Communications in Combinatorics and Optimization
Vol. 5 No. 2, 2020 pp.111-123
DOI: 10.22049/CCO.2019.26598.1123

CCO
Commun. Comb. Optim.

Weak signed Roman domination in graphs

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Received: 25 June 2019; Accepted: 11 November 2019
Published Online: 13 November 2019

Abstract: A weak signed Roman dominating function (WSRDF) of a graph G with vertex set $V(G)$ is defined as a function $f : V(G) \to \{-1, 1, 2\}$ having the property that $\sum_{x \in N[v]} f(x) \geq 1$ for each $v \in V(G)$, where $N[v]$ is the closed neighborhood of v. The weight of a WSRDF is the sum of its function values over all vertices. The weak signed Roman domination number of G, denoted by $\gamma_{wsR}(G)$, is the minimum weight of a WSRDF in G. We initiate the study of the weak signed Roman domination number, and we present different sharp bounds on $\gamma_{wsR}(G)$. In addition, we determine the weak signed Roman domination number of some classes of graphs.

Keywords: Domination, signed Roman domination, weak signed Roman domination

AMS Subject classification: 05C69

1. Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [5]. Specifically, let G be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the *order* and the *size* of the graph G, respectively. The *open neighborhood* of vertex v is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d_G(v) = d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. A graph G is *regular* or *r-regular* if $\delta(G) = \Delta(G) = r$. The *complement* of a graph G is denoted by $\overline{G}$. Let K_n be the complete graph of order n, C_n the cycle of order n, P_n the path of order n, and $K_{p,q}$ the complete bipartite graph with partite sets X and Y, where $|X| = p$ and $|Y| = q$. Let $S(r, s)$ be the *double star* with exactly two adjacent vertices u and v that are not leaves such that u is adjacent to $r \geq 1$ leaves and v is adjacent to $s \geq 1$ leaves.

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A set S of vertices of G is called a *dominating set* if $N[S] = \bigcup_{v\in S} N[v] = V(G)$. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G. In this paper we continue the study of signed Roman domination in graphs (see, for example, [1–4, 6–11]).

A *signed Roman dominating function* (SRDF) on a graph G is defined in [1] as a function $f : V(G) \to \{-1, 1, 2\}$ having the property that $f(N[v]) = \sum_{x\in N[v]} f(x) \geq 1$ for each $v \in V(G)$ and if $f(u) = -1$, then the vertex u must have a neighbor w with $f(w) = 2$. The weight of a signed Roman dominating function is the value $f(V(G)) = \sum_{u\in V(G)} f(u)$. The *signed Roman domination number* $\gamma_{sR}(G)$ is the minimum weight of a signed Roman dominating function on G.

A *weak signed Roman dominating function* (WSRDF) of a graph G is defined as a function $f : V(G) \to \{-1, 1, 2\}$ having the property that $f(N[v]) = \sum_{x\in N[v]} f(x) \geq 1$ for each $v \in V(G)$. The weight of a WSRDF is the value $\omega(f) = f(V(G)) = \sum_{u\in V(G)} f(u)$. The *weak signed Roman domination number* of G, denoted by $\gamma_{wsR}(G)$, is the minimum weight of a WSRDF in G. A $\gamma_{wsR}(G)$-function is a WSRDF of weight $\gamma_{wsR}(G)$. For a WSRDF f on G, let $V_i = \{v \in V(G) : f(v) = i\}$ for $i = -1, 1, 2$. A WSRDF f can be represented by the ordered partition $f = (V_{-1}, V_1, V_2)$. The definitions lead to $\gamma_{wsR}(G) \leq \gamma_{sR}(G)$. Therefore each lower bound of $\gamma_{wsR}(G)$ is also a lower bound of $\gamma_{sR}(G)$.

Our purpose in this work is to initiate the study of the weak signed Roman domination number. We present basic properties and sharp bounds for the weak signed Roman domination number of a graph. In particular, we show that many lower bounds on $\gamma_{sR}(G)$ are also valid for $\gamma_{wsR}(G)$. In addition we prove $\gamma_{wsR}(G) \geq (3n - 4m)/2$ for graphs G without isolated vertices of order n and size m, and we characterize the graphs achieving equality. Furthermore, we show that the difference $\gamma_{sR}(G)-\gamma_{wsR}(G)$ can be arbitrarily large, and we determine the weak signed Roman domination number of some classes of graphs.

