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On the edge-connectivity of C_4-free graphs

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Dedicated to Lutz Volkmann on the occasion of his 75th birthday.

Abstract: Let G be a connected graph of order n and minimum degree $\delta(G)$. The edge-connectivity $\lambda(G)$ of G is the minimum number of edges whose removal renders G disconnected. It is well-known that $\lambda(G) \leq \delta(G)$, and if $\lambda(G) = \delta(G)$, then G is said to be maximally edge-connected. A classical result by Chartrand gives the sufficient condition $\delta(G) \geq \frac{n-1}{2}$ for a graph to be maximally edge-connected. We give lower bounds on the edge-connectivity of graphs not containing 4-cycles that imply that for graphs not containing a 4-cycle Chartrand's condition can be relaxed to $\delta(G) \geq \sqrt{\frac{n}{2}+1}$, and if the graph also contains no 5-cycle, or if it has girth at least six, then this condition can be relaxed further, by a factor of approximately $\sqrt{2}$. We construct graphs to show that for an infinite number of values of n both sufficient conditions are best possible apart from a small additive constant.

Keywords: edge-connectivity, maximally edge-connected

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1. Introduction

Let G be a finite graph. The minimum number of edges whose removal renders G disconnected is called the *edge-connectivity* of G and denoted by $\lambda(G)$. The *degree* of a vertex v, $\deg(v)$, is the number of vertices v is adjacent to in G, and $\delta(G)$ denotes the *minimum degree* of G, i.e., the smallest of all degrees of vertices of G.
It was first observed by Whitney [19] that $\lambda(G) \leq \delta(G)$. If for a graph G this inequality holds with equality, that is, if $\lambda(G) = \delta(G)$, then G is said to be *maximally edge-connected*. The first sufficient condition for graphs to be maximally edge-connected is a classical result due to Chartrand [4], who showed that if a graph G on n vertices satisfies

$$\delta(G) \geq \frac{n-1}{2}, \tag{1}$$

then G is maximally edge-connected. Subsequently it has been shown that the degree condition (1) can be relaxed in the sense that if some vertices have degree less than $\frac{n-1}{2}$ but others have sufficiently large degrees to make up for the small degree vertices, then maximal edge-connectivity is still guaranteed. Lesniak [15] showed that if G satisfies $\deg(u) + \deg(v) \geq n - 1$ for every pair of non-adjacent vertices, then G is maximally edge-connected. Bollobás [2], Dankelmann and Volkmann [8] and Hellwig and Volkmann [13] pursued this idea further by giving degree sequence conditions for maximally edge-connected graphs. There are also many degree sequence conditions for maximally edge-connected digraphs, see, for example, [13] and [18]. Sufficient conditions for maximally edge-connected graphs in terms of the inverse degree were given in [6].

It has also be shown that (1) can be relaxed, sometimes considerably, if only graphs from various graph classes are considered. This approach was taken by Volkmann [17], who showed that (1) can be relaxed for p-partite graphs to

$$n(G) \leq 2\left\lfloor \delta \frac{p-1}{p} \right\rfloor - 1. \tag{2}$$

In [7] it was shown that (2) guarantees maximal edge-connectivity not only for p-partite graphs, but for all graphs not containing a complete subgraph on $p+1$ vertices. Setting $p = 2$ in (2) yields the condition $\delta(G) \geq \frac{n+1}{4}$, which was proved in [16]. A generalisation of (2) for $p = 2$ to bipartite digraphs, in the vein of the above-mentioned result by Lesniak, was given by Balbuena and Carmona [1]. Condition (2) was also relaxed to degree sequence conditions for maximally edge-connected graphs without complete subgraphs on $p + 1$ vertices, see [9]. Hellwig and Volkmann [14] give an excellent survey of these and many other sufficient conditions for maximally edge-connected graphs and digraphs.

