On the edge geodetic and edge geodetic domination numbers of a graph

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Abstract: In this paper, we study both concepts of geodetic dominating and edge geodetic dominating sets and derive some tight upper bounds on the edge geodetic and the edge geodetic domination numbers. We also obtain attainable upper bounds on the maximum number of elements in a partition of a vertex set of a connected graph into geodetic sets, edge geodetic sets, geodetic dominating sets and edge geodetic dominating sets, respectively.

Keywords: domination number, (edge) geodetic number, (edge) geodetic domination number

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1. Introduction and preliminaries

By a graph, we mean a finite, undirected graph with neither loops nor multiple edges. We denote the vertex set and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. In a graph $G$, for a subset $S \subseteq V(G)$ the subgraph induced by $S$ is the graph $\langle S \rangle$ with vertex set $S$ and edge set $\{xy \in E(G) \mid x, y \in S\}$. We write $K_n$ for the complete graph of order $n$, $K_{m,n}$ for the complete bipartite graph with partite sets of order $m$ and $n$, $P_n$ for the path on $n$ vertices, and $C_m$ for the cycle of length $m$.

For any vertex $x$ of a graph $G$, $N_G(x)$ denotes the set of all neighbors of $x$ in $G$, $N_G[x] = N_G(x) \cup \{x\}$ and the degree of $x$ is $\deg(x, G) = |N_G(x)|$. The minimum and maximum degrees of a graph $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a subset $A \subseteq V(G)$, let $N_G[A] = \cup_{x \in A} N_G[x]$. If $e = uv$ is an edge of a graph $G$ with $\deg(u) = 1$ and $\deg(v) > 1$, then we call $e$ a pendant edge, $u$ a leaf and $v$ a support vertex. Let $L(G)$ and $S(G)$ be the sets of all leaves and all support vertices of a graph $G$, respectively. The corona $\text{cor}(G)$ of a graph $G$ is constructed from $G$, where for each vertex $v \in V(G)$, a new vertex $v'$ and a pendant edge $vv'$ are added.
For two graphs $G_1$ and $G_2$, the Cartesian product $G_1 \square G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and $(x_1, x_2)(y_1, y_2) \in E(G_1 \square G_2)$ if and only if $x_1y_1 \in E(G_1)$ and $x_2 = y_2$, or $x_2y_2 \in E(G_2)$ and $x_1 = y_1$.

Let $\mathcal{P}$ be a property of vertex subsets of a graph $G$ and let $\mu(G)$ denote the minimum cardinality of sets with property $\mathcal{P}$. Any set with property $\mathcal{P}$ of cardinality $\mu(G)$ is called a $\mu$-set of $G$.

A vertex in a graph $G$ dominates itself and its neighbors. A set of vertices $D$ in a graph $G$ is a dominating set if each vertex of $G$ is dominated by some vertex of $D$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set of $G$. Cockayne and Hedetniemi [3] defined the domatic number $d(G)$ of a graph $G$ to be the maximum number of elements in a partition of $V(G)$ into dominating sets. A dominating set $D$ of $G$ is called an efficient dominating set (an ED-set) of $G$ if the distance between any two vertices in $D$ is at least three. Not all graphs have ED-sets. If $G$ has an ED-set, then any ED-set is a $\gamma$-set of $G$ [1]. For a comprehensive introduction to the theory of domination in graphs we refer the reader to Haynes et al. [10].

The distance $d_G(u,v)$ between two vertices $u$ and $v$ in $G$ is the length of a shortest $u - v$ path or $\infty$, if no such path exists. The diameter of $G$, denoted by $diam(G)$, is the maximum distance between two vertices in $G$. A $u - v$ path of length $d_G(u,v)$ is called a $u - v$ geodesic. Any geodesic of length $diam(G)$ is called a diametral path. Two vertices $u$ and $v$ of $G$ are called antipodal if $d(u,v) = diam(G)$. A vertex $w$ is said to lie on a $u - v$ geodesic $P$ if $w$ is an internal vertex of $P$. The closed interval $I_G[u,v]$ consists of $u$, $v$ and all vertices lying on some $u - v$ geodesic of $G$. For a set $S$ of vertices, let the interval $I_G[S]$ of $S$ be the union of the intervals $I_G[u,v]$ over all pairs of vertices $u$ and $v$ in $S$. Note that a vertex $w$ belongs to $I_G[u,v]$ if and only if $d_G(u,v) = d_G(u,w) + d_G(w,v)$. A set $S \subseteq V(G)$ is a geodetic set of $G$ if $I_G[S] = V(G)$. Harary et al. [9] define the geodetic number $g(G)$ of a graph $G$ as the minimum cardinality of a geodetic set. It is $NP$-complete to decide for a given chordal or chordal bipartite graph $G$ and a given integer $k$ whether $G$ has a geodetic set of cardinality at most $k$ ([4]).

Let $G$ be a connected graph and $u,v \in V(G)$. The set $I_G^c[u,v]$ consists of all edges of $G$ lying in any $u - v$ geodesic in $G$. If $S \subseteq V(G)$, then the set $I_G^c[S]$ denotes the union of all $I_G^c[u,v]$, where $u,v \in S$. A subset $S$ of $V(G)$ is an edge geodetic set of $G$ if $I_G^c[S] = E(G)$. Clearly, any edge geodetic set of a graph is geodetic. The edge geodetic number of $G$, denoted by $g_e(G)$, is the minimum cardinality of an edge geodetic set of $G$ (Santhakumaran and John [16]).

A subset $S$ of $V(G)$ is a geodetic dominating set (a GD-set for short) in $G$ if $S$ is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set of $G$ is its the geodetic domination number, and is denoted by $\gamma_g(G)$. The study of the geodetic domination was initiated by Escuadro et al. [5] in 2011. Some other interesting results can also be found in Hansberg and Volkmann [8].

The notion of edge geodetic domination was introduced by Arul Paul Sudhahar et al. [19]. A set of vertices $S$ of a graph $G$ is an edge geodetic dominating set (an EGD-set) if it is both an edge geodetic set and a dominating set of $G$. The minimum
cardinality among all the EGD sets of $G$ is called the edge geodetic domination number (EGD-number) and is denoted by $\gamma_{ge}(G)$.