2. Preliminary results and first bounds

In this section we present basic properties and some first bounds on the weak signed Roman domination number. The definitions lead to the first observation immediately

Observation 1. *If $f = (V_{-1}, V_1, V_2)$ is a WSRDF of a graph G of order n, then the following holds.*

(a) $|V_{-1}| + |V_1| + |V_2| = n$.

(b) $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$.

(c) Every vertex of V_{-1} is dominated by one vertex of V_2 or two vertices of V_1.

(d) $V_1 \cup V_2$ is a dominating set of G.

Proposition 1. If G is graph of order n, then $\gamma_{wsR}(G) \leq n$, with equality if and only if $G = \overline{K_n}$.

Proof. Define the function $f : V(G) \to \{-1, 1, 2\}$ by $f(v) = 1$ for each $v \in V(G)$. Then f is a WSRDF on G of weight n and thus $\gamma_{wsR}(G) \le n$. If $G = \overline{K_n}$, then it easy to see that $\gamma_{wsR}(G) = n$.
Conversely, assume that $\Delta(G) \ge 1$. If $\delta(G) = 1$, then there exists a vertex v with $d(v) = 1$. If u is a neighbor of v, then define the function $f : V(G) \to \{-1, 1, 2\}$ by $f(v) = -1$, $f(u) = 2$ and $f(x) = 1$ for $x \in V(G) \setminus \{u, v\}$. It is straightforward to verify that f is a WSRDF on G of weight $n-1$ and thus $\gamma_{wsR}(G) \le n-1$. If $\delta(G) \ge 2$, then define $f(w) = -1$ for an arbitrary vertex $w \in V(G)$ and $f(x) = 1$ for $x \in V(G) \setminus \{w\}$. Then f is a WSRDF on G of weight $n-2$ and thus $\gamma_{wsR}(G) \le n-2$. □

Theorem 2. If G is graph of order n, then $\gamma_{wsR}(G) \ge 2\gamma(G) - n$, with equality if and only if $G = \overline{K_n}$.

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{wsR}(G)$-function. Then it follows from Observation 1 that

$$\gamma_{wsR}(G) = |V_1| + 2|V_2| - |V_{-1}| = 2|V_1| + 3|V_2| - n \ge 2|V_1 \cup V_2| - n \ge 2\gamma(G) - n, \quad (1)$$

and the desired inequality is proved. Clearly, if $G = \overline{K_n}$, then $\gamma_{wsR}(G) = n = 2\gamma(G) - n$. Now assume that G contains at least one edge. Using Proposition 1, we observe that $\gamma_{wsR}(G) \le n-1$ and therefore $|V_{-1}| \ge 1$. If $|V_2| \ge 1$, then it follows from (1) that

$$\gamma_{wsR}(G) = 2|V_1| + 3|V_2| - n > 2|V_1 \cup V_2| - n \ge 2\gamma(G) - n.$$

Therefore assume now that $|V_2| = 0$. Let $u \in V_{-1}$, and let $x, y \in V_1$ be two neighbors of u. The condition $f(N[x]) \ge 1$ shows that x has a neighbor in $V_1 \setminus \{x\}$. Furthermore, since every vertex of V_{-1} has at least two neighbors in V_1, we conclude that $V_1 \setminus \{x\}$ is a dominating set of G. Hence we deduce from (1) that

$$\gamma_{wsR}(G) = 2|V_1| - n > 2\gamma(G) - n.$$

□

Proposition 2. If G is graph of order n with minimum degree $\delta \ge 2$, then $\gamma_{wsR}(G) \le n - 2\lfloor \delta/2 \rfloor$.

