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A note on the Roman domatic number of a digraph

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Abstract: A *Roman dominating function* on a digraph D with vertex set $V(D)$ is a labeling $f\colon V(D) \to \{0,1,2\}$ such that every vertex with label 0 has an in-neighbor with label 2. A set $\{f_1, f_2, \dots, f_d\}$ of Roman dominating functions on D with the property that $\sum_{i=1}^{d} f_i(v) \le 2$ for each $v \in V(D)$, is called a *Roman dominating family* (of functions) on D. The maximum number of functions in a Roman dominating family on D is the *Roman domatic number* of D, denoted by $d_R(D)$. In this note, we study the Roman domatic number in digraphs, and we present some sharp bounds for $d_R(D)$. In addition, we determine the Roman domatic number of some digraphs. Some of our results are extensions of well-known properties of the Roman domatic number of undirected graphs.

Keywords: Digraphs, Roman dominating function, Roman domination number, Roman domatic number

AMS Subject classification: 05C69, 05C20

1. Introduction

In this paper, D is a simple digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. The order $|V|$ of D is denoted by $n = n(D)$. We write $d_D^+(v) = d^+(v)$ for the *out-degree* of a vertex v and $d_D^-(v) = d^-(v)$ for its *in-degree*. The *minimum* and *maximum in-degree* and *minimum* and *maximum out-degree* of D are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. If uv is an arc of D, then we also write $u \to v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. For a vertex v of a digraph D, we denote the set of in-neighbors and out-neighbors of v by $N_D^-(v) = N^-(v)$ and $N_D^+(v) = N^+(v)$, respectively. In addition, $N^-[v] = N^-(v) \cup \{v\}$ and $N^+[v] = N^+(v) \cup \{v\}$. If $X \subset V(D)$, then

$$N_D^+[X] = N^+[X] = \bigcup_{v \in X} N^+[v].$$

We write K_n^* for the *complete digraph* of order n. An *oriented cycle* in a digraph is also called a *cycle*. For notation and graph theory terminology in general we follow [4].

A subset $S \subseteq V(D)$ is a *dominating set* if $N^+[S] = V(D)$. The *domination number* $\gamma(D)$ is the minimum cardinality of a dominating set of D. The domination number for digraphs was introduced by Lee [8]. A *domatic partition* is a partition of $V(D)$ into dominating sets, and the *domatic number* $d(D)$ is the largest number of sets in a domatic partition.

A *Roman dominating function* (RDF) on a digraph D is defined in [7, 12] as a function $f\colon V(D) \to \{0,1,2\}$ satisfying the condition that every vertex v with $f(v) = 0$ has an in-neighbor u with $f(u) = 2$. The *weight* of an RDF f is the value $\omega(f) = \sum_{v\in V(D)} f(v)$. The *Roman domination number* of a digraph D, denoted by $\gamma_R(D)$, is the minimum taken over the weights of all Roman dominating functions on D. A $\gamma_R(D)$-*function* is a Roman dominating function on D with weight $\gamma_R(D)$. A Roman dominating function $f\colon V \to \{0,1,2\}$ can be represented by the ordered partition (V_0, V_1, V_2) (or (V_0^f, V_1^f, V_2^f) to refer to f) of $V(D)$, where $V_i = \{v \in V(D)\colon f(v) = i\}$. In this representation, the weight of f is $\omega(f) = |V_1| + 2|V_2|$. Since $V_1^f \cup V_2^f$ is a dominating set when f is an RDF, and since placing weight 2 at the vertices of a dominating set yields an RDF, we observe that

$$\gamma(D) \le \gamma_R(D) \le 2\gamma(D).$$

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct Roman dominating functions on D with $\sum_{i=1}^{d} f_i(v) \le 2$ for each $v \in V(D)$, is called in [14] a *Roman dominating family* (of functions) on D. The maximum number of functions in a Roman dominating family (RD family) on D is the *Roman domatic number* of D, denoted by $d_R(D)$. The Roman domatic number is well-defined and $d_R(D) \ge 1$ for all digraphs D since the set consisting of any RDF forms an RD family on D (see [14]).