In this paper we follow the latter approach and consider C_4-free graphs, that is, graphs not containing a 4-cycle. While lower bounds on the connectivity of C_4-free graphs were studied in [5], no lower bounds on the edge-connectivity of such graphs appear to be known. We give a lower bound on the edge-connectivity of C_4-free graphs in terms of order and minimum degree which implies that in graphs with no 4-cycle the condition $\delta(G) \geq \frac{n-1}{2}$ in (1) can be replaced by the much weaker condition $\delta(G) \geq \sqrt{\lfloor \frac{n}{2} \rfloor} + 1$ to guarantee that G is maximally edge-connected. We show that for graphs that are C_4-free and C_5-free, this condition can be improved to $\delta(G) \geq \frac{1}{2}\sqrt{n-1} + \frac{3}{2}$, and for graphs that are C_3-free, C_4-free and C_5-free, i.e., for graphs of girth at least 6, this in turn can be improved slightly to $\delta(G) \geq \frac{1}{2}\sqrt{n - \frac{7}{4}} + \frac{3}{4}$. We construct graphs that show that for an infinite number of values of n these conditions are best possible apart from a small additive constant.

The notation we use is as follows. The vertex and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. We use $n(G)$ for the *order* of G, i.e., the number of vertices of G. A subset $S \subseteq E(G)$ is an *edge-cut* of G if $G - S$, the graph obtained by deleting all edges of S from G, is disconnected. The *edge-connectivity* of G is defined as the minimal cardinality of an edge-cut of G and denoted by $\lambda(G)$. For each vertex

$v \in V(G)$, the *open neighbourhood* $N(v)$ of v is the set of all vertices adjacent to v, while $N[v] = N(v)$ is the *closed neighbourhood* of v, and $\deg(v) = |N(v)|$ is the *degree* of v. We denote by $\delta(G)$ the *minimum degree* of G, i.e., the smallest of the degrees of vertices of G. The *distance* $d(u, v)$ between two vertices of a graph G is the minimum number of edges on a (u, v)-path. The *second neighbourhood* of v, denoted by $N^2[v]$, is the set of vertices whose distance from v is not more than two. The *girth* of G is the length of a shortest cycle. For $n \in \mathbb{N}$ with $n \geq 3$ we denote the cycle on n vertices by C_n. We say that a graph is C_k-free if it does not contain C_4 as a subgraph.

2. Graphs containing no C_4

In our proofs we make use of the following result by Hamidoune [12]. If S is an edge-cut of G and G_1 a component of $G - S$, then we say that a vertex of G_1 is an *interior vertex* of G_1 if it is not incident with any edge of S.

Lemma 1. (Hamidoune [12]) *Let G be a connected graph containing an edge-cut S with $|S| < \delta(G)$. Then every component of $G - S$ contains an interior vertex.* □

Theorem 1. *Let G be a C_4-free graph of order n and minimum degree δ. If G is not maximally edge-connected, then*

$$\lambda(G) \geq \delta^2 - \delta + 1 + \epsilon_\delta - \lfloor \frac{n}{2} \rfloor,$$

where ϵ_δ is 0 if δ is even, and 1 if δ is odd.

Proof. Let $S \subseteq E(G)$ be a minimum edge-cut. Since G is not maximally edge-connected, we have $|S| < \delta(G)$. Let G_1 and G_2 be the two components of $G - S$ and let U and W be their respective vertex sets. Without loss of generality we may assume that $|U| \leq |W|$, so $|U| \leq \lfloor \frac{n}{2} \rfloor$. By Lemma 1, G_1 has an interior vertex $u \in U$. Let $u_1, u_2, \ldots, u_d$ be its neighbours. Since G is C_4-free, each u_i has at most one neighbour in $N(u)$. If d is odd, then it follows by the handshake lemma that at most $d - 1$ of the u_i can have a neighbour in $N(u)$, so there exists $j \in \{1, 2, \ldots, d\}$ such that u_j has no neighbour in $N(u)$. Also, since G is C_4-free, the sets $N(u_i) - N[u]$, $i = 1, 2, \ldots, d$, are pairwise disjoint. Hence, by $d \geq \delta$,

$$|N^2[u]| \geq 1 + d + d(\delta - 2) + \epsilon_d \geq \delta^2 - \delta + 1 + \epsilon_\delta. \tag{3}$$

Since $|U| \leq \lfloor \frac{n}{2} \rfloor$ it follows that

$$|N^2[u] \cap W| \geq |N^2[u]| - |U| \geq \delta^2 - \delta + 1 + \epsilon_\delta - \left\lfloor \frac{n}{2} \right\rfloor.$$