Proof. Let $t = \lfloor \delta/2 \rfloor$, and let $A = \{v_1, v_2, \ldots, v_t\}$ be a set of t vertices of G. Define the function $f : V(G) \to \{-1, 1, 2\}$ by $f(x) = -1$ for $x \in A$ and $f(x) = 1$ for $x \in V(G) \setminus A$. Then

$$f(N[w]) \ge -t + (\delta + 1 - t) = \delta + 1 - 2t = \delta + 1 - 2\lfloor \delta/2 \rfloor \ge 1$$

for each $w \in V(G)$. Therefore f is a WSRDF on G of weight $n - 2t$ and thus $\gamma_{wsR}(G) \le n - 2t$. □

Proposition 3. If G is graph of order n, then $\gamma_{wsR}(G) \geq \Delta + 2 - n$.

Proof. Let w be a vertex of maximum degree, and let f be a $\gamma_{wsR}(G)$-function. Then the definitions imply the desired bound as follows:

$$\begin{aligned}\gamma_{wsR}(G) &= \sum_{x \in V(G)} f(x) = \sum_{x \in N[w]} f(x) + \sum_{x \in V(G) \setminus N[w]} f(x) \\ &\geq 1 + \sum_{x \in V(G) \setminus N[w]} f(x) \geq 1 - (n - (\Delta(G) + 1)) = \Delta(G) + 2 - n.\end{aligned}$$

□

The proof of the next proposition is identically with the proof of Proposition 2 in [1] and is therefore omitted.

Proposition 4. Let $f = (V_{-1}, V_1, V_2)$ be a WSRDF of a graph G of order n, $\Delta = \Delta(G)$ and $\delta = \delta(G)$. Then the following holds.

(a) $(2\Delta + 1)|V_2| + \Delta|V_1| \geq (\delta + 2)|V_{-1}|$.

(b) $(2\Delta + \delta + 3)|V_2| + (\Delta + \delta + 2)|V_1| \geq (\delta + 2)n$.

(c) $(\Delta + \delta + 2)\omega(f) \geq (\delta - \Delta + 2)n + (\delta - \Delta)|V_2|$.

(d) $\omega(f) \geq (\delta - 2\Delta + 1)n/(2\Delta + \delta + 3) + |V_2|$.

As an immediate consequence of Proposition 4 (c), we obtain a lower bound on the weak signed Roman domination number of regular graphs.

Corollary 1. If G is an r-regular graph of order n, then $\gamma_{wsR}(G) \geq \lceil n/(r + 1) \rceil$.

In the case that G is not regular, Proposition 4 (c) and (d) lead to the following lower bound.

Corollary 2. Let G be a graph of order n, maximum degree Δ and minimum degree δ. If $\delta < \Delta$, then

$$\gamma_{wsR}(G) \geq \frac{-2\Delta + 2\delta + 3}{2\Delta + \delta + 3} n.$$

Proof. Multiplying both sides of the inequality in Proposition 4 (d) by $\Delta - \delta$ and adding the resulting inequality to the inequality in Proposition 4 (c), we yield the desired lower bound. □

Example 1. *Let $p \geq 2$ be an integer, and let $v_1, v_2, \ldots, v_p$ be the vertex set of the complete graph K_p. Now let H be the graph consisting of K_p such that each vertex v_i is adjacent to $2p-1$ leaves for $1 \leq i \leq p$. Define the function $f : V(H) \to \{-1, 1, 2\}$ by $f(v_i) = 2$ for $1 \leq i \leq p$ and $f(x) = -1$ otherwise. Then f is a WSRDF on H of weight*

$$3p - 2p^2 = \frac{-2\Delta(H) + 2\delta(H) + 3}{2\Delta(H) + \delta(H) + 3} n(H).$$

Therefore Corollary 2 shows that $\gamma_{wsR}(H) = 3p - 2p^2$ and thus Corollary 2 is sharp.

Since f is also a signed Roman dominating function on H, this example also demonstrates that the inequality

$$\gamma_{sR}(G) \geq \frac{-2\Delta + 2\delta + 3}{2\Delta + \delta + 3} n,$$

which can be found in [1] and which follows from Corollary 2, is sharp too.

3. Special classes of graphs

In this section, we determine the weak signed Roman domination number for special classes of graphs.

Proposition 5. If $n \geq 1$, then $\gamma_{wsR}(K_n) = 1$.

