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Paired-domination game played in graphs*

Teresa W. Haynes[1,2] and Michael A. Henning[2]

[1]Department of Mathematics and Statistics, East Tennessee State University
Johnson City, TN 37614-0002 USA
haynes@etsu.edu

[2]Department of Pure and Applied Mathematics, University of Johannesburg
Auckland Park, 2006 South Africa
mahenning@uj.ac.za

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Dedicated to Prof. Dr. Lutz Volkmann, on the occasion of his 75th birthday.

Abstract: In this paper, we continue the study of the domination game in graphs introduced by Brešar, Klavžar, and Rall [SIAM J. Discrete Math. 24 (2010) 979–991]. We study the paired-domination version of the domination game which adds a matching dimension to the game. This game is played on a graph G by two players, named Dominator and Pairer. They alternately take turns choosing vertices of G such that each vertex chosen by Dominator dominates at least one vertex not dominated by the vertices previously chosen, while each vertex chosen by Pairer is a vertex not previously chosen that is a neighbor of the vertex played by Dominator on his previous move. This process eventually produces a paired-dominating set of vertices of G; that is, a dominating set in G that induces a subgraph that contains a perfect matching. Dominator wishes to minimize the number of vertices chosen, while Pairer wishes to maximize it. The game paired-domination number $\gamma_{\rm gpr}(G)$ of G is the number of vertices chosen when Dominator starts the game and both players play optimally. Let G be a graph on n vertices with minimum degree at least 2. We show that $\gamma_{\rm gpr}(G) \le \frac{4}{5}n$, and this bound is tight. Further we show that if G is (C_4, C_5)-free, then $\gamma_{\rm gpr}(G) \le \frac{3}{4}n$, where a graph is (C_4, C_5)-free if it has no induced 4-cycle or 5-cycle. If G is 2-connected and bipartite or if G is 2-connected and the sum of every two adjacent vertices in G is at least 5, then we show that $\gamma_{\rm gpr}(G) \le \frac{3}{4}n$.

Keywords: Paired-domination game; Paired-domination number; Domination game

AMS Subject classification: 05C69

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1. Introduction

The domination game in graphs was first introduced by Brešar, Klavžar, and Rall [7] and extensively studied afterwards in [4–6, 8, 9, 12, 17, 32, 33] and elsewhere. Before formally defining the domination game, we briefly describe basic concepts needed throughout the paper. For notation and graph theory terminology not defined herein, we in general follow [31]. We denote the *degree* of a vertex v in a graph G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from the context. The minimum degree among the vertices of G is denoted by $\delta(G)$. A vertex of degree 1 is called a *leaf*. A *cycle component* and a *path component* of a graph is a component in the graph that is isomorphic to a cycle and a path, respectively. A set S of edges in a graph G are *independent* if no two edges in S are incident to the same vertex. A *matching* in a graph G is a set of independent edges in G. A *perfect matching* M in G is a matching in G such that every vertex of G is incident to an edge of M.

A vertex *dominates* itself and its neighbors. A *dominating set* of a graph G is a set S of vertices of G such that every vertex in G is dominated by a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. We call a dominating set of cardinality $\gamma(G)$ in G a γ*-set of* G. The notion of domination and its variations in graphs and has been studied a great deal; a rough estimate says that it occurs in more than 6000 papers to date. Fundamental concepts of domination in graphs can be found in [22].

A dominating set S with the additional property that the subgraph $G[S]$ induced by S contains a perfect matching M (not necessarily induced) is a *paired-dominating set* of G. Two vertices joined by an edge of M are said to be *paired*. The *paired-domination number* $\gamma_{\rm pr}(G)$ of G is the minimum cardinality of a paired-dominating set in G. Haynes and Slater [23] introduced the concept of paired-domination in graphs as a model for assigning backups to guards for security purposes. A recent survey of paired-domination in graphs can be found in [15].

The *domination game* played on a graph G consists of two players, *Dominator* and *Staller*, who take turns choosing a vertex from G. Each vertex chosen must dominate at least one vertex not dominated by the vertices previously chosen. The game ends when the set of vertices chosen becomes a dominating set in G. Dominator wishes to minimize the number of vertices chosen, while Staller wishes to end the game with as many vertices chosen as possible. The *game domination number* $\gamma_g(G)$ of G is the number of vertices chosen when Dominator starts the game and both players play optimally.

In this paper we introduce and study the paired version of the domination game. The *paired-domination game*, played on a graph G consists of two players called *Dominator* and *Pairer* who take turns choosing a vertex from G. In this version of the game, each vertex chosen by Dominator must dominate at least one vertex not dominated by the vertices previously chosen, while each vertex chosen by Pairer must be a neighbor of the vertex chosen by Dominator on his previous move that has not previously been chosen. The vertex played by Pairer, together with the vertex played on the previous move by Dominator, are said to be *partners*. A vertex is *unpaired* if it is chosen in a move played by Dominator but does not have a partner. This process eventually produces a paired-dominating set of vertices of G, in which the partners form a matching in the subgraph induced by the set. Dominator wishes to minimize the number of vertices chosen, while Pairer wishes to maximize it. The *game paired-domination number* $\gamma_{\rm gpr}(G)$ of G is the number of vertices chosen when Dominator starts the game and both players play optimally. The sequence of moves of the two players will be denoted $d_1, p_1, d_2, p_2, d_3, p_3, \ldots$; that is, the ith vertex played by Dominator is the vertex d_i and the ith vertex played by Pairer is the vertex p_i. We note that the vertices d_i and p_i are adjacent, and are partners.

