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t-Pancyclic arcs in tournaments

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Dedicated to Prof. Dr. Lutz Volkmann, on the occasion of his 75th birthday.

Abstract: Let T be a non-trivial tournament. An arc is *t-pancyclic* in T, if it is contained in a cycle of length ℓ for every $t \leq \ell \leq |V(T)|$. Let $p^t(T)$ denote the number of t-pancyclic arcs in T and $h^t(T)$ the maximum number of t-pancyclic arcs contained in the same Hamiltonian cycle of T. Moon (*J. Combin. Inform. System Sci.*, **19** (1994), 207-214) showed that $h^3(T) \geq 3$ for any non-trivial strong tournament T and characterized the tournaments with $h^3(T) = 3$. In this paper, we generalize Moon's theorem by showing that $h^t(T) \geq t$ for every $3 \leq t \leq |V(T)|$ and characterizing all tournaments which satisfy $h^t(T) = t$. We also present all tournaments which fulfill $p^t(T) = t$.

Keywords: Tournament, pancyclicity, t-pancyclic arc

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1. Terminology and introduction

In this paper we consider only finite and simple digraphs. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. Denote $|V(D)|$ the *order* of D. If xy is an arc of D, then we write $x \to y$ and say x *dominates* y. More generally, if X and Y are two disjoint subdigraphs of D (or subsets of $V(D)$) such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y and denote it by $X \to Y$. In

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addition, we denote the set of arcs from X to Y by $A(X,Y)$. Let $W \subseteq V(D)$. Then $D[W]$ is a subdigraph of D induced by W and $D - W = D[V(D)\backslash W]$.
A *strong component* H of a digraph D is a maximal subdigraph of D such that for any two distinct vertices $x, y \in V(H)$, the subdigraph H contains a path from x to y and a path from y to x. A digraph D is *strong* if it has only one strong component. A *reductor* of D is a smallest subdigraph X such that $D - V(X)$ is not strong.
A path from x to y is called an (x,y)-path. A cycle of length ℓ is said to be an ℓ-cycle. A path (resp. cycle) in D is a *Hamiltonian path* (resp. *Hamiltonian cycle*) if it contains all the vertices of D. An arc of a digraph D is *t-pancyclic* ($t \geq 3$) if it is contained in an ℓ-cycle for every $t \leq \ell \leq |V(D)|$. Instead of 3-pancyclic we just say *pancyclic*. It is immediate that each s-pancyclic arc is also t-pancyclic for $s \leq t \leq |V(D)|$. If $xy \in A(D)$ is t-pancyclic in D, then yx is t-pancyclic in D^{-1}, where $D^{-1} = (V(D), \{yx \mid xy \in A(D)\})$ is the *converse digraph* of D.
The number of pancyclic (resp. t-pancyclic) arcs in a digraph D is denoted by $p(D)$ (resp. $p^t(D)$) and $h(D)$ (resp. $h^t(D)$) is the maximum number of pancyclic (resp. t-pancyclic) arcs belonging to the same Hamiltonian cycle of D.
A *tournament* T is a digraph with exactly one arc between every pair of distinct vertices. A tournament without any cycles is called *transitive*.

In 1994, Moon [4] showed that every non-trivial strong tournament contains at least three pancyclic arcs. Actually, he proved a somewhat stronger result:

Theorem 1. (Moon [4]) *Let T be a strong tournament with order $n \geq 3$. Then $h(T) \geq 3$ with equality holding if and only if $T \in \mathcal{P}_3$, where $\mathcal{P}_3$ is the set of tournaments T containing a vertex v such that $T - v$ is a transitive tournament with a unique Hamiltonian path $t_1 t_2 \ldots t_{n-1}$ and $\{t_i, \ldots, t_{n-1}\} \to v \to \{t_1, \ldots, t_{i-1}\}$ for some $2 \leq i \leq n-1$.*

Further results on pancyclicity in tournaments can be found in [1], [5]-[6]. In this paper we consider the number of t-pancyclic arcs for $t \geq 3$ instead of pancyclic arcs in tournaments. According to the definitions of $p^t(D)$ and $h^t(D)$, we immediately have $p^t(D) \geq h^t(D)$ and $h^t(D) \leq |V(D)|$. Moreover, if D contains a unique Hamiltonian cycle, which therefore has to contain all t-pancyclic arcs, then $p^t(D) = h^t(D)$. Note that all tournaments of $\mathcal{P}_3$ contain exactly one Hamiltonian cycle. So $p(T) = h(T) = 3$ for $T \in \mathcal{P}_3$.
In the next section we generalize Theorem 1 by showing that $h^t(T) \geq t$ for every $3 \leq t \leq |V(T)|$ and characterizing all tournaments which satisfy $h^t(T) = t$. Additionally, we present all tournaments which fulfill $p^t(T) = t$.