Proof. According to Proposition 3, we have $\gamma_{wsR}(K_n) \geq 1$. If n is even, then assign to one vertex the weight 2, to $n/2$ vertices the weight -1 and to the remaining $(n-2)/2$ vertices the weight 1. On the other hand, if n is odd, then assign to $(n+1)/2$ vertices the weight 1 and to the remaining $(n-1)/2$ vertices the weight -1. In both cases, we produce a WSRDF of weight 1, and thus $\gamma_{wsR}(K_n) \leq 1$, and so $\gamma_{wsR}(K_n) = 1$. □

Proposition 6. If $n \geq 2$, then $\gamma_{wsR}(K_{1,n-1}) = 1$, unless $n = 3$, in which case $\gamma_{wsR}(K_{1,2}) = 2$.

Proof. Let w be the central vertex and $v_1, v_2, \ldots, v_{n-1}$ be the leaves of $K_{1,n-1}$. According to Proposition 3, we have $\gamma_{wsR}(K_{1,n-1}) \geq 1$. If $n = 2p$ is even, then define f by $f(w) = 2$, $f(v_i) = -1$ for $1 \leq i \leq p$ and $f(v_i) = 1$ for $p+1 \leq i \leq 2p-1$. Then f is a WSRDF on $K_{1,n-1}$ of weight 1 and thus $\gamma_{wsR}(K_{1,n-1}) = 1$ in this case.
If $n = 2p+1$ is odd, then it is easy to see that $\gamma_{wsR}(K_{1,2}) = 2$. Let now $p \geq 2$. Define f by $f(w) = 2$, $f(v_i) = -1$ for $1 \leq i \leq p+1$, $f(v_i) = 1$ for $p+2 \leq i \leq 2p-1$ and $f(v_{2p}) = 2$. Then f is a WSRDF on $K_{1,n-1}$ of weight 1 and thus $\gamma_{wsR}(K_{1,n-1}) = 1$ also in this case. □

Propositions 5 and 6 show that Proposition 3 is sharp. Since the function f in the proof of Proposition 6 is also an SRDF on $K_{1,n-1}$, we deduce the following corollary, which corrects Observation 5 in [1].

Corollary 3. If $n \geq 2$, then $\gamma_{sR}(K_{1,n-1}) = 1$, unless $n = 3$, in which case $\gamma_{sR}(K_{1,2}) = 2$.

Proposition 7. If C_n is a cycle of length $n \geq 3$, then $\gamma_{wsR}(C_n) = \lceil n/3 \rceil$ when $n \equiv 0, 1 \pmod 3$ and $\gamma_{wsR}(C_n) = \lceil n/3 \rceil + 1$ when $n \equiv 2 \pmod 3$.

Proof. Let $C_n = v_1 v_2 \dots v_n v_1$. Applying Corollary 1, we observe that $\gamma_{wsR}(C_n) \geq \lceil n/3 \rceil$.
Let first $n = 3p$ for an integer $p \geq 1$. Define $f(v_{3i}) = -1$ and $f(v_{3i-2}) = f(v_{3i-1}) = 1$ for $1 \leq i \leq p$. Then f is a WSRDF on C_n of weight $p = n/3$ and thus $\gamma_{wsR}(C_n) \leq p$. Therefore $\gamma_{wsR}(C_n) = \lceil n/3 \rceil$ in this case.
Let second $n = 3p+1$ for an integer $p \geq 1$. Define $f(v_{3i}) = -1$, $f(v_{3i-2}) = f(v_{3i-1}) = 1$ for $1 \leq i \leq p$ and $f(v_{3p+1}) = 1$. Then f is a WSRDF on C_n of weight $p+1 = \lceil n/3 \rceil$ and thus $\gamma_{wsR}(C_n) \leq \lceil n/3 \rceil$. Therefore $\gamma_{wsR}(C_n) = \lceil n/3 \rceil$ in this case.
Finally, let $n = 3p + 2$ for an integer $p \geq 1$, and let f be a $\gamma_{wsR}(C_n)$-function. If $f(x) \geq 1$ for each $x \in V(C_n)$, then $\gamma_{wsR}(C_n) \geq n \geq \lceil n/3 \rceil + 1$. Let next, without loss of generality, $f(v_1) = -1$. Then $f(v_{3p+1}), f(v_{3p+2}) \geq 1$ and we obtain

$$\gamma_{wsR}(C_n) = f(V(C_n)) = f(v_{3p+1}) + f(v_{3p+2}) + \sum_{i=1}^{p} f(N[v_{3i-1}]) \geq p + 2 = \left\lceil \frac{n}{3} \right\rceil + 1.$$