The paired-domination game belongs to the growing family of *competitive optimization* games on graphs and hypergraphs. As remarked in [24], broadly speaking, "competitive optimization" describes a process in which multiple agents with conflicting goals collaboratively produce some special structure in an underlying host graph/hypergraph. In the paired-domination game, that structure is a paired-dominating set, and the players' goals are completely antithetical: while Pairer wants to maximize the size of a paired-dominating set constructed during the game, Dominator wants to minimize it. Thus, the paired-domination game is a competitive optimization variant of the well-studied paired-domination problem on graphs.

As remarked in [10, 24] and elsewhere, one of the first and best-known competitive optimization parameters is the *game chromatic number*, which was introduced by Brams for planar graphs (cf. [18]) and independently by Bodlaender [1] for general graphs; it has seen extensive study, see the survey [36]. Recently, work has been done on competitive optimization variants of list-colouring [3] and its more studied related version called paintability as introduced in [35] (for further references see Section 8 of [36]), matching [14, 19], domination [7, 24], total domination [16, 25–29], disjoint domination [12], Ramsey theory [13, 20, 21], transversals in hypergraphs [10, 11] and more [2].

If a graph G does not contain a graph F as an induced subgraph, then we say that G is *F-free*. We say that G is *(C_4, C_5)-free* if G is both C_4-free and C_5-free; that is, if G has no induced 4-cycle and no induced 5-cycle. By *contracting* two vertices x and y in G, we mean replacing the vertices x and y by a new vertex v_{xy} and joining v_{xy} to all vertices that were adjacent to x or y in G.

In this paper, we introduce and study the paired-domination version of the domination game. In Section 2, we present some preliminary observations. In Section 3, we present some known results on the domination number that we will need in proving our main results, which are stated in Section 4. Thereafter we present proofs of our main results in Section 5. We close in Section 6 with two conjectures that we have yet to settle.

2. Preliminary Observations

As observed earlier, upon completion of the paired-domination game played on a graph G the resulting set of played vertices is a paired-dominating set of G. Hence, the paired-domination of a graph is at most its game paired-domination number.

Observation 1. For every graph G with no isolated vertex, $\gamma_{\rm pr}(G) \leq \gamma_{\rm gpr}(G)$.

Haynes and Slater [23] were the first to observe that if G is a connected graph of order $n \geq 3$, then $\gamma_{\rm pr}(G) \leq n-1$. That this bound is tight may be seen by subdividing each edge of a star $K_{1,r}$, where $r \geq 1$, precisely once. The graph G obtained from an arbitrary connected graph H by attaching a pendant edge to each vertex has game paired-domination number equal to its order, since whenever Dominator plays a vertex of H, Pairer responds by playing its leaf neighbor in G. This implies the following observation.

Observation 2. If G is a graph of order n with no isolated vertex, then $\gamma_{\rm gpr}(G) \leq n$, and this bound is tight.

3. Known Results

In this section, we present some known results on the domination number in graphs with minimum degree at least 2. If we restrict the minimum degree to be at least 2 and the order to be at least 6, then Haynes and Slater [23] established the following upper bound on the paired-domination number.

Theorem 3. ([23]) *If G is a connected graph of order $n \geq 6$ with $\delta(G) \geq 2$, then $\gamma_{\rm pr}(G) \leq \frac{2}{3}n$.*

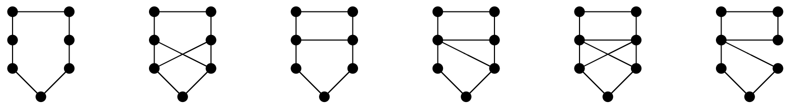
In [30] the authors define two types of reducible graphs and use these reductions to define a family $\mathcal{F}$ of graphs. For completeness, we repeat the definitions here.

Definition 1. ([30]) If there is a path $v_1u_1u_2v_2$ on four vertices in a graph G such that $d(u_1) = d(u_2) = 2$ in G, then the graph obtained from G by contracting v_1 and v_2 and deleting $\{u_1, u_2\}$ is called a *type-1 G-reducible graph*.

Definition 2. ([30]) If there is a path $x_1w_1w_2w_3x_2$ on five vertices in a graph G such that $d(w_2) = 2$ and $N(w_1) = N(w_3) = \{x_1, x_2, w_2\}$ in G, then the graph obtained from G by deleting $\{w_1, w_2, w_3\}$ and adding the edge x_1x_2 if the edge is not already present in G is called a *type-2 G-reducible graph*.

Definition 3. ([30]) Let $\mathcal{F}_4$ be a set of graphs only containing one element, namely the 4-cycle C_4. Thus, $\mathcal{F}_4 = \{C_4\}$. For every $i > 4$ with $i \equiv 1 \pmod 3$, we define the family $\mathcal{F}_i$ as follows. A graph G belongs to $\mathcal{F}_i$ if and only if $\delta(G) \geq 2$ and there is a type-1 or a type-2 G-reducible graph that belongs to $\mathcal{F}_{i-3}$.

The six graphs in the family $\mathcal{F}_7$ are shown in Figure 1.



(a) (b) (c) (d) (e) (f)

Figure 1. **The family $\mathcal{F}_7$.**

Definition 4. ([30]) Let $\mathcal{F}_{\le 13} = \mathcal{F}_4 \cup \mathcal{F}_7 \cup \mathcal{F}_{10} \cup \mathcal{F}_{13}$.

Definition 5. ([30]) A vertex x in a graph G is a *bad-cut-vertex* of G if $G - x$ contains a component C_x, which is an induced 4-cycle such that x is adjacent to at least one but at most three vertices on C_x. Let $\mathrm{bc}(G)$ denote the number of bad-cut-vertices in G.