2. Main Results

The following important lemma will be used frequently in the proofs of our main results. The parts (1)-(3) and (8)-(9) of Lemma 1 can be seen in [3], the other parts (4)-(7) are very easy, so we omit their proofs here.

Lemma 1. *Let T be a non-trivial strong tournament and X a reductor of T. Then the following statements hold.*

(1) *There is a unique sequence $T_1, T_2, \ldots, T_m$ ($m \geq 2$) of the strong components of $T - V(X)$ satisfying $T_i \to T_j$ for every $1 \leq i < j \leq m$. We call it a* strong decomposition *of $T - V(X)$. Similarly, there is a strong decomposition $X_1, X_2, \ldots, X_\ell$ ($\ell \geq 1$) of X.*

(2) *Every vertex of X dominates a vertex of T_1 and is dominated by a vertex of T_m.*

(3) *Every arc from T_m to X_1 is pancyclic and every arc from X_ℓ to T_1 is also pancyclic.*

(4) *Each arc in X that lies on a Hamiltonian path of X is 4-pancyclic.*

(5) *If $m \geq 4$, then every arc from T_i to T_{i+1} is 5-pancyclic for $i = 2, 3, \ldots, m-2$.*

(6) *If $m \geq 3$ and $|V(T_1)| = 1$ (resp. $|V(T_m)| = 1$), then every arc in $A(T_1, T_2)$ (resp. $A(T_{m-1}, T_m)$) is 4-pancyclic.*

(7) *If $|V(T_i)| \geq 3$ for some $1 < i < m$, then every arc, which lies on a Hamiltonian cycle of T_i, is 5-pancyclic in T.*

(8) *If $|V(T_i)| \geq 4$ for some $1 < i < m$, then every t-pancyclic arc in T_i is also t-pancyclic in T for $3 \leq t \leq |V(T_i)|$.*

(9) *If $|V(T_i)| = 3$ for some $1 < i < m$, then at least two arcs of T_i are pancyclic in T.*

Building upon the results above, we can prove the first main result, which is a generalization of the first part of Theorem 1.

Theorem 2. *Let T be a strong tournament with order $n \geq 3$. Then*

$$h^t(T) \geq t$$

for every $3 \leq t \leq n$.

Proof. We prove this theorem by induction on n. For $n = 3$, T is a 3-cycle, and clearly, $h^3(T) = 3$. For $n = 4$, it is easy to check that $h^3(T) = 3$ and $h^4(T) = 4$. Suppose now $n \geq 5$ and it is true for all strong tournaments with less than n vertices. By Theorem 1, $h^3(T) \geq 3$, and clearly, $h^n(T) = n$. So we only need to consider the cases $t = 4, 5, \ldots, n-1$.
Let $X_1, X_2, \ldots, X_\ell$, $\ell \geq 1$, be the strong decomposition of a reductor X of T and $T_1, T_2, \ldots, T_m, m \geq 2$, be the strong decomposition of $T - V(X)$ with $n_i = |V(T_i)|$ for $1 \leq i \leq m$. Because of $n \geq 5$ we have $|V(T) \backslash V(X)| \geq 3$. Note that every component T_i, if not consisting of a single vertex, contains a Hamiltonian cycle $C_i = t_1^i t_2^i \ldots t_{n_i}^i t_1^i$ for $1 \leq i \leq m$.
Let C be a Hamiltonian cycle in T of the form $w_1 Q w_2 P$, where $w_1 \in A(X_\ell, T_1)$, $w_2 \in A(T_m, X_1)$, P is a Hamiltonian path of X and Q is a Hamiltonian path of $T - V(X)$. To prove this theorem we only need to find at least t arcs on C which are

t-pancyclic in T for $t = 4, 5, \dots, n-1$. Note that the two arcs w_1 and w_2 on C are always pancyclic in T by Lemma 1 (3).
Below we give a claim concerning the number of t-pancyclic arcs in T_1.

