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Characterization of signed paths and cycles admitting minus dominating function

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Abstract: Let $G = (V, E, \sigma)$ be a finite signed graph. A function $f : V \to \{-1, 0, 1\}$ is a minus dominating function (MDF) of G if $f(u) + \sum_{v \in N(u)} \sigma(uv)f(v) \geq 1$ for all $u \in V$. In this paper we characterize signed paths and cycles admitting an MDF.

Keywords: signed graphs, minus domination, minus dominating function

AMS Subject classification: 05C22

1. Introduction

Signed graphs are graphs whose edges are labelled positive or negative. This concept introduced by Harary [4] assumed an important place in the field of social psychology after the publication of the work by Cartwright and Harary [2] wherein they used a signed graph to model relations between persons. Based on the idea of dominating functions Dunbar et al. [3] introduced the concept of minus dominating function in a graph that involved assigning of the values $-1, 0$ or 1 to its vertices subject to certain conditions. Their motivation arose from the context of modelling networks of people or organizations in which the responses are positive, negative or neutral. In this direction Acharya [1] initiated a study on minus domination in signed graphs based on the observation that although all simple graphs admit a minus dominating function, the case is different for signed graphs. From the perspective of a network of people, minus domination in signed graphs takes into consideration the response of the people as well as the nature of their mutual interconnection, viewed either as positive or negative.

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Throughout this paper, we will consider only simple, finite signed graphs. A signed graph in short, a *sigraph* is an ordered triple $S = (V, E, \sigma)$ where $G = (V, E)$ is a simple graph referred to as the underlying graph of S and $\sigma : E(G) \to \{-1, 1\}$ is a function called *signature* of S. The edges that are assigned the value 1 are *positive edges* and others *negative edges*. A sigraph with at least one negative edge is called a *pure sigraph*.

In the pictorial representation of a signed graph, positive edges are drawn as solid line segments and negative edges as dashed line segments as shown in Figure 1. For any signed graph S, by $E^+(S)$ and $E^-(S)$ we mean the set of positive and negative edges respectively. For a vertex v of S, $d^+(v)$ and $d^-(v)$ denote the number of positive and negative edges incident on v respectively, and $\deg(v) = d^+(v) + d^-(v)$. Given a set of negative (positive) edges of a signed graph S, the maximal connected subgraph induced by the negative (positive) edges is called the negative (positive) section of S. Two negative sections are said to be consecutive if all the edges between them are positive.

2. Minus dominating function in signed graphs

Acharya [1] introduced the concept of minus domination in signed graphs as follows:

Definition 1. Given a finite sigraph $S = (V, E, \sigma)$, a function $f : V \to \{-1, 0, 1\}$ is a *minus dominating function* (MDF) of S if $f(N[u]) \geq 1$ where

$$f(N[u]) = f(u) + \sum_{v \in N(u)} \sigma(uv) f(v)$$

for all $u \in V$.

Definition 1 is equivalent to the definition of a minus dominating function of a finite graph as given in [3]. As observed by Acharya [1], not all signed graphs admit a minus dominating function (MDF). The signed graph given in Figure 1 is one such example.

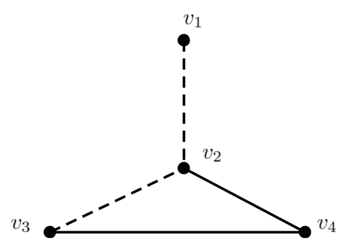

Figure 1. **A signed graph not admitting MDF**

Remark 1. For any sigraph $S = (V, E, \sigma)$ admitting an MDF f, $\sigma(uv)f(v) \leq 1$ for every pair of adjacent vertices u and v.

In the following lemma we present few observations from [1] that are useful in examining whether a given signed graph admits an MDF.

Lemma 1. *Let $S = (V, E, \sigma)$ be any signed graph having a negative pendant edge uv where $\deg(u) = 1$. Then for any minus dominating function $f : V \to \{-1, 0, 1\}$ of S, the following hold.*

(i) $f(u) \in \{0, 1\}$,

(ii) If $f(u) = 0$ *then* $f(v) = -1$,

(iii) If $f(u) = 1$ *then* $f(v) \in \{0, -1\}$.

Next results give properties of signed graphs admitting an MDF.