As remarked in [30], there are 28076 non-isomorphic graphs in the family $\mathcal{F}_{\le 13}$, and 41 of these graphs possess bad-cut-vertices. Let

$$\mathcal{F} = \{G \in \mathcal{F}_{\le 13} \mid \mathrm{bc}(G) = 0\};$$

that is, $\mathcal{F}$ consists of the 28035 non-isomorphic graphs in the family $\mathcal{F}_{\le 13}$ that do not have a bad-cut-vertex. We note that $\mathcal{F}_4 \cup \mathcal{F}_7 \subset \mathcal{F}$. We shall need the following properties of graphs in the family $\mathcal{F}$.

Lemma 1. ([30]) *If $G \in \mathcal{F}$ has order n, and u and v are arbitrary distinct vertices in G, then the following holds.*

1. *$\gamma(G) = \frac{1}{3}(n + 2)$.*
2. *There is a γ-set of G containing both u and v.*

In 1989, McCuaig and Shepherd [34] presented the classical result that the domination number of a connected graph with minimum degree at least 2 is at most two-fifths its order except for seven exceptional graphs. These seven exceptional graphs are precisely the graphs in the family $\mathcal{F}_4 \cup \mathcal{F}_7$. Hence the McCuaig-Shepherd result can be stated as follows:

Theorem 4. ([34]) *If G is a connected graph of order n with $\delta(G) \ge 2$ and $G \notin \mathcal{F}_4 \cup \mathcal{F}_7$, then $\gamma(G) \le \frac{2}{5}n$.*

We shall also need the following structural result given in [30].

Theorem 5. ([30]) *If $G \notin \mathcal{F}$ is a connected graph of order n with $\delta(G) \ge 2$, then the following holds.*

1. *If (C_4, C_5)-free, then $\gamma(G) \le \frac{3}{8}n$.*

2. *If G is bipartite and* $\mathrm{bc}(G) = 0$*, then* $\gamma(G) \leq \frac{3}{8}n$*.*

3. *If G is* 2*-connected and bipartite, then* $\gamma(G) \leq \frac{3}{8}n$*.*

4. *If G is* 2*-connected and* $d_G(u) + d_G(v) \geq 5$ *for every two adjacent vertices u and v, then* $\gamma(G) \leq \frac{3}{8}n$*.*

4. Main Results

In view of Observation 2 it is only of interest to determine upper bounds on the game paired-domination number of a graph with minimum degree at least 2. By Observation 2 and Theorem 3, the best we can hope for is an upper bound of two-thirds the order of the graph. We show that this is not possible. However, we prove that Dominator always has a strategy that will finish the game in at most four-fifths the order of the graph. A proof of Theorem 6 is given in Section 5.

Theorem 6. *If G is a connected graph on n vertices with* $\delta(G) \geq 2$*, then* $\gamma_{\mathrm{gpr}}(G) \leq \frac{4}{5}n$*, and this bound is tight.*

If we impose certain structural restrictions on the graph, then the $\frac{4}{5}$-upper bound on the game paired-domination number given in Theorem 6 can be improved to a $\frac{3}{4}$-upper bound. We state this result formally as follows. A proof of Theorem 7 is given in Section 5.

Theorem 7. *If G is a connected graph on n vertices with* $\delta(G) \geq 2$*, then the following holds.*

1. *If G is* (C_4, C_5)*-free, then* $\gamma_{\mathrm{gpr}}(G) \leq \frac{3}{4}n$*.*

2. *If G is bipartite and* $\mathrm{bc}(G) = 0$*, then* $\gamma_{\mathrm{gpr}}(G) \leq \frac{3}{4}n$*.*

3. *If G is* 2*-connected and bipartite, then* $\gamma_{\mathrm{gpr}}(G) \leq \frac{3}{4}n$*.*

4. *If G is* 2*-connected and* $d_G(u) + d_G(v) \geq 5$ *for every two adjacent vertices u and v, then* $\gamma_{\mathrm{gpr}}(G) \leq \frac{3}{4}n$*.*

5. Proof of Main Results

In this section we present a proof of our main results, namely Theorem 6 and Theorem 7. For this purpose, we shall need the following lemma.

Lemma 2. *If* $G \in \mathcal{F}$ *has order n, then* $\gamma_{\mathrm{gpr}}(G) \leq \frac{2}{3}(n-1)$*.*

Proof. Let $G \in \mathcal{F}$ have order n. Let $\gamma = \gamma(G)$. We note that $n \geq 4$ and $n \equiv 1 \pmod 3$. Further by Lemma 1, $\gamma = \frac{1}{3}(n+2)$. Recall that we denote the sequence of moves of the two players by $d_1, p_1, d_2, p_2, d_3, p_3, \ldots$ where d_i and p_i are the ith moves played by Dominator and Pairer, respectively. We note that the vertices d_i and p_i are partners. In particular, the vertex p_i is a neighbor of the vertex d_i.

Dominator's strategy is to choose as his first vertex an arbitrary vertex, say u, of G. Let v be the first vertex chosen by Pairer. Thus, $d_1 = u$ and $p_1 = v$. By Lemma 1, there is a γ-set, D say, of G that contains both vertices u and v. Thus, $\{u, v\} \subseteq D$ and $|D| = \gamma$. If $\gamma = 2$, then $D = \{u, v\}$ and $\gamma_{\rm gpr}(G) = 2 \le \frac{2}{3}(n-1)$ noting that $n \ge 4$. Hence, we may assume that $\gamma \ge 3$, for otherwise the desired result holds. Let $D = \{u, v\} \cup D'$, where $|D'| = \gamma - 2$ and $D' = \{v_1, \ldots, v_{\gamma-2}\}$.