Since u is an interior vertex of G_1, the vertices of the set $N^2[u] \cap W$ are all at distance exactly two from u. Therefore, each vertex in $N^2[u] \cap W$ has a neighbour in $N(u)$, and thus in U, to which it is joined by an edge in S. It follows that

$$|S| \geq |N^2[u] \cap W| \geq \delta^2 - \delta + 1 + \epsilon_\delta - \left\lfloor \frac{n}{2} \right\rfloor,$$

and by $\lambda(G) = |S|$ the theorem follows. □

Now $\delta^2 - \delta + 1 + \epsilon_\delta - \lfloor \frac{n}{2} \rfloor \geq \delta$ holds if and only if $(\delta - 1)^2 + \epsilon_\delta \geq \lfloor \frac{n}{2} \rfloor$. Hence we have the following corollaries.

Corollary 1. *Let G be a C_4-free graph of order n. If*

$$(\delta(G) - 1)^2 + \epsilon_\delta \geq \left\lfloor \frac{n}{2} \right\rfloor,$$

then G is maximally edge-connected.

Corollary 2. *Let G be a C_4-free graph of order n. If*

$$\delta(G) \geq \sqrt{\frac{n - \epsilon_n}{2}} + 1,$$

then G is maximally edge-connected.

In Example 1 we construct graphs which show that Corollary 2 is sharp apart from a small additive constant for infinitely many values of n. The construction makes use of a C_4-free graph, the graph H_q in Example 1 below, first constructed by Erdös and Rényi [10] and independently Brown [3].
We need the following notation from linear algebra. If q is a prime power, then we denote the field of order q by $GF(q)$, and the vector space of all 3×1 column vectors with entries in $GF(q)$ by $GF(q)^3$. For $\underline{x} \in GF(q)^3$ we denote the transpose of $\underline{x}$ by $\underline{x}^t$. If $U \subseteq GF(q)^3$, then the subspace generated by U is denoted by $\langle U \rangle$. Two vectors $\underline{x}, \underline{y} \in GF(q)^3$ are orthogonal if $\underline{x}^t \underline{y} = 0$ over $GF(q)$, where $\underline{x}^t$ is the transpose of $\underline{x}$, and $\underline{x}^t \underline{y}$ denotes the dot product of $\underline{x}$ and $\underline{y}$. The orthogonal complement of U, i.e., the subspace of all vectors $\underline{x} \in GF(q)^3$ with $\underline{x}^t \underline{y} = 0$ for all $\underline{y} \in U$, is denoted by $U^\perp$. We write $\underline{x}^\perp$ for $\{\underline{x}\}^\perp$ and $\langle \underline{x} \rangle$ for $\langle \{\underline{x}\} \rangle$. We make use of the fact from linear algebra that the orthogonal complement of a k-dimensional subspace of $GF(q)^3$ is a $(3 - k)$-dimensional subspace of $GF(q)^3$.

Example 1. Let q be an odd prime power. Define the graph H_q as follows. The vertices of H_q are the one-dimensional subspaces of the vector space $GF(q)^3$. Since each one-dimensional subspace of $GF(q)^3$ contains $q - 1$ non-zero vectors and since any two distinct one-dimensional subspaces share only the zero-vector, H_q has $\frac{q^3-1}{q-1} = q^2 + q + 1$

vertices. Two vertices $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ of H_q are adjacent if $\underline{x}$ and $\underline{y}$, as vectors of $GF(q)^3$, are orthogonal. If $\langle \underline{x} \rangle$ is a one-dimensional subspace of $GF(q)^3$, then the orthogonal complement $\langle \underline{x} \rangle^\perp$ is a subspace of $GF(q)^3$ of dimension two containing $q^2 - 1$ non-zero vectors. Hence, if $\underline{x} \notin \underline{x}^\perp$, i.e., if $\underline{x}^t \underline{x} \neq 0$, then $\deg(\underline{x}) = \frac{q^2-1}{q-1} = q+1$, and if $\underline{x} \in \underline{x}^\perp$, i.e., if $\underline{x}^t \underline{x} = 0$, then $\deg(\underline{x}) = \frac{(q^2-1)-(q-1)}{q-1} = q$. One can show that $GF(q)^3$ always contains a self-orthogonal vector, hence $\delta(H_q) = q$. Now let G_q be the graph obtained from two disjoint copies of H_q by adding an edge joining two vertices of degree $q+1$ in distinct copies of H_q. Clearly, G_q is C_4-free, $n(G_q) = 2q^2 + 2q + 2$, $\delta(G)_q) = q$, and $\lambda(G_1) = 1$. Moreover, with $n = n(G_q)$,