The paper is organized as follows. The next section consists of some known results which will be useful in proving our main results. In Section 3, we give some general results and sharp bounds for the geodetic domination number and the edge geodetic domination number. To present our results obtained in Section 4 we need the following definitions. For every connected $n$-order graph $G$ we define the geodetic partitionable number (edge geodetic partitionable number, geodetic domatic number, edge geodetic domatic number, respectively), denoted $gp(G)$ ($gep(G)$, $gd(G)$, $ged(G)$, respectively), to be the maximum number of elements in a partition of $V(G)$ into geodetic sets (edge geodetic sets, geodetic dominating sets, edge geodetic dominating sets, respectively). In Section 4 we give upper bounds on these four parameters and present some families of graphs, that achieve these bounds.

2. Known results

A vertex $v$ is an extreme vertex of a graph $G$ if the subgraph induced by its neighbours is complete. The set of all extreme vertices of $G$ is denoted by $\text{Ext}(G)$. A vertex $v$ in a connected graph $G$ is said to be a semi-extreme vertex if it has a neighbor, say $u$, with $N[v] \subseteq N[u]$. The set of all semi-extreme vertices of $G$ is denoted by $\text{Se}(G)$.

**Theorem A.** Let $G$ be a connected $n$-order graph, $n \geq 2$.

(i) [9] $2 \leq g(G) \leq n$ and $g(G) = n$ if and only if $G = K_n$.

(ii) [16] $2 \leq g(G) \leq ge(G) \leq n$.

(iii) [9] Each geodetic set of $G$ contains all extreme vertices of $G$.

(iv) [18] Each semi-extreme vertex of $G$ is contained in every edge geodetic set of $G$. The equality $ge(G) = n$ holds if and only if all vertices of $G$ are semi-extreme.

(v) [5] $2 \leq \max\{g(G), \gamma(G)\} \leq \gamma g(G) \leq n$.

(vi) [19] $2 \leq ge(G) \leq \gamma_{ge}(G) \leq n$ and $\gamma g(G) \leq \gamma_{ge}(G)$.

(vii) [17] If $S = \{u, v\}$ is an edge geodetic set of a connected graph $G$, then $u$ and $v$ are antipodal vertices of $G$.

A vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. The vertex cover number of $G$, denoted by $\alpha(G)$, is the minimum cardinality among all vertex covers of $G$.

**Theorem B.** [14] Let $G$ be a connected graph with $\delta(G) \geq 2$. Then $\gamma(G) = \alpha(G)$ if and only if $G$ is bipartite such that for every pair $x$ and $y$ of distance 2 of the smaller partite set, there exist at least two common neighbors of $x$ and $y$ of degree 2.
A subset $X$ of $V(G)$ is called independent if its vertices are mutually non-adjacent. The independence number $\beta_0(G)$ is the largest cardinality among all independent sets of a graph $G$.

**Theorem C.** [7] Let $G$ be a graph without isolated vertices. A subset $I$ of $V(G)$ is independent if and only if $V(G) - I$ is a vertex cover of $G$. In particular, $\beta_0(G) = |V(G)| - \alpha(G)$.

**Theorem D.** Let $G$ be a connected $n$-order graph, $n \geq 2$.

(i) [12] $\gamma(G) \leq n/2$;

(ii) [6, 13] $\gamma(G) = n/2$ if and only if either $G = C_4$ or $G$ is the corona $cor(H)$ for any connected graph $H$.

**Theorem E.** [3] If $G$ is a connected $n$-order graph, then $d(G) \leq \delta(G) + 1$ and $d(G)\gamma(G) \leq n$.

### 3. Edge geodetic domination

We begin with a result on graphs $G$ with $g_e(G) = 2$.

**Proposition 1.** If $G$ is a graph with $g_e(G) = 2$, then $G$ is bipartite.

**Proof.** Let $S = \{u, v\}$ be a $g_e$-set of $G$, $P$ a $u-v$ geodesic of length $k$, and $N^i(u)$ the set of all vertices at distance $i$ from $u$, $i = 0, 1, \ldots, k$. Clearly $N^0(u), N^1(u), \ldots, N^k(u)$ form a partition of $V(G)$. Since $S$ is an edge geodetic set of $G$, each $N^i(u)$ is independent. But then the union of all $N^i(u)$ with $i$ even and the union of all $N^i(u)$ with $i$ odd are both independent and form a partition of $V(G)$. Thus, $G$ is bipartite. $\square$

**Corollary 1.** If $G$ is a connected graph having a cycle of odd length, then $g_e(G) \geq 3$.

A characterization of all connected graphs whose edge geodetic domination number is 2 follows.

**Theorem 1.** Let $G$ be a connected $n$-order graph, $n \geq 2$. Then $\gamma g_e(G) = 2$ if and only if either $G = K_2$ or $G = K_{2,n-2}, n \geq 3$, or $G = (V_1, V_2; E)$ is a bipartite graph having vertices $v_1 \in V_1$ and $v_2 \in V_2$ such that $N(v_1) = V_2 - \{v_2\}$, $N(v_2) = V_1 - \{v_1\}$, and no vertex in $V(G) - \{v_1, v_2\}$ is a leaf.

**Proof.** Necessity: Let $\gamma g_e(G) = 2$ and $S = \{v_1, v_2\}$ a $g_e$-set of $G$. Then $G = (V_1, V_2; E)$ is bipartite (by Proposition 1) and $d(v_1, v_2) = diam(G)$ because of Theorem A(vii). Hence if $diam(G) \leq 2$, then clearly either $G = K_2$ or $G = K_{2,n-2}, n \geq 3$. Since $\{v_1, v_2\}$ is a dominating set of $G$, $diam(G) \leq 3$. So, let $d(v_1, v_2) = 3$. As $S$ is
an edge geodetic set of $G$, clearly no vertex in $V(G) - S$ is a leaf, $v_1$ is adjacent to all vertices in $V_i - \{v_2\}$ and $v_2$ is adjacent to all vertices in $V_j - \{v_1\}$, where $\{i,j\} = \{1,2\}$.  