Dominator now orders the vertices in D' and plays the vertices $v_1, v_2, \ldots, v_{\gamma-2}$ sequentially when it is his turn to move, provided none of these vertices were previously chosen by Pairer on one of her earlier moves and provided that each vertex played by Dominator dominates at least one new vertex at that stage of the game. If, however, one of these vertices were already chosen by Pairer on one of her previous moves or if such a vertex does not dominate at least one new vertex at that stage of the game, then Dominator plays the next available vertex in the ordering $v_1, v_2, \ldots, v_{\gamma-2}$ that has not yet been played and that dominates at least one new vertex at that stage of the game.

In the case when Dominator plays all vertices in D', the sequence of moves $d_2, d_3, \ldots, d_{\gamma-1}$ of Dominator correspond to the sequence of moves $v_1, v_2, \ldots, v_{\gamma-2}$; that is, $d_{i+1} = v_i$ for $i \in [\gamma - 2]$. In the case when a vertex of D' has already been played by Pairer or a vertex in D' cannot be played by Dominator since it dominates no new vertex at that stage of the game, Dominator plays at most $\gamma - 3$ vertices from the set D' and therefore at most $\gamma - 2$ vertices in total. Hence, Dominator's strategy of playing the vertices $v_1, v_2, \ldots, v_{\gamma-2}$ sequentially when it is his turn to move, if possible, guarantees that the game will finish after at most $2(\gamma - 1)$ moves. Hence, $\gamma_{\rm gpr}(G) \le 2(\gamma - 1) = 2(\frac{1}{3}(n+2) - 1) = \frac{2}{3}(n-1)$. □

We are now in a position to present a proof of Theorem 6. Recall its statement.

Theorem 6 *If G is a connected graph on n vertices with $\delta(G) \ge 2$, then $\gamma_{\rm gpr}(G) \le \frac{4}{5}n$, and this bound is tight.*

Proof. If $G \in \mathcal{F}_4 \cup \mathcal{F}_7$, then noting that $\mathcal{F}_4 \cup \mathcal{F}_7 \subset \mathcal{F}$, Lemma 2 implies that $\gamma_{\rm gpr}(G) \le \frac{2}{3}(n-1) < \frac{4}{5}n$. Hence, we may assume that $G \notin \mathcal{F}_4 \cup \mathcal{F}_7$, for otherwise the desired upper bound follows. With this assumption, Theorem 4 implies that $\gamma(G) \le \frac{2}{5}n$.

We now prove that Dominator always has a strategy that will finish the game in at most four-fifths the order of the graph. Dominator's strategy is to choose an arbitrary γ-set, D say, of G. Let $D = \{v_1, v_2, \ldots, v_\gamma\}$, where $\gamma = \gamma(G)$. Dominator now orders the vertices in D and plays the vertices $v_1, v_2, \ldots, v_\gamma$ sequentially when it is his turn to move, provided none of these vertices were previously chosen by Pairer on one of her earlier moves. If, however, one of these vertices has already been chosen by Pairer, then Dominator simply plays the next available vertex in the ordering $v_1, v_2, \ldots, v_\gamma$ that has not yet been played and that dominates at least one new vertex at that stage of the game.

In the case when Dominator plays all vertices in D, the sequence of moves $d_1, d_2, \ldots, d_\gamma$ of Dominator correspond to the sequence of moves $v_1, v_2, \ldots, v_\gamma$; that is, $d_i = v_i$ for $i \in [\gamma]$. In the case when a vertex of D has already been played by Pairer or a vertex in D cannot be played by Dominator since it dominates no new vertices at that stage of the game, Dominator plays at most $\gamma - 1$ vertices in the course of the game. Therefore, Dominator's strategy of playing the vertices $v_1, v_2, \ldots, v_\gamma$ sequentially when it is his turn to move, if possible, guarantees that the game will finish after at most 2γ moves. Hence, $\gamma_{\rm gpr}(G) \le 2\gamma \le 2 \cdot \frac{2}{5}n = \frac{4}{5}n$. This establishes the desired upper bound.

We prove next that the bound is tight. Let $\mathcal{G}$ be the family of connected graphs with minimum degree at least 2 constructed as follows. For $k \ge 1$, let G_k be a graph obtained from the vertex disjoint union of k 5-cycles by selecting any two non-adjacent vertices from each cycle and designating them as *gluing vertices* and then adding any number of edges joining gluing vertices so that the resulting graph G_k is connected. We note that G_k is a graph on $n = 5k$ vertices with $\delta(G) \ge 2$. When $k = 4$, an example of a graph G_4 constructed in this way is illustrated in Figure 2, where the gluing vertices are depicted by solid (darkened) vertices. Let $\mathcal{G}$ be the family of all such graphs G_k where $k \ge 1$.

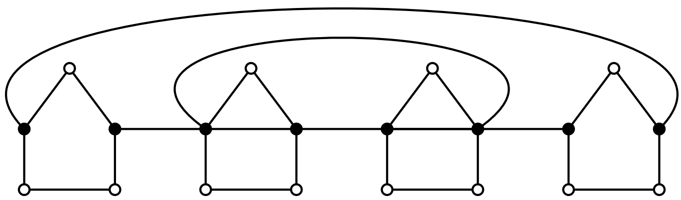


Figure 2. **A graph G_4 in the family $\mathcal{G}$**

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Let G be an arbitrary graph in $\mathcal{G}$ and let G have order n. Thus, $G = G_k$ for some integer $k \ge 1$. Pairer adopts the following strategy. Let $C\colon v_1 v_2 \ldots v_5 v_1$ be an arbitrary 5-cycle used in the construction of the graph G, where v_1 and v_3 are the gluing vertices selected from C. Pairer waits for Dominator to be the first to play a vertex from the cycle C. Let v be the first move that Dominator plays from the cycle C.