Otherwise, define $g(v_{3i}) = -1$, $g(v_{3i-2}) = g(v_{3i-1}) = 1$ for $1 \leq i \leq p$ and $g(v_{3p+1}) = g(v_{3p+2}) = 1$. Then g is a WSRDF on C_n of weight $p + 2 = \lceil n/3 \rceil + 1$ and thus $\gamma_{wsR}(C_n) \leq \lceil n/3 \rceil + 1$. Therefore $\gamma_{wsR}(C_n) = \lceil n/3 \rceil + 1$ in this case. □

In [1], the authors have shown that $\gamma_{sR}(C_n) = \lceil 2n/3 \rceil$. Thus we deduce from Proposition 7 that the difference $\gamma_{sR}(G) - \gamma_{wsR}(G)$ can be arbitrarily large.

Proposition 8. Let P_n be a path of order $n \geq 1$. Then $\gamma_{wsR}(P_2) = 1$, $\gamma_{wsR}(P_n) = \lceil n/3 \rceil$ when $n \equiv 1 \pmod 3$ and $\gamma_{wsR}(P_n) = \lceil n/3 \rceil + 1$ when $n \equiv 0, 2 \pmod 3$ and $n \geq 3$.

Proof. Let $P_n = v_1 v_2 \dots v_n$, and let f be a $\gamma_{wsR}(P_n)$-function.
First assume that $n = 3p + 1$ for an integer $p \geq 0$. If $p = 0$, then the result is trivial. If $p \geq 1$, then we observe that $f(v_1) + f(v_2) \geq 1$ and $f(v_{3p}) + f(v_{3p+1}) \geq 1$. This leads to

$$\begin{aligned} \gamma_{wsR}(P_n) &= f(V(P_n)) = f(v_1) + f(v_2) + f(v_{3p}) + f(v_{3p+1}) + \sum_{i=1}^{p-1} f(N[v_{3i+1}]) \\ &\geq 2 + p - 1 = p + 1 = \left\lceil \frac{n}{3} \right\rceil. \end{aligned}$$

Next define $g(v_{3i+1}) = -1$ for $0 \le i \le p$, $g(v_2) = g(v_{3p}) = 2$ and $g(v_{3i}) = g(v_{3i+2}) = 1$ for $1 \le i \le p-1$. Then g is a WSRDF on P_n of weight $p+1 = \lceil n/3 \rceil$ and thus $\gamma_{wsR}(P_n) \le \lceil n/3 \rceil$. Therefore $\gamma_{wsR}(P_n) = \lceil n/3 \rceil$ in this case.
Second assume that $n = 3p$ for an integer $p \ge 1$. If $f(v_1) + f(v_2) + f(v_3) \ge 2$, then we obtain

$$\begin{aligned}\gamma_{wsR}(P_n) &= f(V(P_n)) = f(v_1) + f(v_2) + f(v_3) + \sum_{i=1}^{p-1} f(N[v_{3i+2}]) \\ &\ge 2 + p - 1 = p + 1 = \left\lceil \frac{n}{3} \right\rceil + 1.\end{aligned}$$

Next assume that $f(v_1) + f(v_2) + f(v_3) = 1$. This is only possible when $f(v_1) = f(v_2) = 1$ and $f(v_3) = -1$. Then $f(v_4) \ge 1$, and it follows that

$$\begin{aligned}\gamma_{wsR}(P_n) &= f(V(P_n)) = f(v_1) + f(v_2) + f(v_3) + f(v_4) + \sum_{i=2}^{p-1} f(N[v_{3i}]) + f(N[v_{3p}]) \\ &\ge 2 + p - 2 + 1 = p + 1 = \left\lceil \frac{n}{3} \right\rceil + 1.\end{aligned}$$