Our purpose in this paper is to improve some results given in [14], and to determine the Roman domatic number of some classes of digraphs. Some of our results are extensions of well-known properties of the Roman domatic number $d_R(G)$ of graphs G.

The following known results are interesting.

Theorem 1 ([14]). *If D is a digraph of order n, then $\gamma_R(D) \cdot d_R(D) \le 2n$.*

Theorem 2 ([14]). *If D is a digraph of order $n \ge 2$, then $\gamma_R(D) + d_R(D) \le n + 2$.*

2. Roman domatic number of digraphs

The following upper bound on the Roman domatic number can be found in [14].

Theorem 3. *For every digraph D,*

$$d_R(D) \leq \delta^-(D) + 2$$

and this bound is sharp.

We will prove the following extension of Theorem 3.

Theorem 4. *For every digraph D,*

$$d_R(D) \leq \delta^-(D) + 2.$$

Moreover, if $d_R(D) = \delta^-(D) + 2$, then the set of vertices of minimum in-degree is an independent set.

Proof. If $d_R(D) \leq 2$, then the bound is immediate. Let now $d_R(D) \geq 3$ and let $\{f_1, f_2, \ldots, f_d\}$ be an RD family on D such that $d = d_R(D)$. Assume that v is a vertex of minimum in-degree $\delta^-(D)$. Since the equality $\sum_{x \in N^-[v]} f_i(x) = 1$ holds for at most two indices $i \in \{1, 2, \ldots, d\}$, we have

$$2d - 2 \leq \sum_{i=1}^{d} \sum_{x \in N^-[v]} f_i(x) = \sum_{x \in N^-[v]} \sum_{i=1}^{d} f_i(x) \leq \sum_{x \in N^-[v]} 2 = 2(\delta^-(D) + 1).$$

This implies the desired bound $d_R(D) \leq \delta^-(D) + 2$. Moreover, if $d_R(D) = \delta^-(D) + 2$, then the following holds for every vertex v of minimum in-degree.

(i) There exist precisely two indices $j, k \in \{1, 2, \ldots, d\}$ such that

$$\sum_{x \in N^-[v]} f_j(x) = \sum_{x \in N^-[v]} f_k(x) = 1;$$

(ii) For every index $i \in \{1, 2, \ldots, d\} \setminus \{j, k\}$: $\sum_{x \in N^-[v]} f_i(x) = 2$;

(iii) For every vertex $x \in N^-[v]$ and every index $i \in \{1, 2, \ldots, d\}$: $\sum_{i=1}^{d} f_i(x) = 2$.

In particular, (i) implies that $f_j(v) = f_k(v) = 1$ and $f_j(x) = f_k(x) = 0$ for every $x \in N^-(v)$, (ii) implies that for every index $i \in \{1, 2, \ldots, d\} \setminus \{j, k\}$ there exists a unique vertex $x_i \in N^-(v)$ such that $f_i(x_i) = 2$, and (iii) implies that $f_i(y) = 0$ for every vertex $x_i \neq y \in N^-[v]$. This is impossible if D contains two adjacent vertices of minimum in-degree. □

The *complement* $\overline{D}$ of a digraph D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u, v the arc uv belongs to $\overline{D}$ if and only if uv does not belong to D. A digraph D is *in-regular* when $\delta^-(D) = \Delta^-(D)$ and *out-regular* when $\delta^+(D) = \Delta^+(D)$. Xie, Hao and Wei [14] proved the following Nordhaus-Gaddum type inequality.

Theorem 5. *If D is a digraph of order $n \geq 2$, then*

$$d_R(D) + d_R(\overline{D}) \leq n + \epsilon$$

with $\epsilon = 1$ when D is out-regular, $\epsilon = 2$ when D is not in-regular and $\epsilon = 3$ otherwise.