**Sufficiency.** In all three cases $\{v_1,v_2\}$ is a $\gamma g_e$-set of $G$. So, $\gamma g_e = 2$. □

Let $\mu \in \{g,g_e,\gamma g,\gamma g_e\}$. We say that a graph $G$ is a $\mu$-excellent if each its vertex belongs to some $\mu$-set. An $n$-crown graph $H_{n,n}$ is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching, $n \geq 3$. By the above theorem we immediately obtain the following characterization of $\gamma g_e$-excellent graphs $G$ with $\gamma g_e(G) = 2$.

**Corollary 2.** All $\gamma g_e$-excellent graphs with the edge geodetic domination number equals 2 are $K_2, K_{2,2} = C_4$ and all crown graphs.

**Proposition 2.** Let $G$ be an $n$-order connected graph, $n \geq 2$. 

(i) Let $g_e(G) < \gamma g_e(G)$. Then for each $g_e$-set $S$ of $G$ the set $V(G) - S$ contains the closed neighborhood of some vertex of $G$. In particular, $g_e(G) \leq \min\{n - \delta(G) - 1, n - 3\}$.

(ii) $\gamma g_e(G) = n$ if and only if $g_e(G) = n$.

(iii) If $G$ has $k \geq 1$ vertices of degree $n - 1$, then $\gamma g_e(G) = n$ when $k \geq 2$, and $\gamma g_e(G) = n - 1$ when $k = 1$.

**Proof.** (i) If $S$ is a $g_e$-set of $G$, then since $g_e(G) < \gamma g_e(G)$, $S$ is not a dominating set of $G$. Hence there is a vertex $x_S$ with $N[x_S] \subseteq V(G) - S$. Clearly $x_S$ is not a leaf and then $\gamma g_e(G) = |S| \leq n - \text{deg}(x_S) - 1 \leq \min\{n - \delta(G) - 1, n - 3\}$.

(ii) If $g_e(G) = n$, then $\gamma g_e(G) = n$ because Theorem A(vi). So let $\gamma g_e(G) = n$. Then by (i), $g_e(G) \geq \gamma g_e(G)$ and by Theorem A(vi), the equality $\gamma g_e(G) = g_e(G) = n$ follows.

(iii) Let $x$ be a vertex of degree $n - 1$ in $G$. Hence each vertex of $V(G) - \{x\}$ is semi-extreme. By Theorem A(iv), (a) exactly one of $V(G) - \{x\}$ and $V(G)$ is a $g_e$-set of $G$, and (b) $\gamma g_e(G) = n$ when $k \geq 2$. So, let $x$ be the only vertex of $G$ of degree $n - 1$ and $xy \in E(G)$. Since $\text{deg}(y) < n - 1$, there is $z \in V(G) - \{x,y\}$ non-adjacent to $y$. But then $z, x, y$ is a $z - y$ geodesic that contains $xy$. Thus $\gamma g_e(G) = n - 1$. □

**Remark 1.** The bound in Proposition 2 is tight. Indeed, (a) for the $(r + s + 1)$-order graph $G$ obtained from the stars $K_{1,r}$ and $K_{1,s}$ having a leaf in common, where $r, s \geq 2$, is fulfilled $\delta(G) = 1$ and $g_e(G) = (r + s + 1) - 3$, and (b) for a graph $H$, having a cut-vertex $x$ such that $\text{deg}(x) = r \geq 3$ and $H - x$ has $r$ components each of which is a complete graph of order more than $r$, is fulfilled $\delta(H) = r$ and $g_e(H) = |V(H)| - r - 1 < |V(H)| - 3$.

**Theorem 2.** Let $G$ be a connected $n$-order graph, $n \geq 2$. Then $2 - n/2 \leq g_e(G) - \gamma(G) \leq \gamma g_e(G) - \gamma(G) \leq n - 1$. Let $\mu \in \{g_e, \gamma g_e\}$. Then (a) $2 - n/2 = \mu(G) - \gamma(G)$ if and only if $G \in \{P_2, P_4, C_4\}$, and (b) $\mu(G) - \gamma(G) = n - 1$ if and only if $G$ has at least 2 vertices of degree $n - 1$.}
Proof. By Theorems A and D we know that $2 \leq \mu(G) \leq n$ and $1 \leq \gamma(G) \leq n/2$. Hence $2 - n/2 \leq \mu(G) - \gamma(G) \leq n - 1$. First note that $\mu(G) - \gamma(G) = n - 1$ if and only if both $\mu(G) = n$ and $\gamma(G) = 1$ hold, which by Proposition 2 is equivalent to $G$ has at least 2 vertices of degree $n - 1$. Second clearly $\mu(G) - \gamma(G) = 2 - n/2$ if and only if both $\mu(G) = 2$ and $\gamma(G) = n/2$ are valid, which is equivalent to $G \in \{P_2, P_4, C_4\}$, because of Theorem D(ii).

Let $\mu, \nu \in \{\gamma, g, g_e, \gamma g, \gamma g_e\}$. In a graph $G$, $\mu(G)$ is strongly equal to $\nu(G)$, written $\mu(G) \equiv \nu(G)$, if each $\mu$-set of $G$ is a $\nu$-set of $G$ and vice versa.

Observation 3. If $T$ is a nontrivial tree, then (a) the set of all leaves in $T$ is the unique $g$-set of $T$ and the unique $g_e$-set of $T$, (b) $g(T) \equiv g_e(T)$, (c) each $\gamma g$-set of $T$ is a $\gamma g_e$-set of $T$ and vice versa, and (d) $\gamma g(T) \equiv \gamma g_e(T)$.

Example 1. Let $D$ be a $\gamma$-set of $C_n$. Clearly $\gamma g_e(C_4) = 2 = \gamma(C_4)$ and if $n = 3, 5$ then $\gamma g_e(C_n) = 3 > \gamma(G)$.

To continue, we need the following obvious observation.

Observation 4. Let $G$ be a connected graph of order $n \geq 2$. Then $\gamma g_e(G) = \min\{|S \cup D| : S$ is an EG-set and $D$ is a dominating set of $G\} \leq \gamma(G) + g_e(G)$. A realization result concerning the numbers $\gamma(G), g(G), g_e(G), \gamma g(G)$ and $\gamma g_e(G)$ follows.