Now define $g(v_{3i}) = -1$ for $1 \le i \le p$, $g(v_{3p-1}) = 2$ and $g(v_{3i-2}) = g(v_{3i-1}) = 1$ for $1 \le i \le p-1$ and $g(v_{3p-2}) = 1$. Then g is a WSRDF on P_n of weight $p+1 = \lceil n/3 \rceil + 1$ and thus $\gamma_{wsR}(P_n) \le \lceil n/3 \rceil + 1$. Therefore $\gamma_{wsR}(P_n) = \lceil n/3 \rceil + 1$ in this case.
Finally, asume that $n = 3p + 2$ for an integer $p \ge 0$. Clearly, $\gamma_{wsR}(P_2) = 1$. Let now $p \ge 1$. If $f(v_1) + f(v_2) + f(v_3) \ge 2$, then we obtain

$$\begin{aligned}\gamma_{wsR}(P_n) &= f(V(P_n)) \\ &= f(v_1) + f(v_2) + f(v_3) + \sum_{i=1}^{p-1} f(N[v_{3i+2}]) + f(v_{3p+1}) + f(v_{3p+2}) \\ &\ge 2 + p - 1 + 1 = p + 2 = \left\lceil \frac{n}{3} \right\rceil + 1.\end{aligned}$$

Next assume that $f(v_1) + f(v_2) + f(v_3) = 1$. Then $f(v_1) = f(v_2) = 1$ and $f(v_3) = -1$ and hence $f(v_4), f(v_5) \ge 1$. We conclude that

$$\begin{aligned}\gamma_{wsR}(P_n) &= f(V(P_n)) = f(v_1) + f(v_2) + f(v_3) + f(v_4) + f(v_5) + \sum_{i=2}^{p} f(N[v_{3i+1}]) \\ &\ge 3 + p - 1 = p + 2 = \left\lceil \frac{n}{3} \right\rceil + 1.\end{aligned}$$

Conversely define $g(v_{3i}) = -1$ for $1 \le i \le p$ and $g(v_{3i-2}) = g(v_{3i-1}) = 1$ for $1 \le i \le p+1$. Then g is a WSRDF on P_n of weight $p+2 = \lceil n/3 \rceil + 1$ and thus $\gamma_{wsR}(P_n) \le \lceil n/3 \rceil + 1$. Therefore $\gamma_{wsR}(P_n) = \lceil n/3 \rceil + 1$ in this case. □

If G is 1-regular of order n, then $\gamma_{wsR}(G) = n/2$. Corollary 1 implies $\gamma_{wsR}(G) \geq \lceil n/3 \rceil$ when G is 2-regular, and it follows from Proposition 7 that $\gamma_{wsR}(C_n) = \lceil n/3 \rceil$ when $n \equiv 0, 1 \pmod 3$. Therefore Corollary 1 is tight if $r = 1, 2$. By Proposition 5, the lower bound of Corollary 1 is also tight if $r = n-1$. In [1], the authors have construct cubic graphs H of order n with the property that $\gamma_{sR}(H) = n/4$. Since $\gamma_{sR}(G) \geq \gamma_{wsR}(G) \geq \lceil n/4 \rceil$ for each cubic graph, Corollary 1 implies $\gamma_{sR}(H) = \gamma_{wsR}(H) = n/4$. Therefore Corollary 1 is also tight for $r = 3$. Next we will show that Corollary 1 is tight for $r = n - 2$ and $r = n - 3$.

Proposition 9. If G is an $(n-2)$-regular graph of order $n \geq 4$, then $\gamma_{wsR}(G) = 2$.