Proposition 1. *Let (G, σ) be a signed graph admitting an MDF f. If uvw is a path in G such that $\deg_G(u) = 1$ and $\deg_G(v) = 2$, then $\sigma(uv) = 1$.*

Proof. Suppose, to the contrary, that $\sigma(uv) = -1$. By Lemma 1 (item (i)) we have $f(u) \in \{0, 1\}$. If $f(u) = 0$, then Lemma 1 (item (ii)) implies that $f(v) = -1$ and so $f(N[v]) = f(v) + \sigma(vu)f(u) + \sigma(vw)f(w) = -1 + -1 \times 0 + \sigma(vw)f(w) < 1$ as $\sigma(vw)f(w) \leq 1$, thus contradicting the fact that f is an MDF of G. Assume that $f(u) = 1$. It follows from Lemma 1 (item (iii)) that $f(v) \in \{0, -1\}$. If $f(v) = 0$, then $f(N[v]) = f(v) + \sigma(vu)f(u) + \sigma(vw)f(w) = 0 + (-1) \times 1 + \sigma(vw)f(w) < 1$ which is impossible as f is an MDF. If $f(v) = -1$, then we have $f(N[v]) = -1 + (-1) \times 1 + \sigma(vw)f(w) < 1$, a contradiction again. Thus $\sigma(uv) = 1$. □

Proposition 2. *Let (G, σ) be a signed graph admitting an MDF f and let u_1uvv_1 (possibly $u_1 = v_1$) be a path in G such that $\deg_G(u) = \deg_G(v) = 2$. If $\sigma(uv) = -1$ and $\sigma(uu_1) = \sigma(vv_1) = 1$, then $f(u) = f(v)$.*

Proof. Suppose, to the contrary, that $f(u) \neq f(v)$. Assume without loss of generality that $f(v) < f(u)$. Hence, we have $f(u) = 1$ and $f(v) = 0$ or $f(u) = 0$ and $f(v) = -1$ or $f(u) = 1, f(v) = -1$. In the first case $f(N[v]) = 0 + f(v_1) - 1 < 1$ for any value of $f(v_1) \in \{-1, 0, 1\}$, whereas in the second case we have $f(N[v]) = -1 + 0 + f(v_1) < 1$ and in the third case we have $f(N[v]) = -1 - 1 + f(v_1) < 1$. Thus, in all the cases we get a contradiction to the fact that f is an MDF and hence the proof is complete. □

Proposition 3. *Let (G, σ) be a signed graph admitting an MDF f and let v_1vv_2 be a path in G such that $\deg_G(v) = 2$, $\deg_G(v_1), \deg_G(v_2) \leq 2$ and $\sigma(vv_1) = \sigma(vv_2) = -1$. Then $f(v) = -1$.*

Proof. By Proposition 1, v_1 and v_2 are not pendant vertices. Therefore, $\deg_G(v_1) = \deg_G(v_2) = 2$. Suppose, to the contrary, that $f(v) \neq -1$. Then $f(v) \in \{0, 1\}$. We consider the following cases.

Case 1. $v_1v_2 \notin E(G)$ and v_1 and v_2 have no common neighbor different from v.
Let v_0 be the second neighbor of v_1 and v_3 be the second neighbor of v_2. We distinguish the following subcases.

Subcase 1.1. $f(v) = 0$.
Since f is an MDF, $f(N[v]) = f(v) + \sigma(vv_1)f(v_1) + \sigma(vv_2)f(v_2) \geq 1$ and we get

$$f(v_1) + f(v_2) \leq -1. \tag{1}$$

The possible pair of values of $f(v_1)$ and $f(v_2)$, satisfying (1) are $0, -1$ and $-1, -1$. Without loss of generality let us consider two situations.

(a) $f(v_1) = 0$ and $f(v_2) = -1$ (the case $f(v_1) = -1$ and $f(v_2) = 0$ is similar).
Since f is an MDF, we must have $f(N[v_2]) = f(v_2) + \sigma(v_2v)f(v) + \sigma(v_2v_3)f(v_3) = -1 + (-1) \times 0 + \sigma(v_2v_3)f(v_3) \geq 1$ or $\sigma(v_2v_3)f(v_3) \geq 2$ which is not possible.

(b) $f(v_1) = -1$ and $f(v_2) = -1$.
As in the previous case since f is an MDF, we must have $f(N[v_2]) = f(v_2) + \sigma(v_2v)f(v) + \sigma(v_2v_3)f(v_3) = -1 + 0 + f(v_3)\sigma(v_2v_3) \geq 1$ yielding $\sigma(v_2v_3)f(v_3) \geq 2$, which again is not possible by Remark 1.