Claim 1. If $n_1 \geq 3$, then there is a Hamiltonian path P_1 of T_1 on which at least $\min\{t-1, n_1-1\}$ *arcs are t-pancyclic in T and P_1 is the first part of Q.*

Proof of Claim 1. By the induction hypothesis for T_1, there is a Hamiltonian cycle in T_1, say $C_1 = t_1^1 t_2^1 \dots t_{n_1}^1 t_1^1$, containing $h^t(T_1) \geq t$ arcs which are t-pancyclic in T_1.
Let x_ℓ be an arbitrary vertex of X_ℓ. By Lemma 1 (2) we may assume without loss of generality that $t_m \to x_\ell \to t_1^1$ for some $t_m \in V(T_m)$.
If $x_\ell \to t_{n_1}^1$, then let $P_1 = t_{n_1}^1 t_1^1 t_2^1 \dots t_{n_1-1}^1$. The two cycles $t_1^1 t_2^1 \cdots t_{n_1-1}^1 t_m x_\ell t_1^1$ and $t_{n_1}^1 t_1^1 t_2^1 \dots t_{n_1-2}^1 t_m x_\ell t_{n_1}^1$ yield that every arc on the Hamiltonian path P_1 of T_1 is contained in an (n_1+1)-cycle. Furthermore, the (n_1+2)-cycle $t_{n_1}^1 t_1^1 t_2^1 \dots t_{n_1-1}^1 t_m x_\ell t_{n_1}^1$ can be successively extended to a Hamiltonian cycle $C = w_1 Q w_2 P$ in T such that P_1 is the first part of Q. So every arc on P_1 is n_1-pancyclic in T. In the case when $t \geq n_1$, we immediately have that every arc on P_1 is t-pancyclic in T; In the other case when $t < n_1$, we deduce that least $t-1$ arcs on P_1 are t-pancylic in T.
If $t_{n_1}^1 \to x_\ell$, then let $P_1 = t_1^1 t_2^1 \dots t_{n_1}^1$ and from the cycles $t_1^1 t_2^1 \dots t_{n_1}^1 x_\ell t_1^1$ and $t_1^1 t_2^1 \dots t_{n_1}^1 t_m x_\ell t_1^1$ we can deduce the same conclusion as above. So we are done. □

Analogously, Claim 1 also holds for T_m. We distinguish the following two cases according to the value of t.

Case 1. $t = 4$.

If $|V(T_1)| = |V(T_m)| = 1$, then $m \geq 3$ and the two arcs of $A(T_1, T_2)$ and $A(T_{m-1}, T_m)$ on Q are 4-pancyclic by Lemma 1 (6). So $h^4(T) \geq 4$.
Assume without loss of generality that $|V(T_1)| \geq 3$. According to Claim 1, at least two arcs of T_1 are 4-pancyclic in T which are contained in the Hamiltonian cycle C. So $h^4(T) \geq 4$.

Case 2. $5 \leq t \leq n-1$.

If $|V(T_1)| = |V(T_m)| = 1$, then $m \geq 3$ and every arc on the Hamiltonian cycle $C = w_1 Q w_2 P$ is t-pancyclic in T by Lemma 1 (3)-(7). So $h^t(T) = n > t$. Assume without loss of generality that $|V(T_1)| \geq 3$.
If $|V(T_1)| \geq t$ or $|V(T_m)| \geq t$, then by Claim 1 we have $h^t(T) \geq t - 1 + |\{w_1, w_2\}| = t + 1$. So assume in the following that $3 \leq |V(T_1)| \leq t-1$ and $1 \leq |V(T_m)| \leq t-1$.
If $|V(T_m)| = 1$, then by Claim 1 and Lemma 1 (3)-(7) only the arc of $A(T_1, T_2)$ on $C = w_1 Q w_2 P$ is possibly not t-pancyclic. So $h^t(T) \geq n - 1 \geq t$.
If $3 \leq |V(T_m)| \leq t-1$, then by Claim 1 and Lemma 1 (3)-(7) only the arcs $e_C \in A(T_1, T_2)$ and $e'_C \in A(T_{m-1}, T_m)$ on $C = w_1 Q w_2 P$ are possibly not t-pancyclic. So $h^t(T) \geq n-2$. For $t \leq n-2$ we are done obviously. For $m = 2$, we are also done

with $e_C = e'_C$ and $h^t(T) \geq n-1 \geq t$. For the remaining case $t = n-1$ and $m \geq 3$, it is easy to see that the arc e_C (resp. e'_C) is on an $(n-1)$-cycle just by skipping one vertex of T_m (resp. T_1). Therefore, $h^t(T) = n > t$. □

To characterize all tournaments with $h^t(T) = t$, we need the following definition.