Proof. Since G is $(n-2)$-regular, the graph is isomorphic to the complete r-partite graph $K_{n_1, n_2, \ldots, n_r}$ with $r \geq 2$ and $n_1 = n_2 = \ldots = n_r = 2$. Corollary 1 implies $\gamma_{wsR}(G) \geq \lceil n/(n-1) \rceil = 2$.
Now let $X_i = \{x_i, y_i\}$ be the partite sets of G for $1 \leq i \leq r$. Define $f(x_i) = f(y_1) = 1$ for $1 \leq i \leq r$ and $f(y_i) = -1$ for $2 \leq i \leq r$. Then f is a WSRDF on G of weight 2 and thus $\gamma_{wsR}(G) \leq 2$. Therefore $\gamma_{wsR}(G) = 2$. □

Example 2. *Let H be the complete r-partite graph with $r \geq 2$ and the partite sets $X_1, X_2, \ldots, X_r$ such that $|X_1| = |\{a, b, u, v\}| = 4$ and $|X_i| = 3$ for $2 \leq i \leq r$. Now let G consisting of H with the additional edges ab and uv. Then G is an $(n-3)$-regular graph of order $n = 3r + 1$. Corollary 1 implies $\gamma_{sR}(G) \geq \gamma_{wsR}(G) \geq \lceil n/(n-2) \rceil = 2$.*
Now let $X_i = \{x_i, y_i, z_i\}$ be the partite sets of G for $2 \leq i \leq r$. Define $f(x_i) = f(a) = f(u) = 2$ for $2 \leq i \leq r$ and $f(b) = f(v) = f(y_i) = f(z_i) = -1$ for $2 \leq i \leq r$. Then f is a WSRDF (even an SRDF) on G of weight 2 and thus $\gamma_{wsR}(G) \leq \gamma_{sR}(G) \leq 2$. Therefore $\gamma_{sR}(G) = \gamma_{wsR}(G) = 2$.

Proposition 10. If $S(r, s)$ is the double star with $r, s \geq 5$, then $\gamma_{wsR}(S(r, s)) = -2$.

Proof. Let u and v be two adjacent vertices of $S(r, s)$ such that u is adjacent to r leaves and v is adjacent to s leaves. If f is a $\gamma_{wsR}(S(r, s))$-function, then the definition implies

$$\omega(f) = f(N[u]) + f(N[v]) - f(u) - f(v) \geq 1 + 1 - 2 - 2 = -2$$

and so $\gamma_{wsR}(S(r, s)) \geq -2$.
Conversely, let $r = 2p + 1$ and $s = 2q + 1$ be odd. Define f by $f(u) = f(v) = 2$. In addition, we assign the weight -1 to $p + 2$ leaves of u, the weight 1 to $p - 1$ leaves of u, the weight -1 to $q + 2$ leaves of v and the weight 1 to $q - 1$ leaves of v. Then f is a WSRDF on $S(r, s)$ of weight -2 and thus $\gamma_{wsR}(S(r, s)) \leq -2$. Therefore $\gamma_{wsR}(S(r, s)) = -2$ in this case.
Let $r = 2p$ and $s = 2q$ be even with $p, q \geq 3$. Define f by $f(u) = f(v) = 2$. In addition, we assign the weight -1 to $p + 2$ leaves of u, the weight 1 to $p - 3$ leaves of u, the weight 2 to one leaf of u, the weight -1 to $q + 2$ leaves of v, the weight 1 to

$q-3$ leaves of v and the weight 2 to one leaf of v. Then f is a WSRDF on $S(r,s)$ of weight -2 and thus $\gamma_{wsR}(S(r,s)) \le -2$. Therefore $\gamma_{wsR}(S(r,s)) = -2$ in this case.
The cases r even and s odd or r odd and s even are similar to the cases above and are therefore omitted. □

Since f is also a signed Roman dominating function on $S(r,s)$ in the proof of Proposition 10, we also have $\gamma_{sR}(S(r,s)) = -2$ when $r,s \ge 5$. The proof of Proposition 10 shows that $\gamma_{wsR}(S(3,s)) = -2$ for $s \ge 5$. In addition, it is straightforward to verify that $-2 \le \gamma_{wsR}(S(r,s)) \le 0$ for $r,s \ge 2$.

Proposition 11. If $p \ge 4$ is an integer, then $\gamma_{wsR}(K_{p,p}) = 4$ and $\gamma_{wsR}(K_{3,3}) = 3$.