Theorem 5. Let $a, b$ and $c$ be positive integers. Then there is a connected graph $G$ such that $\gamma(G) = a$, $g(G) \equiv g_e(G) = b$ and $\gamma g(G) \equiv \gamma g_e(G) = c$ if and only if either $a, b \geq 2$ and $\max\{a, b\} \leq c \leq a + b$, or $a = 1$ and $2 \leq b = c$.

Proof. $\Rightarrow$ Denote by $\mathcal{G}_{a,b,c}$ the class of all graphs $G$ with $\gamma(G) = a$, $g(G) \equiv g_e(G) = b$ and $\gamma g(G) \equiv \gamma g_e(G) = c$. Clearly, $a \geq 1$. By Theorem A, $\max\{g(G), \gamma(G)\} \leq \gamma g(G)$ and $\max\{\gamma g(G), g_e(G)\} \leq \gamma g_e(G)$. Hence for any $G \in \mathcal{G}_{a,b,c}$ are fulfilled: $a \geq 1$, $b \geq 2$ and $\max\{a, b\} \leq c$. On the other hand, Observation 4 implies $c \leq a + b$. If $a \geq 2$, we are done. If $a = 1$, then $b = c = n - 1$ or $b = c = n$ depending of the number of vertices of degree $n - 1$ by Proposition 2.

$\Leftarrow$ Case 1: $a = 1$. Clearly, for the star $K_{1,b}$ with $b \geq 2$ we have $\gamma(K_{1,b}) = 1$. On the other hand, the set of all leaves is the unique $\mu$-set of $K_{1,b}$, for each $\mu \in \{g, g_e, \gamma g, \gamma g_e\}$. Therefore, $K_{1,b} \in \mathcal{G}_{1,b,b}$.

Case 2: $2 \leq a = b = c$. If $T$ is a tree of order $a$, then obviously $cor(T) \in \mathcal{G}_{a,a,a}$. 
Case 3: $2 \leq a < b = c$. Let $H_{a,b}$ be the graph obtained from $K_{1,b}$ by subdividing $a - 1$ edges once. Then the vertex of maximum degree and all leaves nonadjacent to it form a $\gamma$-set of $H_{a,b}$. Hence $H_{a,b}$ is in $C_{a,b,b}$, $a < b$, because of Observation 3.

Case 4: $2 \leq b < a = c$. Take a copy of $K_{1,b-1}$ with the leaves $x_1, x_2, ..., x_{b-1}$ and the support vertex $x$ (if $b = 2$ then let $V(K_{1,1}) = \{x, x_1\}$). Subdivide the edges $xx_i$, $i = 1, 2, ..., b - 1$. Obtain the graph $G_{b,c}$ by taking a copy of the path $P : w_0, w_1, ..., w_{3(a-b)}$, and joining $w_0$ to $x$. Since $U_{b,c}$ is a tree, $U_{b,c}$ is in some $G_{p,q,r}$ (by Observation 3). Since $U_{b,c}$ has $b$ leaves, $q = b$. Finally, $N(x) \cup \{w_3, w_6, ..., w_{3(a-b)}\}$ is clearly a $\gamma$-set of $G$ which is also a geodetic dominating set of $U_{b,c}$. Thus, $p = r = a = c$.

Case 5: $2 \leq \min\{a, b\}$ and $\max\{a, b\} < c \leq a + b - 1$. Take a copy of $K_{1,c-b}$ with the leaves $y_1, y_2, ..., y_{c-b}$ and the support vertex $y$ when $c - b \geq 2$, and $V(K_{1,c-b}) = \{y, y_1\}$ if $c - b = 1$. Subdivide each edge once. Obtain the graph $G_5$ by taking a copy of $K_{1,c-1}$ (with the central vertex $x$ and $x_1, x_2, ..., x_{c-1}$ as leaves) and adding the edges $x_b y_1, x_b + 1 y_2, ..., x_{c-1} y_{c-b}$. If $c < a + b - 1$, then let $G_6$ be a graph obtained from $G_5$ by subdividing each of the edges $xx_{c-a+1}, xx_{c-a+2}, ..., xx_{b-1}$ once.

Let us first consider the case $c = a + b - 1$. Note that $S = \{y, x_1, x_2, ..., x_{c-a}\}$ with $|S| = b$ is the unique $g$-set of $G_5$ and the unique $\gamma_e$-set of $G_5$. Also it is obvious that each $\gamma_g$-set $D$ of $G_5$ is the union of $S$ and some $(c-b)$-cardinality set $\{z_1, z_2, ..., z_{c-b}\}$, where $z_i \in \{y_i, x_{b-1+i}\}$, $i = 1, 2, ..., c - b$. Clearly, $D$ is also a $\gamma_{ge}$-set of $G_5$. It remains to note that $D' = N(y) \cup \{x\}$ is the unique $\gamma$-set of $G_5$ and $|D'| = a$.

Second, assume that $c < a + b - 1$. Now $S_1 = \{y, x_1, x_2, ..., x_{b-1}\}$ with $|S_1| = b$ is the unique $g$-set of $G_6$ and the unique $\gamma_e$-set of $G_6$. Note also that each $\gamma_g$-set $D_1$ of $G_6$ has the form $S_1 \cup \{z_1, z_2, ..., z_{c-b}\}$, where $z_i \in \{y_i, x_{b-1+i}\}$, $i = 1, 2, ..., c - b$. It is obvious that $D_1$ is also a $\gamma_{ge}$-set of $G_6$ and $|D_1| = c$. Finally, each $\gamma$-set $D''$ of $G_6$ has the form $N(y) \cup \{x\} \cup \{u_{c-a+1}, ..., u_{b-1}\}$, where $u_i \in N[x_i]$, $i = c - a + 1, ..., b - 1$. Clearly, $|D''| = a$.