If $v = v_1$, then Pairer responds to Dominator's move by playing the vertex v_2. Since the vertex v_4 is not yet dominated, Dominator must play a second move in C, namely one of the vertices v_3, v_4 or v_5 in order to dominate v_4. If Dominator plays v_3 or v_5, then Pairer responds by playing v_4, while if Dominator plays v_4, then Pairer responds by playing v_3. Analogously, if $v = v_3$, Pairer guarantees that at least four vertices will be played from C. If $v = v_2$, then Pairer responds to Dominator's move by playing the vertex v_1, and as before she can guarantee that at least four vertices will be played from C.

If $v = v_4$, then Pairer responds to Dominator's move by playing the vertex v_5. Since the vertex v_2 is not yet dominated, Dominator must play a second move in C, namely one of the vertices v_1, v_2 or v_3 in order to dominate v_1. If Dominator plays v_1 or v_3, then Pairer responds by playing v_2, while if Dominator plays v_2, then Pairer responds by playing v_1 (or v_3). In this way, Pairer has a strategy that will force at least four vertices to be played from C. Since C is an arbitrary 5-cycle used in the construction of the graph G, Pairer has a strategy that will force at least $4k$ vertices to be played in G. Thus, $\gamma_{\rm gpr}(G) \ge 4k = \frac{4}{5}n$. As observed earlier, $\gamma_{\rm gpr}(G) \le \frac{4}{5}n$. Consequently, $\gamma_{\rm gpr}(G) = \frac{4}{5}n$. Since G was an arbitrary graph in the family $\mathcal{G}$, every graph in $\mathcal{G}$ has game paired-domination number equal to four-fifths its order. This establishes tightness of the $\frac{4}{5}n$-upper bound. $\Box$

Next we present a proof of Theorem 7. Recall its statement.

Theorem 7 *If G is a connected graph on n vertices with $\delta(G) \ge 2$, then the following holds.*

1. *If G is (C_4, C_5)-free, then $\gamma_{\rm gpr}(G) \le \frac{3}{4}n$.*
2. *If G is bipartite and $\mathrm{bc}(G) = 0$, then $\gamma_{\rm gpr}(G) \le \frac{3}{4}n$.*
3. *If G is 2-connected and bipartite, then $\gamma_{\rm gpr}(G) \le \frac{3}{4}n$.*
4. *If G is 2-connected and $d_G(u) + d_G(v) \ge 5$ for every two adjacent vertices u and v, then $\gamma_{\rm gpr}(G) \le \frac{3}{4}n$.*

Proof. Let G be a connected graph on n vertices with $\delta(G) \ge 2$. Suppose that $G \in \mathcal{F}$. By Lemma 2, $\gamma_{\rm gpr}(G) \le \frac{2}{3}(n-1) < \frac{3}{4}n$. Hence, we may assume that $G \notin \mathcal{F}$, for otherwise the desired upper bound follows. With this assumption, Theorem 5 implies that if G is a (C_4, C_5)-free graph or if G is bipartite and $\mathrm{bc}(G) = 0$ or if G is 2-connected and bipartite or if G is 2-connected and $d_G(u) + d_G(v) \ge 5$ for every two adjacent vertices u and v, then $\gamma(G) \le \frac{3}{8}n$. Dominator's strategy is now identical to that presented in the proof of Theorem 6. He chooses an arbitrary γ-set, D say, of G and orders the vertices in D. On each of his moves, Dominator plays the next available vertex in the ordering of vertices of D that has not yet been played and that dominates at least one new vertex at that stage of the game. This strategy of Dominator guarantees that the game will finish after at most 2γ moves. Hence, $\gamma_{\rm gpr}(G) \le 2\gamma \le 2 \cdot \frac{3}{8}n = \frac{3}{4}n$. This completes the proof of the theorem. $\Box$

We show next that the bounds of Theorem 7(a) and 7(b) are tight.

Proposition 1. *There exists an infinite family $\mathcal{B}_{\rm bip}$ of connected, bipartite, C_4-free graphs with minimum degree at least 2 such that if $F \in \mathcal{B}_{\rm bip}$ has order n, then $\gamma_{\rm gpr}(F) = \frac{3}{4}n$.*

Proof. Let $\mathcal{B}_{\rm bip}$ be the family of connected, bipartite, C_4-free graphs with minimum degree at least 2 constructed as follows. For $k \ge 1$, let B_k be a bipartite graph obtained from the vertex disjoint union of k 8-cycles by selecting one vertex from each cycle and designating it as a *gluing vertex* and then adding any number of edges joining gluing vertices so that the subgraph induced by the gluing vertices is a connected, C_4-free, bipartite graph. We call the subgraph of B_k induced by the k selected gluing vertices an *underlying graph* of B_k. When $k = 4$, an example of a graph B_4 constructed in this way is illustrated in Figure 3, where the gluing vertices are depicted by four large solid vertices and the underlying graph is a path P_4. Let $\mathcal{B}_{\text{bip}}$ be the family of all such graphs B_k where $k \geq 1$.