$$\delta(G_q) = \sqrt{\frac{n}{2} - \frac{3}{4}} - \frac{1}{2},$$

so $\delta(G_q)$ differs from the term $\sqrt{n/2} + 1$ in Corollary 2 by less than 2. Hence Corollary 2 is sharp apart from a small additive constant.

3. Graphs containing neither C_4 nor C_5

We now show that Theorem 1 can be strengthened if G contains neither 4-cycles nor 5-cycles.

Theorem 2. *Let G be a graph of order n and minimum degree $\delta \geq 3$ that contains neither C_4 nor C_5 as a subgraph. If G is not maximally edge-connected, then*

$$\lambda(G) \geq 2\delta^2 - 5\delta + 5 + 2\epsilon_\delta - \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. Let G_1 G_2, U, W, $u, u_1, \ldots, u_d$ and d be as in the proof of Theorem 1.
We first show that there exists $i \in \{1, 2, \ldots, d\}$ such that u_i is an interior vertex of G_1. Suppose to the contrary that each of $u_1, u_2, \ldots, u_d$ is incident with an edge in S. Then $|S| \geq d \geq \delta$, a contradiction. Hence u has a neighbour, without loss of generality u_1, that is also an interior vertex. We prove that

$$|N^2[u] \cup N^2[u_1]| \geq 2\delta^2 - 5\delta + 5 + 2\epsilon_\delta. \tag{4}$$

We consider two cases, depending on whether u and u_1 have a common neighbour or not.

Case 1: $N(u) \cap N(u_1) = \emptyset$.
As in the proof of Theorem 1 we show that

$$|N^2[u]| \geq 1 + d + d(\delta - 2) + \epsilon_d = 1 + d(\delta - 1) + \epsilon_d. \tag{5}$$

Similarly, if d_1 is the degree of u_1, we show that $|N^2[u_1]| \geq 1 + d_1(\delta - 1) + \epsilon_{d_1}$. We now bound $|N^2[u] \cup N^2[u_1]|$ from below using inclusion-exclusion. Since G contains no cycle of length five, there is no vertex in G that is at distance exactly two from both, u and u_1. Hence the intersection $N^2[u] \cap N^2[u_1]$ is contained in, and in fact equal to, $N[u] \cup N[v]$, and so $|N^2[u] \cap N^2[u_1]| = d + d_1$. Therefore,

$$\begin{aligned} |N^2[u] \cup N^2[u_1]| &= |N^2[u]| + |N^2[u_1]| - |N^2[u] \cap N^2[u_1]| \\ &= |N^2[u]| + |N^2[u_1]| - |N[u] \cup N[u_1]| \\ &\geq 2 + (d+d_1)(\delta-1) + \epsilon_d + \epsilon_{d_1} - d - d_1 \\ &= 2 + (d+d_1)(\delta-2) + \epsilon_d + \epsilon_{d_1} \\ &\geq 2(\delta-1)^2 + 2\epsilon_\delta, \end{aligned}$$

and (4) follows since $2(\delta-1)^2 \geq 2\delta^2 - 5\delta + 5$.

Case 2: $N(u) \cap N(u_1) \neq \emptyset$.

Without loss of generality we may assume that u_2 is a common neighbour of u and u_1. Since G contains no 4-cycle, u_2 is the only common neighbour of u and u_1, and furthermore, u_2 has no neighbour in $(N(u) \cup N(u_1)) - (\{u, u_1\}$. Denote the degree of u_2 by d_2. Since G contains no cycle of length five, every vertex of G that is at distance exactly two from both, u and u_1, is adjacent to u_2, so there are exactly $d_2 - 2$ such vertices. We now bound $|N^2[u]|$ from below. Vertices u_1 and u_2 have exactly $d_1 - 2$ and $d_2 - 2$ neighbours, respectively, at distance exactly two from u. As in the proof of Theorem 1, each of the vertices $u_3, u_4, \ldots, u_d$ have at least $\delta - 2$ neighbours at distance exactly two from u, and if d is odd, then one of these $d-2$ vertices has at least $\delta - 1$ neighbours at distance exactly two from u. Hence