Case 6: $2 \leq a, b$ and $a + b = c$. Let $x$ and $y$ be antipodal vertices of a copy of a cycle $C_6$. Add new vertices $x_1, x_2, ..., x_{a-1}$ and join each of them to the vertex $x$ to obtain the graph $H$. Define the graph $G$ as obtained from $H$ by taking a copy of the path on $3(a-2) + 1$ vertices $y_0, y_1, ..., y_{3(a-2)}$ and joining $y_0$ to the vertex $y$. Clearly, (a) the set $A = \{x, y, y_2, y_3, ..., y_{3(a-2)-1}\}$ is the unique $\gamma$-set of $G$ and $|A| = a$, (b) the set $B = \{x_1, x_2, ..., x_{b-1}, y_{3(a-2)}\}$ has $b$ elements and it is the unique $g$-set of $G$ and the unique $\gamma_{ge}$-set of $G$. Let $C$ be a $\gamma_g$-set of $G$. Then, it contains all vertices in $B$, necessarily. On the other hand, it is easy to observe that such a $\gamma_g$-set must have at least $a$ vertices of $G - C$ in order to be a dominating set in $G$. Therefore, $|C| \geq a + b = c$. Moreover, $S_2 = B \cup \{x, y\} \cup \bigcup_{i=0}^{a-3} \{y_{3i}\}$ if $a \geq 3$, and $S_2 = B \cup \{x, y\}$ if $a = 2$ is a geodetic dominating set in $G$ of cardinality $c$. So, $\gamma_g(G) = c$. Finally, it is obvious that $C$ is an EGD-set of $G$ as well, which implies that $\gamma_g(G) \equiv \gamma_{ge}(G)$.

Theorem 5 shows that the bound in Observation 4 is attainable.

Observation 6. Let $G$ be a connected $n$-order graph, $n \geq 2$, $S$ an edge geodetic set of $G$ and $F$ a dominating set of $G - N[S]$. Then $S \cup F$ is an EGD-set of $G$. In particular, $\gamma_{ge}(G) \leq \gamma_e(G) + \gamma(G - N[S])$. 

**Remark 2.** Clearly \( \{x_1, x_n\} \) is the unique \( g_e \)-set of \( P_n : x_1, x_2, \ldots, x_n \) and if \( 2 \leq n \leq 4 \) then \( \{x_1, x_n\} \) is the unique \( \gamma g_e \)-set of \( P_n \). Let \( n \geq 5 \) and \( S \) be a \( \gamma g_e \)-set of \( P_n \). Then we can choose \( S \) so that \( x_2, x_3, x_4, x_5 \notin S \). Hence \( S \setminus \{x_1, x_n\} \) is a \( \gamma \)-set of \( P_n \). Thus \( \gamma g_e(P_n) = g_e(P_n) + \gamma(P_n - \{x_1, x_2, x_4, x_5\}) \), which shows that the bound in Observation 6 is attainable and in addition \( \gamma g_e(P_n) = 2 + \lceil (n - 4)/3 \rceil = \lceil (n + 2)/3 \rceil \).

In the next theorem we give some conditions under which \( g_e(G) \leq \alpha(G) \) is valid for a graph \( G \).

**Theorem 7.** Let \( G \) be an \( n \)-order connected graph, \( n \geq 2 \).

(i) If \( F \subseteq V(G) \) is a vertex cover of \( G \) which contains all extreme vertices of \( G \), then \( F \) is a geodetic dominating set of \( G \).

(ii) If \( G \) has no extreme vertices, then (a) each vertex cover of \( G \) is a GD-set of \( G \) and (b) \( \gamma g(G) \leq \alpha(G) = n - \beta_0(G) \).

(iii) If \( U \) is a vertex cover of \( G \) which contains all semi-extreme vertices of \( G \), then \( U \) is an EGD-set of \( G \).

(iv) If \( G \) has no semi-extreme vertices then (a) each vertex cover of \( G \) is an EGD-set of \( G \) and (b) \( \gamma g_e(G) \leq \alpha(G) = n - \beta_0(G) \).

**Proof.** (i) By Theorem C, \( V(G) - F \) is independent. As \( F \) contains all extreme vertices of \( G \), each vertex \( x \) in \( V(G) - F \) has 2 nonadjacent neighbors both belonging in \( F \), say they are \( y_x \) and \( z_x \). But then \( F \) is a dominating set and \( y_x, x, z_x \) is geodesic. This shows that \( F \) is a geodetic dominating set of \( G \).

(ii) The required immediately follows by (i) and Theorem C.

(iii) The set \( V(G) - U \) is independent because of Theorem C. Hence \( U \) is a dominating set of \( G \). Let \( xy \) be an arbitrary edge of \( G \) with \( x \in U \) and \( y \in V(G) - U \). Since all semi-extreme vertices are in \( U \), the vertex \( y \) has a neighbor, say \( z_y \in U \), which is nonadjacent to \( x \). Since \( x, y, z_y \) is geodesic, \( U \) is an edge geodetic dominating set of \( G \).

(iv) Immediately by (iii) and Theorem C. \( \square \)

**Corollary 3.** Let \( G \) be a connected \( n \)-order triangle-free graph with \( \delta(G) \geq 2 \). Then each vertex cover of \( G \) is an EGD-set of \( G \). In particular, \( \gamma g_e(G) \leq \alpha(G) = n - \beta_0(G) \).

**Corollary 4.** Let \( G = (X, Y; E) \) be a connected bipartite graph. Then both \( X \cup L(G) \) and \( Y \cup L(G) \) are edge geodetic dominating sets of \( G \). In particular, if \( \delta(G) \geq 2 \) then \( \gamma g_e(G) \leq \min\{|X|, |Y|\} \).

Two edges in a graph \( G \) are independent if they have no a common endpoint. A matching in \( G \) is a set of (pairwise) independent edges. A maximal matching is a matching \( M \) of a graph \( G \) with the property that if any edge not in \( M \) is added to \( M \), it is no longer a matching.
Corollary 5. Let $G$ be a triangle-free graph with minimum degree $\delta(G) \geq 2$. Then $\gamma_{ge}(G) \leq 2 \min\{|M| \mid M$ is a maximal matching of $G\}$.

Proof. Let $M$ be an arbitrary maximal matching of a graph $G$. The maximality of $M$ shows that $V(G) - V(M)$ is an independent set, where $V(M)$ is the set of all vertices incident with an edge of $M$. By Theorem C, $V(M)$ is a vertex cover of $G$ and the required follows immediately by Corollary 3.