As an application of Theorem 4, we shall prove the following improvement of Theorem 5

Theorem 6. *If D is a digraph of order n, then*

$$d_R(D) + d_R(\overline{D}) \leq n + 1. \tag{1}$$

Proof. If, without loss of generality, $D = K_n^*$, then $d_R(D) = n$ and $d_R(\overline{D}) = 1$ and the inequality holds. So assume that $D \neq K_n^*$ and $\overline{D} \neq K_n^*$.
If D and $\overline{D}$ contain adjacent vertices of minimum in-degree, then let x be a vertex such that $d_D^-(x) = \delta^-(D)$. By Theorem 4, it follows that

$$d_R(D) + d_R(\overline{D}) \leq (\delta^-(D) + 1) + (\delta^-(\overline{D}) + 1) \leq d_D^-(x) + d_{\overline{D}}^-(x) + 2 = n + 1.$$

If, without loss of generality, $\overline{D}$ contains adjacent vertices of minimum in-degree and D does not contain adjacent vertices of minimum in-degree, then D is not in-regular and thus $\delta^-(D) < \Delta^-(D)$. Let x and y be two vertices such that $d_D^-(x) = \delta^-(D)$ and $d_D^-(y) = \Delta^-(D)$. Using Theorem 3, we obtain

$$\begin{aligned} d_R(D) + d_R(\overline{D}) &\leq (\delta^-(D) + 2) + (\delta^-(\overline{D}) + 1) \leq d_D^-(x) + d_{\overline{D}}^-(y) + 3 \\ &< d_D^-(y) + d_{\overline{D}}^-(y) + 3 = n + 2 \end{aligned}$$

and thus $d_R(D) + d_R(\overline{D}) \leq n + 1$.
It remains to check the case that neither D nor $\overline{D}$ contain adjacent vertices of minimum in-degree. Then both digraphs are not in-regular. First we assume that $\delta^-(D) \leq \Delta^-(D) - 2$ or $\delta^-(\overline{D}) \leq \Delta^-(\overline{D}) - 2$, say $\delta^-(D) \leq \Delta^-(D) - 2$. Then let x and y be two vertices such that $d_D^-(x) = \delta^-(D)$ and $d_D^-(y) = \Delta^-(D)$. Again by Theorem 4, it follows that

$$\begin{aligned} d_R(D) + d_R(\overline{D}) &\leq (\delta^-(D) + 2) + (\delta^-(\overline{D}) + 2) \leq d_D^-(x) + d_{\overline{D}}^-(y) + 4 \\ &\leq d_D^-(y) + d_{\overline{D}}^-(y) + 2 = n + 1. \end{aligned}$$

So assume that $\delta^-(D) = \Delta^-(D) - 1$ and $\delta^-(\overline{D}) = \Delta^-(\overline{D}) - 1$. If $d_R(D) = \delta^-(D) + 2$ and $d_R(\overline{D}) = \delta^-(\overline{D}) + 2$, then let $A = \{v\colon d_D^-(v) = \delta^-(D)\}$ and $B = \{v\colon d_{\overline{D}}^-(v) = \delta^-(\overline{D})\}$. Note that $A \cup B = V(D)$. Since A is independent in D, it follows that $\overline{D}[A]$

is complete. Likewise, B is independent in $\overline{D}$ and $D[B]$ is complete. Let $d = d_R(D)$ and $\{f_1, f_2, \ldots, f_d\}$ be an RD family on D. Note that $f_i(a) \le 1$ for every vertex $a \in A$ and thus, $f_i(b) = 2$ for at least one vertex $b \in B$ for every index i. It follows that $d \le |B| \le \delta^-(D) + 1$, a contradiction. Hence, without loss of generality, $d_R(D) \le \delta^-(D) + 1$, and thus

$$\begin{aligned} d_R(D) + d_R(\overline{D}) &\le (\delta^-(D) + 1) + (\delta^-(\overline{D}) + 2) \le d^-_D(x) + d^-_{\overline{D}}(y) + 3 \\ &\le d^-_D(y) + d^-_{\overline{D}}(y) + 2 = n + 1. \end{aligned}$$