Definition 1. Let H^n be the strong tournament on n vertices with a Hamiltonian path $P = x_1x_2 \ldots x_n$ such that $x_j \to x_i$ for all $3 \leq i+2 \leq j \leq n$. Instead of H^n we often write H^n_P or $H^n_{x_1}$ to mark the path P or its initial vertex x_1.

Lemma 2. *Let T be a strong tournament of order $n \geq 3$ and $x \in V(T)$. Then $T = H^n_x$ if and only if for every Hamiltonian path of T with initial vertex x there is no path of length $n-2$ from x to the end vertex of such Hamiltonian path.*

Proof. The necessity is clear and we prove the sufficiency by using induction on n. If $n = 3$, then T is a 3-cycle and therefore $T = H^3$. If $n = 4$, then let $P = x_1x_2x_3x_4$ be a Hamiltonian path of T with $x_1 = x$. Since there is no (x_1, x_4)-path of length 2, we have $x_3 \to x_1$ and $x_4 \to x_2$. If $x_1 \to x_4$, then $P' = x_1x_4x_2x_3$ is another Hamiltonian path starting at x_1, but $x_1x_2x_3$ is an (x_1, x_3)-path of length 2, a contradiction. So $x_4 \to x_1$ and $T = H^4_x$. Assume $n \geq 5$ and the claim holds for all strong tournaments with less than n vertices.
Let $P = x_1x_2 \ldots x_n$ be a Hamiltonian path in T with $x_1 = x$. As there is no (x_1, x_n)-path of length $n-2$, we have $x_{i+2} \to x_i$ for all $1 \leq i \leq n-2$. Consider the strong subdigraph $T - x_1$ of T. For any Hamiltonian path Q of $T - x_1$ starting at x_2, there is no path S of length $n-3$ from x_2 to the end vertex of Q. As otherwise we can extend S and Q to $S' = x_1S$ and $P' = x_1Q$, a contradiction. Therefore, $T - x_1 = H^{n-1}_{x_2}$. If there exists an index $i \in \{4, \ldots, n\}$ such that $x \to x_i$, then $xx_i \ldots x_nx_2 \ldots x_{i-1}$ is a Hamiltonian path of T and $xx_i \ldots x_nx_3 \ldots x_{i-1}$ is an (x, x_{i-1})-path of length $n-2$, a contradiction. So $T = H^n_x$. □

Now we are ready to generalize the second part of Theorem 1 by Moon.

Theorem 3. *Let T be a strong tournament with order n and $t \geq 4$. Then $h^t(T) = t$ if and only if $n = t$ or $T = H^{t+1}$.*

Proof. First we assume $n = t$ or $T = H^{t+1}$. If $n = t$, then the desired result is obvious. If $T = H^{t+1} = H^{t+1}_Q$ with $Q = x_1 \ldots x_{t+1}$, then this tournament has exactly one Hamiltonian cycle and every arc of Q is contained in the cycles $x_1 \ldots x_tx_1$ or $x_2 \ldots x_{t+1}x_2$ and therefore t-pancyclic. By Lemma 2, there is no (x_1, x_{t+1})-path of length $t-1$, and therefore, the arc $x_{t+1}x_1$ cannot be contained in any t-cycle. So $h^t(T) = t$.
To prove the other direction, let $X_1, X_2, \ldots, X_\ell,\ \ell \geq 1$, be the strong decomposition of a reductor X of T and $T_1, T_2, \ldots, T_m,\ m \geq 2$, be the strong decomposition of $T - V(X)$ with $n_i = |V(T_i)|$ for $1 \leq i \leq m$. Like in the proof of Theorem 2 we

distinguish two cases $t = 4$ and $t > 4$. In both cases we assume $h^t(T) = t$ and $n > t$. Therefore we always have $n \geq 5$ and $|V(T)\backslash V(X)| \geq 3$. Again we consider a Hamiltonian cycle $C = w_1 Q w_2 P$ of T, where $w_1 \in A(X_\ell, T_1)$, $w_2 \in A(T_m, X_1)$, P is a Hamiltonian path of X and Q is a Hamiltonian path of $T - V(X)$. Note that w_1, w_2 are always pancyclic in T, and Claim 1 in the proof of Theorem 2 also holds here.

Case 1. $t = 4$.