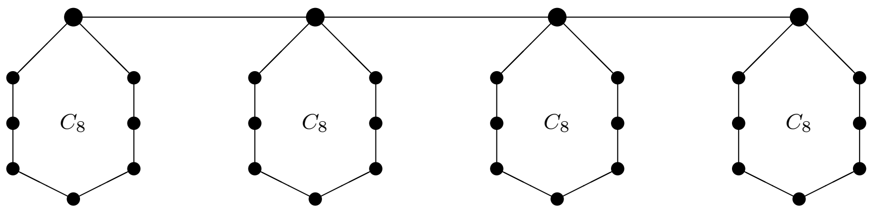

Figure 3. **A graph B_4 in the family $\mathcal{B}_{\text{bip}}$.**

Let F be an arbitrary graph in $\mathcal{B}_{\text{bip}}$ and let F have order n. Thus, $F = B_k$ for some integer $k \geq 1$, and so $n = 8k$. Pairer adopts the following strategy. Let $C\colon v_1v_2\ldots v_8v_1$ be an arbitrary 8-cycle used in the construction of the graph F. By construction, exactly one vertex from C is the gluing vertex. Renaming vertices, if necessary, we may assume that v_8 is the gluing vertex. Thus, the vertex v_i has degree 2 in both C and F for all $i \in [7]$.

We show that Pairer has a strategy to force six vertices on the cycle C to be played. Pairer waits for Dominator to be the first to play a vertex from the cycle C. Let $d_1, d_2, d_3, \ldots$ be the sequence of moves that Dominator plays on C, and let $p_1, p_2, p_3, \ldots$ be the response of Pairer to Dominator's moves; that is, when Dominator plays the vertex d_i, then Pairer responds by playing the vertex p_i.

Suppose that $d_1 = v_8$. Pairer responds by playing $p_1 = v_1$. If $d_2 = v_3$, then Pairer plays $p_2 = v_2$. If $d_2 \in \{v_2, v_4\}$, then Pairer plays $p_2 = v_3$. If $d_2 = v_6$, then Pairer plays $p_2 = v_7$. If $d_2 \in \{v_5, v_7\}$, then Pairer plays $p_2 = v_6$. In all cases, at least one vertex on C has yet to be dominated, forcing Dominator to play a third vertex on the cycle C. Pairer responds by playing her third vertex on the cycle C. Thus, at least six vertices on the cycle C are played. Suppose that $d_1 \in \{v_1, v_7\}$. Pairer responds by playing $p_1 = v_8$, and identical arguments as before show that Pairer can force at least six vertices on C to be played.

Suppose that $d_1 = v_2$. Pairer responds by playing $p_1 = v_1$. If $d_2 = v_7$, then Pairer plays $p_2 = v_6$. If $d_2 \in \{v_6, v_8\}$, then Pairer plays $p_2 = v_7$. If $d_2 = v_4$, then Pairer plays $p_2 = v_3$. If $d_2 \in \{v_3, v_5\}$, then Pairer plays $p_2 = v_4$. In all cases, at least one vertex on C has yet to be dominated, forcing Dominator to play a third vertex on the cycle C. Pairer responds by playing her third vertex on the cycle C. Analogously, if $d_1 = v_6$, then Pairer can force at least six vertices on C to be played.

Suppose that $d_1 = v_3$. Pairer responds by playing $p_1 = v_4$. If $d_2 = v_1$, then Pairer

plays $p_2 = v_2$. If $d_2 \in \{v_2, v_8\}$, then Pairer plays $p_2 = v_1$. If $d_2 = v_6$, then Pairer plays $p_2 = v_5$. If $d_2 \in \{v_5, v_7\}$, then Pairer plays $p_2 = v_6$. In all cases, at least one vertex on C has yet to be dominated, forcing Dominator to play a third vertex on the cycle C. Pairer responds by playing her third vertex on the cycle C. Analogously, if $d_1 = v_5$, then Pairer can force at least six vertices on C to be played. Suppose that $d_1 = v_4$. Pairer responds by playing $p_1 = v_3$, and analogously as in the case when $d_1 = v_3$ and $p_1 = v_4$, Pairer can force at least six vertices on C to be played.
In this way, Pairer has a strategy that will force at least six vertices to be played from C. Since C is an arbitrary 8-cycle used in the construction of the graph F, Pairer has a strategy that will force at least $6k$ vertices to be played in F. Thus, $\gamma_{\rm gpr}(F) \ge 6k = \frac{3}{4}n$. By Theorem 7(a) and 7(b), $\gamma_{\rm gpr}(F) \le \frac{3}{4}n$. Consequently, $\gamma_{\rm gpr}(F) = \frac{3}{4}n$. Since F was an arbitrary graph in the family $\mathcal{B}_{\rm bip}$, every graph in the $\mathcal{B}_{\rm bip}$ has game paired-domination number equal to three-fourths its order. □

We show next that the bound of Theorem 7(c) is tight in that there exists an infinite family of graphs achieving equality in this bound.

Proposition 2. *There exists an infinite family $\mathcal{H}_{\rm 2conn}$ of 2-connected, bipartite graphs such that if $H \in \mathcal{H}_{\rm 2conn}$ has order n, then $\gamma_{\rm gpr}(H) = \frac{3}{4}n$.*

Proof. Let $\mathcal{H}_{\rm 2conn}$ be the family of 2-connected bipartite graphs constructed as follows. For $k \ge 2$, let H_k be the graph obtained from a cycle C_{2k} on $2k$ vertices as follows. Let M be a perfect matching in the cycle. For each edge $e = uv$ in the matching M, duplicate the edge e, subdivide one of the duplicated edges twice and subdivide the other duplicated edge four times. (Hence each edge uv is deleted from H and replaced by an 8-cycle containing u and v as vertices at distance 3 apart on the cycle.) Let H_k denote the resulting graph of order $n = 8k$. The graph H_4 in the family $\mathcal{H}_{\rm 2conn}$ obtained from an 8-cycle is shown in Figure 4. We call the original vertices of the cycle C_{2k} the *gluing vertices* of H_k.