$$|N^2[u]| \geq 1+d+(d_1-2)+(d_2-2)+(d-2)(\delta-2)+\epsilon_{d-2} = d_1+d_2-1+(d-2)(\delta-1)+\epsilon_d,$$

and similarly,

$$|N^2[u]| \geq d + d_2 - 1 + (d_1 - 2)(\delta - 1) + \epsilon_{d_1}.$$

We now bound $|N^2[u] \cup N^2[u_1]|$ from below using inclusion-exclusion. Since G contains no cycle of length five, the only vertices at distance exactly two from both, u and u_1, are the $d_2 - 2$ vertices in $N(u_2) - \{u, u_1\}$. Since every vertex in $N^2[u] \cap N^2[u_1]$ is either a neighbour of u or u_1, or has distance exactly two from both, u and u_1, we have $|N^2[u] \cap N^2[u_1]| = d + d_1 + d_2 - 3$. By inclusion-exclusion we have

$$\begin{aligned} |N^2[u] \cup N^2[u_1]| &= |N^2[u]| + |N^2[u_1]| - |N^2[u] \cap N^2[u_1]| \\ &\geq \Big[d_1 + d_2 - 1 + (d-2)(\delta-1) + \epsilon_d\Big] \\ &\quad + \Big[d + d_2 - 1 + (d_1-2)(\delta-1) + \epsilon_{d_1}\Big] - \Big[d + d_1 + d_2 - 3\Big] \\ &= (d + d_1 - 4)(\delta - 1) + d_2 + 1 + \epsilon_d + \epsilon_{d_1} \\ &\geq (2\delta - 4)(\delta - 1) + \delta + 1 + 2\epsilon_\delta \\ &= 2\delta^2 - 5\delta + 5 + 2\epsilon_\delta, \end{aligned}$$

which is (4).

Since $|U| \leq \lfloor \frac{n}{2} \rfloor$, we have

$$|(N^2[u] \cup N^2[u_1]) \cap W| \geq |N^2[u] \cup N^2[u_1]| - |U| \geq 2\delta^2 - 5\delta + 5 + 2\epsilon_\delta - \left\lfloor \frac{n}{2} \right\rfloor.$$

Since u and u_1 are interior vertices of G_1, the vertices of the set $(N^2[u] \cup N^2[u_1]) \cap W$ are all at distance exactly two from u or u_1. Therefore, each vertex in $(N^2[u] \cup N^2[u_1]) \cap W$ has a neighbour in $N(u) \cup N(u_1)$, and thus in U. It follows that

$$|S| \geq |(N^2[u] \cup N^2[u_1]) \cap W| \geq 2\delta^2 - 5\delta + 5 + 2\epsilon_\delta - \left\lfloor \frac{n}{2} \right\rfloor,$$

as desired. □

Now $2\delta^2 - 5\delta + 5 + 2\epsilon_\delta - \lfloor \frac{n}{2} \rfloor \geq \delta$ holds if and only if $\delta^2 - 3\delta + \frac{5}{2} + \epsilon_\delta \geq \frac{1}{2}\lfloor \frac{n}{2} \rfloor$. Hence we have the following corollaries.

Corollary 3. *Let G be a graph of order n and minimum degree δ that contains neither C_4 nor C_5 as a subgraph. If*

$$\delta^2 - 3\delta + \frac{5}{2} + \epsilon_\delta \geq \left\lfloor \frac{n}{2} \right\rfloor,$$

then G is maximally edge-connected.

Corollary 4. *Let G be a graph of order n and minimum degree δ that contains neither C_4 nor C_5 as a subgraph. If*

$$\delta(G) \geq \frac{3}{2} + \frac{1}{2}\sqrt{n-1},$$

then G is maximally edge-connected.

The following example demonstrates that for an infinite number of values of n, Corollary 4 is sharp apart from a small additive constant. The construction is based on the well-known construction of a projective plane and its incidence graph (see, for example [11]), the graph H'_q below.