If $|V(T_1)| = |V(T_m)| = 1$, then $m \geq 3$ and by Lemma 1 (3) and (6), there are already four 4-pancyclic arcs w_1, w_2, $e_C \in A(T_1, A_2)$ and $e'_C \in A(T_{m-1}, T_m)$ on C. Since $h^4(T) = 4$, by Lemma 1 (4), (8), (9) and $n \geq 5$ we have $|V(X)| = 1$, $|V(T_i)| = 1$ for $i = 2, 3, \dots, m-1$, and $m \geq 4$. Let $V(X) = \{x\}$ and $V(T_i) = \{t_i\}$ for $i = 1, 2, \dots, m$. If $m \geq 5$, then either $t_2 t_3$ is 4-pancyclic in T when $t_3 \to x$ or $t_3 t_4$ is 4-pancyclic when $x \to t_3$. It is a contradiction. So $m = 4$ and $\{t_2, t_4\} \to x \to \{t_1, t_3\}$. This means $T = H^5_{P^*}$ with $P^* = t_3 t_4 x t_1 t_2$.
Assume without loss of generality that $|V(T_1)| \geq 3$. Since $h^4(T) = 4$, by Claim 1 and Lemma 1 it is not difficult to deduce that $|V(T_1)| = 3$, $|V(T_m)| = 1$, $|V(X)| = 1$ and $m = 2$. Let $t_1 t_2 t_3 t_1$ be the Hamiltonian cycle of T_1, $V(X) = \{x\}$, $V(T_2) = \{y\}$ and assume without loss of generality that $x \to t_1$. Then $\{t_2, t_3\} \to x$ and $T = H^5_{P*}$ with $P^* = y x t_1 t_2 t_3$.

Case 2. $5 \leq t \leq n - 1$.

If $|V(T_1)| = |V(T_m)| = 1$, then $m \geq 3$ and every arc on the Hamiltonian cycle $C = w_1 Q w_2 P$ is t-pancyclic by Lemma 1 (3)-(7). That is to say $h^t(T) = n \neq t$, a contradiction. So assume without loss of generality that $|V(T_1)| \geq 3$.
In addition, we have $|V(T_1)|, |V(T_m)| \leq t-1$, as otherwise $h^t(T) \geq t-1+|\{w_1, w_2\}| \geq t+1$ by Claim 1 in the proof of Theorem 2, a contradiction.
Now by Claim 1 and Lemma 1 only the arcs $e_C \in A(T_1, T_2)$ and $e'_C \in A(T_{m-1}, T_m)$ on $C = w_1 Q w_2 P$ are possibly not t-pancyclic. So $n - 1 \geq t = h^t(T) \geq n - 2$.

Subcase 2.1. $3 \leq |V(T_1)|, |V(T_m)| \leq t - 1$.

If $m \geq 3$, then the arc e_C (resp. e'_C) is on cycles of length $n - 1$ and $n - 2$ just by skipping one or two vertices in T_m (resp. T_1). So whenever $t = n - 1$ or $t = n - 2$, e_C and e'_C are t-pancyclic. Therefore $h^t(T) = n \neq t$, a contradiction.
Assume in the following that $m = 2$. Then $e_C = e'_C$ is the unique arc which is not t-pancyclic in T. So $t = h^t(T) = n - 1$ and e_C is not on any $(n - 1)$-cycle. Hence, $|V(X)| = 1$, as otherwise, e_C is on an $(n - 1)$-cycle by skipping one vertex of $V(X)$ on C.
Let $V(X) = \{x\}$ and $C_i = t^i_1 t^i_2 \dots t^i_{n_i} t^i_1$ be a Hamiltonian cycle of T_i for $i = 1, 2$. By Lemma 1 (2) we may assume without loss of generality that $t^2_{n_2} \to x \to t^1_1$.
In T_1, for any Hamiltonian path with the initial vertex t^1_1, there is no $(n_1 - 2)$-path from t^1_1 to the end vertex of such Hamiltonian path, as otherwise e_C lies on an $(n-1)$-