Subcase 1.2. $f(v) = 1$.
As in the Subcase 1.1, we can see that

$$f(v_1) + f(v_2) \leq 0. \tag{2}$$

If $f(v_2) = -1$, then it follows from $f(N[v_2]) = f(v_2) + \sigma(v_2v)f(v) + \sigma(v_2v_3)f(v_3) = -1 - 1 + \sigma(v_2v_3)f(v_3) \geq 1$ that $\sigma(v_2v_3)f(v_3) \geq 3$ which is not possible. If $f(v_2) = 0$, then as above we get $\sigma(v_2v_3)f(v_3) \geq 2$ which again is not possible. Assume that $f(v_2) = 1$. Then by (2) the only possible value of $f(v_1)$ is -1. Now $f(N[v_1]) \geq 1$ implies that $f(v_1) + \sigma(v_1v)f(v) + \sigma(v_1v_0)f(v_0) = -1 - 1 + \sigma(v_1v_0)f(v_0) \geq 1$ or $\sigma(v_1v_0)f(v_0) \geq 3$, which is not possible as observed in the earlier cases. Thus (2) is satisfied for no value of $f(v_2) \in \{-1, 0, 1\}$.

Case 2. v_1 and v_2 are adjacent in G.
Then G is the triangle C_3. Consider the following subcases.

Subcase 2.1. $f(v) = 0$.
As in the Case 1, we can see that

$$f(v_1) + f(v_2) \leq -1 \tag{3}$$

and that the possible pair of values of $f(v_1)$ and $f(v_2)$, satisfying (3) are $0, -1$ and $-1, -1$. Without loss of generality let us consider two situations.

(c) $f(v_1) = 0$ and $f(v_2) = -1$.
Since f is an MDF, we must have $f(N[v_2]) = f(v_2) + \sigma(v_2v)f(v) + \sigma(v_2v_1)f(v_1) = -1 + (-1) \times 0 + \sigma(v_2v_1) \times 0 \geq 1$ which is not possible.

(d) $f(v_1) = -1$ and $f(v_2) = -1$.
Again we must have $f(N[v_2]) = f(v_2) + \sigma(v_2v)f(v) + \sigma(v_2v_1)f(v_1) = -1 + 0 - \sigma(v_2v_1) \geq 1$ which is not possible.

Subcase 2.2. $f(v) = 1$.
Then we must have

$$f(v_1) + f(v_2) \leq 0. \tag{4}$$

If $f(v_2) = -1$, then $f(N[v_2]) = f(v_2) + \sigma(v_2v)f(v) + \sigma(v_2v_1)f(v_1) = -1 - 1 + \sigma(v_2v_1)f(v_1) \geq 1$ which is not possible. If $f(v_2) = 0$, then as above we get $\sigma(v_2v_1)f(v_1) \geq 2$ which again is not possible as f is an MDF. Hence $f(v_2) = 1$. Likewise, we must have $f(v_1) = 1$. But then $f(N[v]) < 1$ which is a contradiction.

Case 3. $v_1v_2 \notin E(G)$ and v_1 and v_2 have a common neighbor w different from v. Using an argument similar to that described in above cases, we get a contradiction. Thus $f(v) = -1$ and the proof is complete. □

Corollary 1. *Let P_n $(n \geq 3)$ be any signed path having a vertex v with $d^-(v) = 2$. Let v_1 and v_2 be the vertices adjacent to v. Then for any MDF f of P_n, $f(v_1) = f(v_2) = -1$. Further, the edges incident with v_1 and v_2 are not pendant.*

Proof. By Proposition 3 we have $f(v) = -1$. We conclude from $f(N[v]) \geq 1$ that $f(v_1) + f(v_2) \leq -2$ which is true only when $f(v_1) = f(v_2) = -1$. Proposition 1 implies that vv_1 and vv_2 are not pendant edges and so $\deg(v_1) = \deg(v_2) = 2$. Let v_3 be the second neighbor of v_2. Suppose, to the contrary, that v_2v_3 is a pendant edge. Then by Proposition 1 the edge v_2v_3 will be positive and we deduce from $f(N[v_3]) \geq 1$ that $f(v_3) \geq 2$ which is not possible. Likewise, we can show that the edge incident with v_1 also is not a pendant edge. □

Remark 2. For any vertex v of a negative section of length more than one in a signed path admitting an MDF f, $f(v) = -1$.

Now we will examine the properties of the negative section of a path admitting an MDF.

Lemma 2. *If P_n is a signed path admitting an MDF f, then no positive pendant edge is incident with a negative section of length more than one.*

Proof. Let S_1 be a negative section of length more than one of a signed path P_n admitting an MDF. Then S_1 does not contain any pendant edge. Suppose, to the contrary, vw is a pendant path in P_n such that $v \in S_1$. Then $d^-(v) = 1$ and by

Corollary 1 we have $f(v) = -1$. Then $f(N[w]) = f(w) - 1 < 1$ contradicting that f is an MDF. □

Theorem 1. *Let S_1 and S_2 be two consecutive negative sections of a signed path P_n admitting an MDF f. Then the following are true:*

(i) If exactly one of the sections S_1 and S_2 is of cardinality one, then the positive section between S_1 and S_2 is of length at least two.