The girth of a graph $G$, denoted by $\text{girth}(G)$, is the length of a shortest cycle contained in $G$.

Theorem 8. Let $G$ be a connected graph with $\delta(G) \geq 2$ and girth at least 6. Then $g_e(G) \leq \gamma_{ge}(G) \equiv \gamma(G) < \alpha(G)$.

Proof. Let $D$ be an arbitrary minimum dominating set of $G$. Suppose that $D$ is not edge geodetic. Then there is an edge $e = xy$, with $x \notin D$, which lies in no shortest path connecting two vertices of $D$. Since $G$ has no leaves, $x$ has at least one more neighbor, say $z$. Clearly, $yz \notin E(G)$ and at most one of $y$ and $z$ is in $D$.

Case 1. $y \in D$. Hence $z \notin D$ and there is a vertex $t \in D - \{y\}$ which is adjacent to $z$. Since the girth of $G$ is at least 6, the path $yz$, $z$, $t$ is a $y-t$ geodesic and $y$, $t \in D$. But this contradicts the choice of $e = xy$.

Case 2. $y \notin D$. Then there are different $u, v \in D$ such that $ux, vy \in E(G)$. As $G$ has girth at least 6, the path $u, x, y, v$ is a $u-v$ geodesic and $u, v \in D$. Again a contradiction.

Thus $D$ is an EGD-set of cardinality $\gamma(G)$. Since $D$ was chosen arbitrarily, and $\gamma_{ge}(G) \geq \gamma(G)$, we immediately obtain $\gamma_{ge}(G) \equiv \gamma(G)$. Finally, Theorem B and $\text{girth}(G) \geq 6$ imply $\gamma(G) \neq \alpha(G)$.

A graph $G$ is a geodetic graph if for every pair of its vertices there is a unique path of minimum length between them [12]. The concept of geodetic graph is a natural generalization of a tree. A subgraph $H$ of a graph $G$ is called an isometric subgraph if for every two vertices of $H$, the distance between them in $G$ equals the distance in $H$. Notice that an isometric subgraph is an induced subgraph.

Theorem 9. Let $G$ be a geodetic graph and $H$ its isometric subgraph with no isolated vertices. Let $F$ be an edge geodetic set of $H$. Then all the following hold.

(i) $S = (V(G) - V(H)) \cup F$ is an edge geodetic set of $G$. If $F$ is a dominating set of $H$, then $S$ is an edge geodetic dominating set of $G$.

(ii) $g_e(G) \leq |V(G)| - |V(H)| + g_e(H) \text{ and } \gamma_{ge}(G) \leq |V(G)| - |V(H)| + \gamma_{ge}(H)$.

Proof. (i) Clearly, to prove that $S = (V(G) - V(H)) \cup F$ is an edge geodetic set of $G$, it is sufficient to show that each edge of $G$ with one end in $V(H - F)$ and the other end in $G - V(H)$ belongs to some $a-b$ geodesic, where $a, b \in S$. So, let $v' \in V(H) - F$
be adjacent to \( x \in V(G) - V(H) \). Let \( P : v_0, v_1, \ldots, v_d \) be a \( v_0 - v_d \) geodesic in \( H \), and hence in \( G \), with \( v' = v_i \in V(P) \) and \( V(P) \cap F = \{v_0, v_d\} \). Assume that neither the path \( P_1 : v_0, v_1, \ldots, v_i = v' = v_1, v_{i+1}, \ldots, v_d \) is geodesic in \( G \). Let \( Q_1 : u_0 = v_0, u_1, \ldots, u_k = x \) and \( Q_2 : x = w_0, w_1, \ldots, w_{l-1}, w_l = v_d \) be geodesics in \( G \). Then \( Q_1, Q_2 : u_0 = v_0, u_1, \ldots, u_k = x = w_0, w_1, \ldots, w_{l-1}, w_l = v_d \) is a \( v_0 - v_d \) walk in \( G \) with length \(|E(Q_1, Q_2)| = |E(Q_1)| + |E(Q_2)| \leq (E(P_1)| - 1) + (|E(P_2)| - 1) = |E(P)|\). Since \( P \) is \( v_0 - v_d \) geodesic, \( Q_1, Q_2 \) is also \( v_0 - v_d \) geodesic. Since \( V(P) \neq V(Q_1, Q_2) \) and \( G \) is geodesic, we arrive at a contradiction. Hence at least one of \( P_1 \) and \( P_2 \) is geodesic in \( G \). Thus, \( S \) is an edge geodetic set of \( G \). The rest is obvious.

(ii) Let \( \mu \in \{g_e, \gamma g_e\} \). Choose \( F \) to be a \( \mu \)-set of \( H \). Then by (i) we immediately obtain \( \mu(G) \leq |V(G)| - |V(H)| + \mu(H) \). \hfill \( \square \)

**Corollary 6.** Let \( G \) be an \( n \)-order geodetic graph, \( n \geq 2 \), with diameter \( d \). Then \( \gamma g_e(G) \leq n - \lfloor 2d/3 \rfloor \) and (16) \( g_e(G) \leq n - d + 1 \).

**Proof.** Let \( H \) be any diametral path in \( G \). Clearly \( H \) is an isometric subgraph of \( G \). Applying Theorem 9 to \( G \) and \( H \) we obtain \( \mu(G) \leq n - (d + 1) + \mu(H) \), where \( \mu \in \{g_e, \gamma g_e\} \). But \( g_e(H) = 2 \) and \( \gamma g_e(H) = \lceil (d + 3)/3 \rceil \) (by Remark 2). Thus, the result immediately follows. \hfill \( \square \)

**Theorem 10.** Let \( G \) be a connected graph, \( \gamma(G) \geq 2 \), \( D \) an ED-set of \( G \) and let \( G - D \) contain no semi-extreme vertices. Then \( D \) is a \( \gamma \)-set of \( G \), \( \gamma(G) = \gamma g(G) \) and \( I^G_e[D] \supseteq E(G) - \{xy \in E(G) \mid x, y \in N(d) \text{ for some } d \in D\} \). In particular, if no element of \( D \) belongs to a triangle, then \( D \) is a \( \gamma g_e \)-set of \( G \) and \( \gamma(G) = \gamma g_e(G) \).