Example 2. Let q be a prime power. Define the graph H'_q as follows. Let A_q (B_q) be the set of all 1-dimensional (2-dimensional) subspaces of the vector space $GF(q)^3$. Let H'_q be the bipartite graph with partite sets A_q and B_q, where $a \in A_q$ and $b \in B_q$ are adjacent if a is a subspace of b in $GF(q)^3$. As in Example 1 we have $|A_q| = |B_q| = q^2 + q + 1$, so $n(H'_q) = 2(q^2+q+1)$. and that H'_q is $(q+1)$-regular since every two-dimensional subspace of $GF(1)^3$ contains exactly $q+1$ one-dimensional subspaces,and every one-dimensional subspace of $GF(1)^3$ is contained in exactly $q+1$ two-dimensional subspaces. H'_q does not contain a 4-cycle since any two vertices $a_1, a_2 \in A_q$ have exactly one common neighbour b, where b is the two-dimensional subspace $\langle a_1 \cup a_2 \rangle$ of $GF(q)^3$.
Now let G'_q be the graph obtained from two disjoint copies of H'_q by adding an edge joining two vertices in distinct copies of H'_q. Since H'_q is bipartite and C_4-free, G'_q contains neither C_4

nor C_5 as a subgraph. We have $n(G'_q) = 4(q^2+q+1)$, $\delta(G'_q) = q+1$, and $\lambda(G'_q) = 1 < \delta(G'_q)$. Moreover,

$$\delta(G'_q) = q + 1 = \sqrt{\frac{n(G'_q) - 3}{4}} - \frac{1}{2},$$

which differs from the value $\frac{3}{2} + \frac{1}{2}\sqrt{n-1}$ by less than 5.

4. Graphs of girth at least 6

We now show that Theorems 1 and 2 can be strengthened further if G contains neither C_3, nor C_4, nor C_5 as a subgraph, i.e., if G has girth at least 6.

Theorem 3. *Let G be a C_4-free graph of order n, minimum degree δ and girth at least 6. If G is not maximally edge-connected, then*

$$\lambda(G) \geq 2\delta^2 - 2\delta + 2 - \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. Let G_1 G_2, U, W, $u, u_1, \ldots, u_d$ and d be as in the proof of Theorem 1. As in the proof of Theorem 2 we may assume that u_1 is an interior vertex of G_1.
We now bound $|N^2[u] \cup N^2[u_1]|$ from below. Since G is C_3-free, the neighbours of u form an independent set, so each u_i has at least $\delta - 1$ neighbours that are at distance two from u. Since G is C_4-free the sets $N(u_i) - \{u\}$ are disjoint for $i = 1, 2, \ldots, d$. Hence

$$|N^2[u]| \geq 1 + d + d(\delta - 1) = d\delta + 1. \tag{6}$$

Similarly, if d_1 is the degree of u_1, we show that $|N^2[u_1]| \geq d_1\delta + 1$. Since G contains no cycle of length five, there is no vertex in G that is at distance exactly two from both, u and u_1. Hence the intersection $N^2[u] \cap N^2[u_1]$ is contained in, and in fact equal to, $N[u] \cup N[v]$. Hence, by inclusion-exclusion,

$$\begin{aligned}
|N^2[u] \cup N^2[u_1]| &= |N^2[u]| + |N^2[u_1]| - |N^2[u] \cap N^2[u_1]| \\
&= |N^2[u]| + |N^2[u_1]| - |N[u] \cup N[u_1]| \\
&\geq (d\delta + 1) + (d_1\delta + 1) - d - d_1 \\
&= (d + d_1)(\delta - 1) + 2 \\
&\geq 2\delta(\delta - 1) + 2.
\end{aligned}$$

Since $|U| \leq \lfloor \frac{n}{2} \rfloor$, we have

$$|(N^2[u] \cup N^2[u_1]) \cap W| \geq |N^2[u] \cup N^2[u_1]| - |U| \geq 2\delta^2 - 2\delta + 2 - \left\lfloor \frac{n}{2} \right\rfloor.$$

