamma_R(T) - 1$. Now suppose that $f(v) = 0$. Thus, $f(s) = 2$ and $f(h) = 0$, according to the choice of f with maximality in $|V_2|$. If s' is a leaf, then $f(s') = 1$ and reassigning the weight 2 to v ; the weight 0 to s and s' ; and the weight 1 to h ; we construct a RDF of weight equal to that of f . In consequence, by proceeding analogously to the case above ($f(v) = 2$) it follows that $\gamma_R(T') \leq \gamma_R(T) - 1$. Conversely, if s' is a support, then $f(s') = 2$ and reassigning the weight 1 to s and h , we produce again a RDF of weight equal to that of f . Observe that again, this new function restricted to $V(T')$ is a RDF on T' and so, $\gamma_R(T') \leq \gamma_R(T) - 1$.

Consequently, by the previous inequalities, and Proposition 1, we obtain that $\gamma_{\{R2\}}(T) \leq \gamma_{\{R2\}}(T') + 1 \leq \gamma_R(T') + 1 \leq \gamma_R(T) = \gamma_{\{R2\}}(T)$. Therefore, we must have equality throughout this inequality chain. In particular, $\gamma_{\{R2\}}(T') = \gamma_R(T')$. Also note that $\gamma_{\{R2\}}(T) = \gamma_{\{R2\}}(T') + 1$. Applying the inductive hypothesis to T' , it follows that $T' \in \mathcal{F}$.

Moreover, a minimum weight near-R2DF relative to s on T' can be extended to a R2DF on T by assigning to h the weight 1. So $\gamma_{\{R2\}}(T) \leq \gamma_{\{R2\}}^n(T', s) + 1$. If s is not a near stable vertex of T' , then $\gamma_{\{R2\}}^n(T', s) < \gamma_{\{R2\}}(T')$, implying that $\gamma_{\{R2\}}(T) \leq \gamma_{\{R2\}}^n(T', s) + 1 < \gamma_{\{R2\}}(T') + 1$, which is a contradiction with the related equality obtained before. Therefore, s is a near stable vertex, and also, a leaf vertex of T' . Since T can be obtained from the tree T' by applying operation F_6 , we get $T \in \mathcal{F}$.

Claim V. If $\delta_T(s) = 2$ and $\delta_T(v) = 2$, then $T \in \mathcal{F}$.

Proof. We consider the tree $T' = T - V(T_v) = T - \{h, s, v\}$. Notice that a R2DF on T' can be extended to a R2DF of T by assigning the weight 2 to s and the weight 0 to v and h , implying that $\gamma_{\{R2\}}(T) \leq \gamma_{\{R2\}}(T') + 2$.

By using Theorem 1, there exists a $\gamma_{\{R2\}}(T)$ -function f with $V_{0,1} = \emptyset$, and without loss of generality, we assume that $|V_2|$ is maximum. Notice that f is a $\gamma_R(T)$ -function as well. On the other hand, it is easy to check that $f(s) = 2$ and $f(h) = f(v) = 0$, implying that f restricted to $V(T')$ is a RDF on T' . Hence $\gamma_R(T') \leq f(V(T')) = \omega(f) - f(N[s]) = \gamma_R(T) - 2$. Consequently, by the previous inequalities and Proposition 1, we obtain that $\gamma_{\{R2\}}(T) \leq \gamma_{\{R2\}}(T') + 2 \leq \gamma_R(T') + 2 \leq \gamma_R(T) = \gamma_{\{R2\}}(T)$. Therefore, we must have equality throughout this inequality chain. In particular, $\gamma_{\{R2\}}(T') = \gamma_R(T')$. Also note that $\gamma_{\{R2\}}(T) = \gamma_{\{R2\}}(T') + 2$. By using a similar reasoning as in the previous case, T' has order at least 3 and applying the inductive hypothesis to T' , it follows that $T' \in \mathcal{F}$.

Moreover, a minimum weight near-R2DF on T' relative to w can be extended to a R2DF on T by assigning to v and h the weight 1 and to s the weight 0. So $\gamma_{\{R2\}}(T) \leq$

$\gamma_{\{R2\}}^n(T', w) + 2$. If w is not a near stable vertex of T' , then $\gamma_{\{R2\}}^n(T', w) < \gamma_{\{R2\}}(T')$, implying that $\gamma_{\{R2\}}(T) \leq \gamma_{\{R2\}}^n(T', w) + 2 < \gamma_{\{R2\}}(T') + 2$, which is a contradiction with the related equality noted before. Therefore, w is a near stable vertex of T' , and since T can be obtained from the tree T' by applying operation F_4 , we finally deduce $T \in \mathcal{F}$, which complete the proof. \square

As an immediate consequence of Lemmas 1 and 2, we have the following characterization concerning one of the main goals of this article.

Theorem 5. *A tree T of order $n \geq 3$ satisfies that $\gamma_{\{R2\}}(T) = \gamma_R(T)$ if and only if $T \in \mathcal{F}$.*

To conclude this work, we next give solution to the second problem which is studied in this work. To this end, we need the following known result.

Theorem 6. [2, 3] *For every tree T , $\gamma_{\{R2\}}(T) = \gamma_{r2}(T)$.*

Hence, as a consequence of Proposition 1, Theorem 5 and Theorem 6, the next characterization follows.

Theorem 7. *A tree T of order $n \geq 3$ satisfies that $\gamma_{r2}(T) = \gamma_R(T)$ if and only if $T \in \mathcal{F}$.*

References

- [1] J.D. Alvarado, S. Dantas, and D. Rautenbach, *Relating 2-rainbow domination to Roman domination*, Discuss. Math. Graph Theory **37** (2017), no. 4, 953–961.
- [2] B. Brešar, M.A. Henning, and D.F. Rall, *Rainbow domination in graphs*, Taiwanese J. Math. **12** (2008), no. 1, 213–225.
- [3] M. Chellali, T.W. Haynes, S.T. Hedetniemi, and A.A. McRae, *Roman $\{2\}$ -domination*, Discrete Appl. Math. **204** (2016), 22–28.
- [4] E.J. Cockayne, P.A. Dreyer Jr, S.M. Hedetniemi, and S.T. Hedetniemi, *Roman domination in graphs*, Discrete Math. **278** (2004), no. 1-3, 11–22.
- [5] O. Favaron, H. Karami, R. Khoeilar, and S.M. Sheikholeslami, *On the Roman domination number of a graph*, Discrete Math. **309** (2009), no. 10, 3447–3451.
- [6] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.
- [7] M.A. Henning, *A characterization of Roman trees*, Discuss. Math. Graph Theory **22** (2002), no. 2, 325–334.
- [8] M.A. Henning and W.F. Klostermeyer, *Italian domination in trees*, Discrete Appl. Math. **217** (2017), 557–564.

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- [9] W.F. Klostermeyer and G. MacGillivray, *Roman, Italian, and 2-domination*, Manuscript (2016).
 - [10] C.-H. Liu and G.J. Chang, *Upper bounds on Roman domination numbers of graphs*, Discrete Math. **312** (2012), no. 7, 1386–1391.
 - [11] I. Stewart, *Defend the Roman empire!*, Sci. Amer. **281** (1999), no. 6, 136–138.
 - [12] I.G. Yero and J.A. Rodríguez-Velázquez, *Roman domination in cartesian product graphs and strong product graphs*, Appl. Anal. Discrete Math. **7** (2013), no. 2, 262–274.