**Proof.** Choose \( x \in V(G) - D \) arbitrarily. Since \( D \) is an ED-set, \( x \) has exactly one neighbor in \( D \), say \( y_x \). Since \( x \) is not semi-extreme, choose arbitrarily \( z \in N(x) - N(y_x) \). Denote by \( t \) the unique neighbor of \( z \) in \( D \). Then \( y_x, x, z, t \) is a geodesic in \( G \). Hence \( D \) is a geodetic set and as \( z \) was chosen arbitrarily, \( I^G_e[D] \supseteq E(G) - \{xy \in E(G) \mid x, y \in N(d) \text{ for some } d \in D\} \). Finally, if no element of \( D \) belongs to a triangle, then \( I^G_e[D] \supseteq E(G) \) implies \( D \) is an EGD-set of \( G \). \hfill \( \square \)

The next result follows immediately by the above theorem.

**Corollary 7.** Let \( G \) be a connected graph without semi-extreme vertices. If all \( \gamma \)-sets of \( G \) are efficient dominating, then (a) \( \gamma(G) \equiv \gamma g(G) \), and (b) if \( G \) has no triangles, then \( \gamma(G) \equiv \gamma g(G) \equiv \gamma g_e(G) \).

4. **Vertex partitions**

Here we present some initial results on the parameters \( gd, g_e, gp, g_e p \).

**Proposition 3.** Let \( G \) be a connected \( n \)-order graph, \( n \geq 2 \). Then
(i) \( g_e d(G) \leq gd(G) \leq \min\{gp(G), d(G)\} \leq \delta(G) + 1 \) and \( g_e d(G) \leq g_e p(G) \leq gp(G) \).

(ii) \( g_e d(G) \gamma (G) \leq gd(G) \gamma (G) \leq n \), \( gp(G) g(G) \leq n \) and \( g_e p(G) g_e (G) \leq n \).

**Proof.** (i) Denote by \( \mathcal{I}(G) \), \( \mathcal{I}_e(G) \), \( \mathcal{P}(G) \), \( \mathcal{I}_e \mathcal{P}(G) \), respectively the family of all geodetic sets of \( G \) (the family of all EG-sets of \( G \), the family of all dominating sets of \( G \), the family of all GD-sets of \( G \), the family of all EGD-sets of \( G \), respectively). Clearly,

\[
\mathcal{I}(G) \supseteq \mathcal{I}_e(G) \quad \text{and} \quad \mathcal{I}_e \mathcal{P}(G) = \mathcal{I}(G) \cap \mathcal{P}(G) \supseteq \mathcal{I}_e(G) \cap \mathcal{P}(G) = \mathcal{I}_e \mathcal{P}(G),
\]

which implies \( g_e d(G) \leq gd(G) \leq gp(G) \), \( g_e d(G) \leq gd(G) \leq d(G) \) and \( g_e d(G) \leq g_e p(G) \leq gp(G) \). The rest immediately follows by Theorem E.

(ii) Let \( \mathcal{P} = [U_1, U_2, ..., U_k] \) be a partition of \( V(G) \) into geodetic sets, or into EG-sets, or into GD-sets, or into EGD-sets. Then \( n = |U_1| + |U_2| + ... + |U_k| \geq k \min\{|U_i| \mid 1 \leq r \leq k\} \), which implies the required inequality.

The next example shows that there is a graph \( G \) for which all inequalities in Proposition 3 become equalities.

**Example 2.** Let \( \mu \in \{gp, g_e p, d, gd, g_e d\} \) and consider a crown graph \( H_{n,n} \), \( n \geq 3 \), with \( V(H_{n,n}) = \{1,2,..,n,1',2',..,n'\} \) and \( E(H_{n,n}) = \{i j' \mid i \neq j\} \). Clearly, all \( \nu \)-sets of \( H_{n,n} \) are \( S_i = \{i,i'\}, i = 1,2,..,n \), where \( \nu = \{\gamma, \gamma g, \gamma g_e, \gamma g_e\} \). Hence \( \nu(H_{n,n}) = 2 \) and since all \( S_i \) form a partition of \( V(H_{n,n}) \), \( \mu(H_{n,n}) = n = \delta(H_{n,n}) + 1 \) and \( \mu(H_{n,n}) \nu(H_{n,n}) = |V(H_{n,n})| \).

Let us consider the Cartesian product \( C_p \square C_q \) of cycles as a \( p \times q \) array of vertices \( \{x_{i,j} \mid 0 \leq i \leq p - 1 \} \) and \( 0 \leq j \leq q - 1 \), with an adjacency \( \mathcal{N}(x_{ij}) = \{x_{i,j-1}, x_{i,j+1}, x_{i-1,j}, x_{i+1,j}\} \), where the first subscript is taken modulo \( p \) and the second subscript is taken modulo \( q \).

**Example 3.** Note that \( \text{diam}(C_{2m} \square C_{2n}) = m + n \), \( x_{i,j} \) and \( x_{i+m,j+n} \) are antipodal, \( g(C_{2m} \square C_{2n}) = 2 \) and all \( g \)-sets of \( C_{2m} \square C_{2n} \) are \( X_{i,j} = \{x_{i,j}, x_{i+m,j+n}\} \) \((2)\). It is easy to see that each \( X_{i,j} \) is also a \( g_e \)-set of \( C_{2m} \square C_{2n} \). Clearly, all \( X_{i,j} \) form a partition of \( V(C_{2m} \square C_{2n}) \). Therefore \( gp(C_{2m} \square C_{2n}) = g_e p(C_{2m} \square C_{2n}) = 2mn \) and \( g_e p(C_{2m} \square C_{2n}) g_e(C_{2m} \square C_{2n}) = gp(C_{2m} \square C_{2n}) g_e(C_{2m} \square C_{2n}) = |V(C_{2m} \square C_{2n})| \).