(ii) If both S_1 and S_2 are of cardinality greater than one, then the positive section between S_1 and S_2 is of length at least three.

Proof. (i) Consider two negative sections S_1 and S_2 of P_n such that one of them say S_1 is of length one and the other having length more than one. Let $e = uv$ be the negative edge of S_1. If possible assume that there exist only one positive edge vw between S_1 and S_2. We conclude from Propositions 2 and 3 that $f(u) = f(v)$ and $f(v_i) = -1$ for all vertices v_i of S_2, but then $f(N[u]) = f(u) - f(v) - 1 = -1 < 1$ which is a contradiction. Hence there exist at least two positive edges between S_1 and S_2.

(ii) Let S_1 and S_2 be two consecutive negative sections of P_n each of length at least two. Suppose, to the contrary, that there exists at most two positive edges between S_1 and S_2. First let there exist exactly one positive edge, say uv between S_1 and S_2. Then $f(u) = f(v) = -1$ by Proposition 2 yielding $f(N[u]) = -1 - 1 + 1 < 1$ which is a contradiction. Now let there exist two positive edges uv and vw between S_1 and S_2. Then $f(u) = f(w) = -1$ and we get the contradiction that $f(N[v]) = f(v) - 1 - 1 < 1$.

□

Now we are ready to characterize all signed paths admitting an MDF.

Theorem 2. *A signed path $P_n, n \geq 2$, admits an MDF if and only if it satisfies the following conditions:*

(i) the pendant edges are positive,

(ii) no positive pendant edge is incident with a negative section of length at least 2,

(iii) there are at least two positive edges between two consecutive negative sections such that one of them has exactly one negative edge and the other is of length more than one and

(iv) there are at least three positive edges between two consecutive negative sections, each of which is of length more than one.

Proof. The necessity follows from Proposition 1, Corollary 1 and Theorem 1. To prove the sufficiency, let P_n be a signed path satisfying the given conditions. Define $f : V(P_n) \to \{-1, 0, 1\}$ such that

$$f(v) = \begin{cases} -1 & \text{if } v \text{ is a vertex of a negative section of length more than 1,} \\ 1 & \text{otherwise.} \end{cases}$$

We will show that the function f is an MDF. That is for any vertex $v \in V(P_n)$, $f(N[v]) \geq 1$. We will examine the value of $f(N[v])$ for any vertex v of P_n by considering the following cases.

Case 1: v is a pendant vertex.
Let w be the vertex of P_n adjacent to v. Then w does not belong to a negative section of length more than one. Hence $f(v) = f(w) = 1$ and $f(N[v]) = 1 + 1 = 2$.

Case 2: The vertex v is not pendant.
Let v_1 and v_2 be the vertices adjacent to v. Let us now consider two sub-cases.

Subcase 2.1. The vertex v does not belong to a negative section of length more than 1. Then $d^-(v)$ is either 0 or 1. If $d^-(v) = 0$, then $f(v_1) = f(v_2) = 1$ and $f(N[v]) = 1 + 1 + 1 = 3$ and when $d^-(v) = 1$, we have $f(N[v]) = 1 + 1 - 1 = 2$.

Subcase 2.2. Vertex v belong to a negative section of length more than one. In this case $d^-(v)$ is either 1 or 2. If $d^-(v) = 2$, then by Corollary 1 we have $f(v_1) = f(v_2) = -1$ and $f(N[v]) = -1 + 1 + 1 = 1$. If $d^-(v) = 1$, then without loss of generality let $\sigma(vv_1) = 1$ and $\sigma(vv_2) = -1$ so that $f(v_1) = 1$ and $f(v_2) = -1$, and we have $f(N[v]) = -1 + 1 + 1 = 1$.
Hence f is an MDF of signed path P_n and the proof is complete. □

After having studied signed paths admitting MDF, the case that follows immediately is that of signed cycles. Note that a cycle can be considered as a closed path. It is not difficult to observe that signed cycles where all the edges are either positive or negative admit MDF. The following theorem gives a characterization of pure signed cycles admitting an MDF.

Theorem 3. *A pure signed cycle C_n admits an MDF if and only if the following holds.*

(i) $\sigma(e) = -1$ *for all* $e \in E(C_n)$ *or*

(ii) All negative sections of C_n *are of length one or*

(iii) C_n *has exactly one negative section of length more than one and a positive section of length at least three or*

(iv) C_n *has more than one negative section such that between any two consecutive negative sections there exist (a) at least one positive edge if both the negative sections are of length one, (b)at least two positive edges if exactly one negative section is of length more than one and (c) at least three positive edges if both the negative sections are of length more than one.*

Proof. The proof follows from a similar argument as in Theorems 1 and 2. □

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