This completes the proof. □

If D is isomorphic to the complete graph K_n^*, then $d_R(D) = n$ and $d_R(\overline{D}) = 1$ and therefore $d_R(D) + d_R(\overline{D}) = n + 1$. This example demonstrates that the bound (1) is sharp.
For out-regular graphs, the following upper bound on the Roman domatic number is valid (see [14]).

Theorem 7. *If D is a δ-out-regular digraph of order $n = p(\delta+1)+r$ with integers $p \ge 1$ and $0 \le r \le \delta$, then*

$$d_R(D) \le \delta + \varepsilon \tag{2}$$

with $\varepsilon = 1$ when $\delta = 0$ or $r = 0$ or $2r = \delta + 1$ and $\varepsilon = 0$ otherwise.

Let $q \ge 4$ be an integer, and let $K^*_{q,q}$ be the complete bipartite digraph with the partite sets $\{u_1, u_2, \ldots, u_q\}$ and $\{v_1, v_2, \ldots, v_q\}$. Define the RDFs $f_1, f_2, \ldots, f_q$ by $f_i(u_i) = f_i(v_i) = 2$ and $f_i(u_j) = f_i(v_j) = 0$ for $j \ne i$ and $1 \le i, j \le q$. Because of $\sum_{i=1}^{q} f_i(x) = 2$ for each vertex x in $K^*_{q,q}$, the set $\{f_1, f_2, \ldots, f_q\}$ is an RD family on $K^*_{q,q}$. Thus $d_R(K^*_{q,q}) \ge q$ and therefore Theorem 7 implies that $d_R(K^*_{q,q}) = q$. This example demonstrates that Theorem 7 is sharp.

If D is a digraph, then we denote by D^{-1} the digraph obtained by reversing all arcs of D. A digraph without cycles of length 2 is called an *oriented graph*. An oriented graph D is a *tournament* when either $xy \in A(D)$ or $yx \in A(D)$ for each pair of distinct vertices $x, y \in V(D)$.
For regular tournaments the following upper bound is valid.

Theorem 8. *If D is a δ-regular tournament of order $n \ge 3$, then*

$$d_R(D) + d_R(D^{-1}) \le n - 1.$$

Proof. Since D is a δ-regular tournament, D^{-1} is also a δ-regular tournament of order $n = 2\delta + 1$. If $\delta \ge 2$, then Theorem 7 leads to

$$d_R(D) + d_R(D^{-1}) \le 2\delta = n - 1.$$

Since this remains valid for $\delta = 1$, the proof is complete. □

The class of round regular tournaments shows that the bound in Theorem 8 is sharp. A tournament D on $V = \{v_0, v_1, \ldots, v_{2p}\}$ is called *round* if for every vertex v_i the out- and in-neighborhood are given by $N^+(v_i) = \{v_{i+1}, v_{i+2}, \ldots, v_{i+p}\}$ and $N^-(v_i) = \{v_{i-1}, v_{i-2}, \ldots, v_{i-p}\}$, where all indices are taken modulo $n = 2p+1$. For $i = 1, 2, \ldots, p$, define $f_i \colon V \to \{0,1,2\}$ by $f_i(v_i) = f_i(v_{i+p+1}) = 2$ and $f_i(v_j) = 0$ for $j \notin \{i, i+p+1\}$. Then $\{f_1, f_2, \ldots, f_p\}$ is an RD family on D. Since D is isomorphic to D^{-1}, it follows that $d_R(D) \geq p$ and $d_R(D^{-1}) \geq p$ and thus $d_R(D) + d_R(D^{-1}) = 2p = n-1$ by Theorem 8.
For arbitrary oriented digraphs we shall prove a slightly weaker upper bound.