**Example 4.** Here we consider the graph \( C_4 \square C_n \), \( n \geq 4 \). It is known that \( \gamma (C_4 \square C_n) = n \) \((11)\). Define the sets \( D_i = \{x_{i,j} \mid j \equiv_2 0\} \cup \{x_{i+2,j} \mid j \equiv_2 1\} \), \( i = 0,1,2,3 \). It is easy to see that all these sets are EGD-sets of cardinality \( n \). Hence \( n = \gamma (C_4 \square C_n) = \gamma g(C_4 \square C_n) = g_e (C_4 \square C_n) \). Since \( D_0, D_1, D_2, D_3 \) form a partition of \( V(G) \), \( d(C_4 \square C_n) = gd(C_4 \square C_n) = g_e d(C_4 \square C_n) = 4 \). Therefore \( d(C_4 \square C_n) \gamma (C_4 \square C_n) = gd(C_4 \square C_n) \gamma g(C_4 \square C_n) = g_e d(C_4 \square C_n) \gamma g_e (C_4 \square C_n) = |V(C_4 \square C_n)| \).

An efficient domination partition (or an ED-partition) of a graph \( G \) is a partition of \( V(G) \) into ED-sets. A graph \( G \) is said to be an efficient domination partitionable graph (or an EDP-graph) if \( G \) has an ED-partition.
Theorem 11. [15] Let G be a graph of order n. Then the following assertions are equivalent.

(i) G is an EDP-graph.

(ii) G is regular and \(d(G) = \delta(G) + 1\).

(iii) \(n = \gamma(G)(\Delta(G) + 1)\) and \(n = d(G)\gamma(G)\).

If G is an EDP-graph, then each its \(\gamma\)-set is efficient dominating.

Theorem 12. Let G be an n-order connected EDP-graph without semi-extreme vertices, \(n \geq 3\).

(i) Then \(\gamma(G) \equiv \gamma g(G)\) and \(d(G) = gd(G)\).

(ii) If G has no triangles, then \(\gamma(G) \equiv \gamma g(G) \equiv \gamma g_c(G)\) and \(d(G) = gd(G) = g_c d(G)\).

Proof. By Theorem 11, all \(\gamma\)-sets of G are efficient dominating. Now Corollary 7 immediately implies \(\gamma(G) \equiv \gamma g(G)\) and if G has no triangles then \(\gamma(G) \equiv \gamma g(G) \equiv \gamma g_c(G)\). Since G is an EDP-graph, \(d(G) = gd(G)\) and if there is no triangles in G, \(d(G) = gd(G) = g_c d(G)\).

Denote by \(\mathbb{Z}_n = \{0,1,...,n-1\}\) the additive group of order n. The generalized Petersen graph \(P(n,k)\), where \(n \geq 3\) and \(k \in \mathbb{Z}_n - \{0\}\), is the graph on the vertex set \(\{x_i,y_i\mid i \in \mathbb{Z}_n\}\) with adjacencies \(x_i,x_{i+1},x_iy_i\), and \(y_iy_{i+k}\) for all i.

Example 5. If \(n \equiv 4\) 0 and \(k\) is odd then \(\gamma(P(n,k)) \equiv \gamma g(P(n,k)) \equiv \gamma g_c(P(n,k))\) and \(d(P(n,k)) = gd(P(n,k)) = g_c d(P(n,k))\).

Proof. A graph \(P(n,k)\) is bipartite if and only if \(n\) is even and \(k\) is odd ([1]), and it is an EDP-graph if and only if \(n \equiv 4\) 0 and \(k\) is odd ([15]). Now the required immediately follows by Theorem 12.

Let \(S\) be a subset of \(\mathbb{Z}_n\) such that \(0 \notin S\) and \(x \in S\) implies \(-x \notin S\). The circulant graph with distance set \(S\) is the graph \(C(n;S)\) with vertex set \(\mathbb{Z}_n\) and vertex \(x\) is adjacent to vertex \(y\) if and only if \(x - y \in S\). It is clear from the definition that \(C(n;S)\) is vertex-transitive and regular of degree \(|S|\).

Example 6. Let \(G = C(n; (2k+1)t; \{1,...,k\} \cup \{n-1,...,n-k\})\), where \(k,t \geq 1\). Then G is an EDP-graph, \(\gamma(G) = t\) and G has only one ED-partition ([15]). By Theorem 12, \(\gamma(G) \equiv \gamma g(G)\) and \(d(G) = gd(G)\).

Example 7. Let \(G = C(n; \{\pm 1, \pm s\})\) where \(2 \leq s \leq n - 2\), \(s \not= n/2\), \(s \equiv_s \pm 2\) and \(5|n\). Then G is an EDP-graph, \(\gamma(G) = n/5\) and G has only one ED-partition ([15]). By Theorem 12, \(\gamma(G) \equiv \gamma g(G)\) and \(d(G) = gd(G)\).
Example 8. Let $G = C_5 \square C_{5k}$, $k \geq 1$. Then $G$ is an EDP-graph, $\gamma(G) \equiv \gamma g(G) \equiv \gamma g_e(G)$ and $d(G) = gd(G) = g_e d(G)$.

Proof. Define the sets $D_i$ as the union of $\{x_{i,j} \mid j \equiv_5 0\}$, $\{x_{1+i,j} \mid j \equiv_5 3\}$, $\{x_{2+i,j} \mid j \equiv_5 1\}$, $\{x_{3+i,j} \mid j \equiv_5 4\}$, and $\{x_{4+i,j} \mid j \equiv_5 2\}$, $i = 0, 1\ldots, 4$. Clearly, all these sets are efficient dominating and they form an ED-partition of $G$. Thus $G$ is an EDP-graph with $\gamma(G) = 5k$ (the fact that $\gamma(G) = 5k$ is known, see [11]). Since $G$ has no triangles, by Theorem 12 we immediately have $\gamma(G) \equiv \gamma g(G) \equiv \gamma g_e(G)$ and $d(G) = gd(G) = g_e d(G)$.

5. Problems

We conclude the paper by two problems.

Problem 1. Let $G$ be a graph and $\mu \in \{gd, g_e d, gp, g_e p\}$. Find a nontrivial characterization of graphs with $\mu(G) \leq 2$.

Problem 2. Let $G$ be an $n$-order graph and $\nu \in \{g, g_e, \gamma g, \gamma g_e\}$. Find results on $\nu$-excellent graphs.

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