Since u and u_1 are interior vertices of G_1, the vertices of the set $(N^2[u] \cup N^2[u_1]) \cap W$ are all at distance exactly two from u or u_1. Therefore, each vertex in $(N^2[u] \cup N^2[u_1]) \cap W$ has a neighbour in $N(u) \cup N(u_1)$, and thus in U. It follows that

$$|S| \geq |(N^2[u] \cup N^2[u_1]) \cap W| \geq 2\delta^2 - 2\delta + 2 - \left\lfloor \frac{n}{2} \right\rfloor,$$

as desired. □

Now $2\delta^2 - 2\delta + 2 - \lfloor \frac{n}{2} \rfloor \geq \delta$ holds if and only if $2\delta^2 - 3\delta + 2 \geq \lfloor \frac{n}{2} \rfloor$. Hence we have the following corollaries.

Corollary 5. *Let G be a graph of order n, minimum degree δ and girth at least six. If*

$$2\delta^2 - 3\delta + 2 \geq \lfloor \frac{n}{2} \rfloor,$$

then G is maximally edge-connected.

Corollary 6. *Let G be a graph of order n, minimum degree δ and girth at least six. If*

$$\delta(G) \geq \frac{1}{2}\sqrt{n - \epsilon_n - \frac{7}{4}} + \frac{3}{4},$$

then G is maximally edge-connected.

Again, Example 2 shows that Corollary 6 is sharp apart from a small additive constant for an infinite number of values of n and δ.

References

[1] C. Balbuena and A. Carmona, *On the connectivity and superconnectivity of bipartite digraphs and graphs*, Ars Combin. **61** (2001), 3–22.

[2] B. Bollobás, *On graphs with equal edge connectivity and minimum degree*, Discrete Math. **28** (1979), no. 3, 321–323.

[3] W.G. Brown, *On graphs that do not contain a thomsen graph*, Canad. Math. Bulletin **9** (1966), no. 3, 281–285.

[4] G. Chartrand, *A graph-theoretic approach to a communications problem*, SIAM J. Appl. Math. **14** (1966), no. 4, 778–781.

[5] P. Dankelmann, A. Hellwig, and L. Volkmann, *On the connectivity of diamond-free graphs*, Discrete Appl. Math. **155** (2007), no. 16, 2111–2117.

[6] ———, *Inverse degree and edge-connectivity*, Discrete Math. **309** (2009), no. 9, 2943–2947.

[7] P. Dankelmann and L. Volkmann, *New sufficient conditions for equality of minimum degree and edge-connectivity*, Ars Combin. **40** (1995), 270–278.

[8] ———, *Degree sequence conditions for maximally edge-connected graphs and digraphs*, J. Graph Theory **26** (1997), no. 1, 27–34.

[9] ———, *Degree sequence conditions for maximally edge-connected graphs depending on the clique number*, Discrete Math. **211** (2000), no. 1-3, 217–223.

[10] P. Erdös and A. Rényi, *On a problem in the theory of graphs (hungarian)*, Magyar Tud. Akad. Mat. Kutató. Int. Közl. **7** (1962), 623–641.

[11] C. Godsil and G. Royle, *Algebraic graph theory*, Springer, New York, 2001.

[12] Y.O. Hamidoune, *A property of a-fragments of a digraph*, Discrete Math. **31** (1980), no. 1, 105–106.

[13] A. Hellwig and L. Volkmann, *Maximally edge-connected digraphs*, Australas. J. Combin. **27** (2003), 23–32.

[14] ______, *Maximally edge-connected and vertex-connected graphs and digraphs: A survey*, Discrete Math. **308** (2008), no. 15, 3265–3296.

[15] L. Lesniak, *Results on the edge-connectivity of graphs*, Discrete Math. **8** (1974), no. 4, 351–354.

[16] L. Volkmann, *Bemerkungen zum p-fachen zusammenhang von graphen*, An. Univ. Bucuresti Mat. **37** (1988), 75–79.

[17] ______, *Edge-connectivity in p-partite graphs*, J. graph theory **13** (1989), no. 1, 1–6.

[18] ______, *Degree sequence conditions for maximally edge-connected oriented graphs*, Applied Math. lett. **19** (2006), no. 11, 1255–1260.

[19] H. Whitney, *Congruent graphs and the connectivity of graphs*, Amer. J. Math. **54** (1932), 150–168.