cycle, a contradiction. By Lemma 2, $T_1 = H^{n_1}_{P_1}$ for $P_1 = t^1_1 t^1_2 \dots t^1_{n_1}$. Using T^{-1}, we can similarly deduce that $T_2 = H^{n_2}_{P_2}$ for $P_2 = t^2_1 t^2_2 \dots t^2_{n_2}$. Now our aim is to show $\{t^1_2, \dots, t^1_{n_1}\} \to x \to \{t^2_1, \dots, t^2_{n_2-1}\}$, and then, $T = H^{t+1}_{P^*}$ for $P^* = P_2 x P_1$.
If $t^2_1 \to x$, then from this cycle $t^1_1 t^1_2 \dots t^1_{n_1} t^2_2 t^2_3 \dots t^2_{n_2}(t^2_1) x t^1_1$ we can see that $e_C = t^1_{n_1} t^2_2$ is on an $(n-1)$-cycle, a contradiction. So $x \to t^2_1$.
If $t^2_2 \to x$ and $n_2 \geq 4$, then from this cycle $t^1_1 t^1_2 \dots t^1_{n_1} t^2_3 \dots (t^2_{n_2}) t^2_1 t^2_2 x t^1_1$ we can see that $e_C = t^1_{n_1} t^2_3$ is on an $(n-1)$-cycle, a contradiction. If $t^2_2 \to x$ and $n_2 = 3$, then from this cycle $t^1_1 \dots t^1_{n_1} t^2_1 t^2_2 (t^2_3) x t^1_1$ we can see that $e_C = t^1_{n_1} t^2_1$ is on an $(n-1)$-cycle, a contradiction. So $x \to t^2_2$.
Successively, we can show that $x \to \{t^2_3, \dots, t^2_{n_2-1}\}$. Considering T^{-1}, we can further deduce that $\{t^1_2, \dots, t^1_{n_1}\} \to x$. Altogether, $T = H^{t+1}_{P^*}$.

Subcase 2.2. $3 \leq |V(T_1)| \leq t-1$ and $|V(T_m)| = 1$.

If $m \geq 3$, then e'_C is t-pancyclic by Lemma 1 (6), and if $m = 2$, then $e_C = e'_C$. All of these yield $t = n-1$ and e_C is not on any $(n-1)$-cycle. So $|V(X)| = 1$, as otherwise, e_C is on an $(n-1)$-cycle by skipping one vertex of $V(X)$ on C. Similarly, we get $m \leq 3$, $|V(T_2)| = 1$ and $X \twoheadrightarrow T_1$. We also have $X \to T_2$ when $m = 3$.
If $m = 3$, then it can be transferred to the case $m = 2$ by choosing another reductor $X' = T_3$, where $T'_1 = T[T_1 \cup V(X)]$, $T'_2 = T_2$ is the strong decomposition of $T - V(X')$. So we only need to consider the case $m = 2$.
Let $V(X) = \{x\}$, $T_2 = \{y\}$ and $C_1 = t^1_1 t^1_2 \dots t^1_{n_1} t^1_1$ be a Hamiltonian cycle of T_1. Assume without loss of generality that $x \to t^1_1$. Then by a similar argument as in Subcase 2.1 we can deduce that $T_1 = H^{n_1}_{P_1}$ with $P_1 = t^1_1 t^1_2 \dots t^1_{n_1}$ and $T = H^{t+1}_{P^*}$ with $P^* = y x t^1_1 t^1_2 \dots t^1_{n_1}$. □

All the previous results deal with the maximum number of t-pancyclic arcs on the same Hamiltonian cycle. As we have a characterisation of all tournaments with $h^t(T) = t$, we naturally look for all tournaments with $p^t(T) = t$. We have seen that tournaments which achieve $h(T) = 3$ have been characterised by Moon [4] and these are the same tournaments with $p(T) = 3$. The important fact is that these tournaments contain exactly one Hamiltonian cycle. As this is also the key in the following theorem, we refer to an earlier work by Douglas [2] which gives valuable information about the structure of tournaments containing exactly one Hamiltonian cycle.

Theorem 4. *Let T be a strong tournament with order n.*

(1) *If $4 \leq t \leq n-1$, then $p^t(T) = t$ if and only if $T = H^{t+1}$;*

(2) *If $t = n$, then $p^t(T) = t$ if and only if there is exactly one Hamiltonian cycle in T.*

Proof. From the definitions of $p^t(T)$ and $h^t(T)$ and Theorem 2, we have $p^t(T) \geq h^t(T) \geq t$.

(1) If $p^t(T) = t$, then from the inequality above we have $h^t(T) = t$. By Theorem 3 and $n \neq t$ we deduce that $T = H^{t+1}$. To prove the other direction, let $T = H^{t+1}$. Then there is exactly one Hamiltonian cycle in T, which implies $p^t(T) = h^t(T) = t$.

(2) Note that every arc on a Hamiltonian cycle of T is n-pancyclic. So this statement obviously holds. □

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