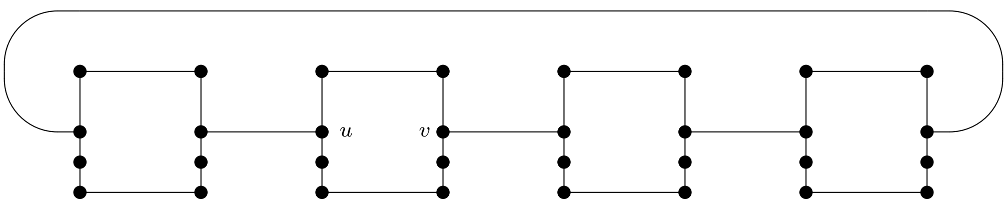

Figure 4. **The graph H_4 in the family $\mathcal{H}_{\rm 2conn}$.**

Let H be an arbitrary graph in $\mathcal{H}_{\rm 2conn}$ and let H have order n. Thus, $H = H_k$ for some integer $k \ge 2$, and so $n = 8k$. Pairer adopts the following strategy. Let $C\colon v_1v_2\dots v_8v_1$ be an arbitrary 8-cycle used in the construction of the graph H. Renaming vertices, if necessary, we may assume that v_3 and v_8 are the gluing vertices of H that belong to the cycle C. We show that Pairer has a strategy to force six vertices on the cycle C to be played. Pairer waits for Dominator to be the first to play a vertex from the cycle C. Let $d_1, d_2, d_3, \ldots$ be the sequence of moves that Dominator plays on C, and let $p_1, p_2, p_3, \ldots$ be the response of Pairer to these moves of Dominator.

Suppose that $d_1 = v_1$. Pairer responds by playing $p_1 = v_2$. If $d_2 \in \{v_3, v_5\}$, then Pairer plays $p_2 = v_4$. If $d_2 \in \{v_6, v_8\}$, then Pairer plays $p_2 = v_7$. If $d_2 = v_4$, then Pairer plays $p_2 = v_3$. If $d_2 = v_7$, then Pairer plays $p_2 = v_8$. In all cases, at least one vertex on C has yet to be dominated, forcing Dominator to play a third vertex on the cycle C. Pairer responds by playing her third vertex on the cycle C. Thus, at least six vertices on the cycle C are played. If $d_1 = v_2$, then Pairer responds by playing $p_1 = v_1$ and, analogously as above, Pairer can force at least six vertices on C to be played.

Suppose that $d_1 = v_3$. Pairer responds by playing $p_1 = v_4$. If $d_2 \in \{v_5, v_7\}$, then Pairer plays $p_2 = v_6$. If $d_2 \in \{v_2, v_8\}$, then Pairer plays $p_2 = v_1$. If $d_2 = v_6$, then Pairer plays $p_2 = v_5$. If $d_2 = v_1$, then Pairer plays $p_2 = v_2$. In all cases, at least one vertex on C has yet to be dominated, forcing Dominator to play a third vertex on the cycle C. Pairer responds by playing her third vertex on the cycle C. Thus, at least six vertices on the cycle C are played. If $d_1 = v_4$, then Pairer responds by playing $p_1 = v_3$ and, analogously as above, Pairer can force at least six vertices on C to be played. If $d_1 \in \{v_7, v_8\}$, then analogously as above when $d_1 \in \{v_3, v_4\}$, Pairer can force at least six vertices on C to be played.

Suppose that $d_1 = v_5$. Pairer responds by playing $p_1 = v_4$. If $d_2 \in \{v_1, v_3\}$, then Pairer plays $p_2 = v_2$. If $d_2 \in \{v_6, v_8\}$, then Pairer plays $p_2 = v_7$. If $d_2 = v_7$, then Pairer plays $p_2 = v_6$. If $d_2 = v_2$, then Pairer plays $p_2 = v_3$. In all cases, at least one vertex on C has yet to be dominated, forcing Dominator to play a third vertex on the cycle C. Pairer responds by playing her third vertex on the cycle C. Thus, at least six vertices on the cycle C are played. If $d_1 = v_6$, then Pairer responds by playing $p_1 = v_7$ and, analogously as above, Pairer can force at least six vertices on C to be played.

In this way, Pairer has a strategy that will force at least six vertices to be played from C. Since C is an arbitrary 8-cycle used in the construction of the graph H, Pairer has a strategy that will force at least $6k$ vertices to be played in H. Thus, $\gamma_{\rm gpr}(H) \ge 6k = \frac{3}{4}n$. By Theorem 7(c), $\gamma_{\rm gpr}(H) \le \frac{3}{4}n$. Consequently, $\gamma_{\rm gpr}(H) = \frac{3}{4}n$. Since H was an arbitrary graph in the family $\mathcal{H}$, every graph in the $\mathcal{H}$ has game paired-domination number equal to three-fourths its order. $\square$

6. Closing Conjectures

In this paper we introduce and study the paired-domination version of the domination game. We show in Theorem 6 that if G is a connected graph on n vertices with $\delta(G) \ge 2$, then $\gamma_{\rm gpr}(G) \le \frac{4}{5}n$, and this bound is tight. We also show that if we impose certain structural restrictions on the graph, then the $\frac{4}{5}$-upper bound can be improved to a $\frac{3}{4}$-upper bound. We pose the following conjecture that we have yet to settle.

Conjecture 1. If G is a bipartite graph of order n with $\delta(G) \geq 2$, then $\gamma_{\rm gpr}(G) \leq \frac{3}{4}n$.

By Theorem 7, we note that Conjecture 1 is true if the bipartite graph G satisfies $\mathrm{bc}(G) = 0$. In particular, if the bipartite graph G is C_4-free or 2-connected, then we note that $\mathrm{bc}(G) = 0$, and therefore the conjecture holds. Hence a counterexample to Conjecture 1, if it exists, must contain at least one bad-cut-vertex.
However, we believe the $\frac{4}{5}n$ upper bound on the paired-domination number in a graph of order n with minimum degree at least 2 can be improved to $\frac{2}{3}n$ if we impose the restriction that the graph has minimum degree at least 3. We state our conjecture formally as follows.