Theorem 9. *If D is an oriented graph of order n, then*

$$d_R(D) + d_R(D^{-1}) \leq n + 1.$$

Proof. For regular tournaments the bound is true by Theorem 8.
If D is an almost-regular tournament, then D^{-1} is also an almost-regular tournament and $\delta^-(D) = \delta^-(D^{-1}) = (n-2)/2$. Furthermore, D and D^{-1} contain adjacent vertices of minimum in-degree. By Theorem 3, it follows that

$$d_R(D) + d_R(D^{-1}) \leq \delta^-(D) + 1 + \delta^-(D^{-1}) + 1 = n.$$

If D is a tournament that is neither regular nor almost-regular, then $\delta^-(D) + \delta^-(D^{-1}) \leq n-3$. Again Theorem 3 implies that

$$d_R(D) + d_R(D^{-1}) \leq \delta^-(D) + 2 + \delta^-(D^{-1}) + 2 \leq n + 1.$$

If D is not a tournament, then it is a subdigraph of a tournament. Since every RD family on D is also an RD family on every superdigraph D' of D, it follows that $d_R(D) \leq d_R(D')$. Using this observation, it is immediate that $d_R(D) + d_R(D^{-1}) \leq n+1$ for every oriented digraph D. This completes the proof. □

3. The Roman domatic number of a graph

A *Roman dominating function* on a graph $G = (V(G), E(G))$ is defined in [10, 13] as a function $f \colon V(G) \to \{0,1,2\}$ satisfying the condition that every vertex v with $f(v) = 0$ is adjacent to at least one vertex u with $f(u) = 2$. The *weight* of an RDF f is the value $\omega(f) = \sum_{v \in V(G)} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G. In [1–3, 5, 6, 9] the reader can find a lot of results on Roman domination.
A set $\{f_1, f_2, \ldots, f_d\}$ of distinct RDFs on G with $\sum_{i=1}^{d} f_i(v) \leq 2$ for each $v \in V(G)$, is called a *Roman dominating family* (of functions) on G. The maximum number of functions in an RD family on G is the *Roman domatic number* of G, denoted by $d_R(G)$.

The *associated digraph* $D(G)$ of a graph G is the digraph obtained from G by replacing each edge of G by a 2-cycle. Since $N^-_{D(G)}[v] = N_G[v]$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 10. If $D(G)$ is the associated digraph of a graph G, then $\gamma_R(D(G)) = \gamma_R(G)$ and $d_R(D(G)) = d_R(G)$.

There are a lot of interesting applications of Observation 10, as for example the following results. Using Theorem 1, we obtain the first one.

Corollary 1 ([11]). *If G is a graph of order n, then $\gamma_R(G) \cdot d_R(G) \leq 2n$.*

Theorem 2 and Observation 10 imply the next corollary immediately.

Corollary 2 ([11]). *If G is a graph of order $n \geq 2$, then $\gamma_R(G) + d_R(G) \leq n + 2$.*

Since $\delta^-(D(G)) = \delta(G)$ and a set of vertices is independent in G if and only if it is independent in $D(G)$, Theorem 4 and Observation 10 lead to the following result.

Corollary 3. *If G is a graph, then $d_R(G) \leq \delta(G) + 2$. Moreover, if $d_R(G) = \delta(G) + 2$, then the set of vertices of minimum degree is an independent set.*

The bound $d_R(G) \leq \delta(G) + 2$ can be found in [11]. Finally, we obtain the following Nordhaus-Gaddum bound in view of Theorem 6 and Observation 10.

Corollary 4. *If G is a graph of order n, then*

$$d_R(G) + d_R(\overline{G}) \leq n + 1.$$

In [11] Corollary 4 was only proved for regular graphs.

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