Conjecture 2. If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma_{\rm gpr}(G) \leq \frac{2}{3}n$.

We remark that if Conjecture 2 is true, then the bound is tight as there exists an infinite family of graphs achieving equality in this bound. In order to construct such a family, by a 3-prism we mean the graph $C_3 \,\Box\, K_2$ shown in Figure 5.

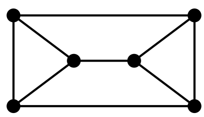

Figure 5. **The 3-prism $C_3 \Box K_2$.**

Proposition 3. *There exists an infinite family $\mathcal{L}$ of graphs with minimum degree 3 such that if $L \in \mathcal{L}$ has order n, then $\gamma_{\rm gpr}(L) = \frac{2}{3}n$.*

Proof. Let $\mathcal{L}$ be the family of connected graphs with minimum degree 3 constructed as follows. Let $(C_3 \,\Box\, K_2)^-$ be the graph obtained from a 3-prism by deleting from it one edge that does not belong to a triangle. We call the two vertices of degree 2 in $(C_3 \,\Box\, K_2)^-$ the *gluing vertices* of $(C_3 \,\Box\, K_2)^-$. For $k \geq 1$, let L_k be obtained from k vertex disjoint copies of $(C_3 \,\Box\, K_2)^-$ by adding any number of edges joining gluing vertices so that the resulting graph is connected and has minimum degree 3. We note that L_k is a graph on $n = 6k$ vertices with $\delta(G) = 3$. We note that L_1 is precisely the 3-prism $C_3 \,\Box\, K_2$. When $k = 4$, an example of a graph L_4 constructed in this way is illustrated in Figure 6, where the gluing vertices are depicted by solid (darkened) vertices. Let $\mathcal{L}$ be the family of all such graphs L_k where $k \geq 1$.
Let L be an arbitrary graph in $\mathcal{L}$ and let L have order n. Thus, $L = L_k$ for some integer $k \geq 1$, and so $n = 6k$. Pairer adopts the following strategy. Consider an arbitrary copy of $G = (C_3 \,\Box\, K_2)^-$ used in the construction of the graph L, where

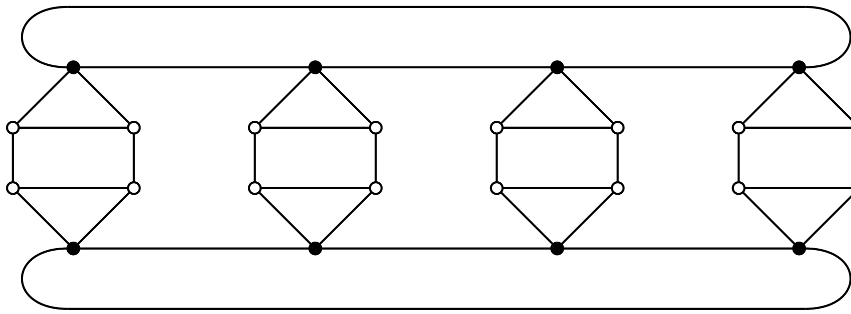

Figure 6. **A graph L_4 in the family $\mathcal{L}$.**

$V(G) = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ and where $G[\{a_1, a_2, a_3\}] = C_3$ and $G[\{b_1, b_2, b_3\}] = C_3$. Further, let a_2b_2 and a_3b_3 be edges of G, and so a_1b_1 was the edge deleted from the 3-prism $C_3 \,\square\, K_2$ when constructing G. We note that a_1 and b_1 are the two gluing vertices of G. We show that Pairer has a strategy to force four vertices of G to be played. Pairer waits for Dominator to be the first to play a vertex from the subgraph G in L. Let $d_1, d_2, \ldots$ be the sequence of moves that Dominator plays in G, and let $p_1, p_2, \ldots$ be the response of Pairer to these moves of Dominator.
Suppose that $d_1 = a_1$. Pairer responds by playing $p_1 = a_2$. In order to dominate the vertex b_3, Dominator must play at least one additional vertex from G. If $d_2 \in \{a_3, b_1, b_2\}$, then Pairer plays $p_2 = b_3$. If $d_2 = b_3$, then Pairer plays $p_2 = b_2$. Thus, at least four vertices from the graph G are played. Suppose that $d_1 \in \{a_2, a_3\}$. Pairer responds by playing $p_1 = a_1$, and, analogously as above, Pairer can force at least four vertices from the graph G to be played. By symmetry, if $d_1 \in \{b_1, b_2, b_3\}$, then Pairer can force at least four vertices from the graph G to be played.
In this way, Pairer has a strategy that will force at least four vertices to be played from the graph G. Since G is an arbitrary copy of $(C_3 \,\square\, K_2)^-$ used in the construction of the graph L, Pairer has a strategy that will force at least $4k$ vertices to be played in L. Thus, $\gamma_{\rm gpr}(L) \geq 4k = \frac{2}{3}n$. However, once all vertices in a copy of $(C_3 \,\square\, K_2)^-$ are dominated, Dominator starts playing in a new copy of $(C_3 \,\square\, K_2)^-$, thereby guaranteeing that the game is complete after at most $4k$ moves. Thus, $\gamma_{\rm gpr}(L) \leq 4k$. Consequently, $\gamma_{\rm gpr}(L) = 4k = \frac{2}{3}n$. Since L was an arbitrary graph in the family $\mathcal{L}$, every graph in the $\mathcal{L}$ has game paired-domination number equal to two-thirds its order. $\square$

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