Proof. In [7], the authors have shown that $\gamma_{sR}(K_{p,p}) = 4$ for $p \ge 3$. Hence it follows that $\gamma_{wsR}(K_{p,p}) \le \gamma_{sR}(K_{p,p}) = 4$. In the case $p=3$, define $g(x_1) = g(x_2) = g(x_3) = 1$, $g(y_1) = g(y_2) = -1$ and $g(y_3) = 2$. Then g is a WSRDF on $K_{3,3}$ of weight 3 and thus $\gamma_{wsR}(K_{3,3}) \le 3$.
Conversely, assume that f is a $\gamma_{wsR}(K_{p,p})$-function, and let $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_p\}$ be the partite sets. Assume first that $f(x_i) \ge 1$ for each $i \in \{1, 2, \ldots, p\}$. Then

$$\gamma_{wsR}(K_{p,p}) = f(N[x_1]) + f(X \setminus \{x_1\}) \ge 1 + p - 1 = p.$$

If $f(y_i) \ge 1$ for $1 \le i \le p$, then we obtain analogously $\gamma_{wsR}(K_{p,p}) \ge p$.
Now assume, without loss of generality, that $f(x_1) = f(y_1) = -1$. Then

$$\gamma_{wsR}(K_{p,p}) = f(N[x_1]) + f(N[y_1]) - f(x_1) - f(y_1) \ge 1 + 1 + 1 + 1 = 4.$$

Therefore $\gamma_{wsR}(K_{p,p}) \ge 4$ and thus $\gamma_{wsR}(K_{p,p}) = 4$ when $p \ge 4$ and $\gamma_{wsR}(K_{3,3}) = 3$. □

4. Further lower bounds

A set $S \subseteq V(G)$ is a 2-*packing* of the graph G if $N[u] \cap N[v] = \emptyset$ for any two distinct vertices $u, v \in S$. The maximum cardinality of a 2-packing in G is the 2-*packing number*, denoted by $\rho(G) = \rho$.

Theorem 3. If G is a graph of order n, minimum degree δ and packing number ρ, then $\gamma_{wsR}(G) \ge \rho(\delta + 2) - n$.

Proof. Let $\{v_1, v_2, \ldots, v_\rho\}$ be a 2-packing of G, and let f be a $\gamma_{wsR}(G)$-function. If we define the set $A = \bigcup_{i=1}^{\rho} N[v_i]$, then since $\{v_1, v_2, \ldots, v_\rho\}$ is a 2-packing, we note that

$$|A| = \sum_{i=1}^{\rho} (d(v_i) + 1) \ge \rho(\delta + 1).$$

This leads to

$$\begin{aligned}\gamma_{wsR}(G) &= \sum_{x\in V(G)} f(x) = \sum_{i=1}^{\rho} f(N[v_i]) + \sum_{x\in V(G)\setminus A} f(x)\\ &\geq \rho + \sum_{x\in V(G)\setminus A} f(x) \geq \rho - n + |A| \geq \rho - n + \rho(\delta+1) = \rho(\delta+2) - n,\end{aligned}$$

and the proof is complete. □

The next example will demonstrate that the bound in Theorem 3 is sharp.

Example 3. *Let F be an arbitrary graph of order $t \geq 1$, and for each vertex $v \in V(F)$ add a vertex-disjoint copy of a complete graph K_s ($s \geq 2$) and identify the vertex v with one vertex of the added complete graph. Let H denote the resulting graph. Furthermore, let $H_1, H_2, \ldots, H_t$ be the added copies of K_s. For $i = 1, 2, \ldots, t$, let v_i be the vertex of H_i that is identified with a vertex of F. We now construct a WSRDF on H as follows. For each $i = 1, 2, \ldots, t$, let $f_i : V(H_i) \to \{-1, 1, 2\}$ be the WSRDF on the complete graph defined as in Proposition 5 such that $f_i(v_i) \geq 1$. As shown in Proposition 5, we have $\omega(f_i) = 1$. Now let $f : V(H) \to \{-1, 1, 2\}$ be the function defined by $f(v) = f_i(v)$ for each $v \in V(H_i)$. Then f is a WSRDF of H of weight t and hence $\gamma_{wsR}(H) \leq t$. Since $n(H) = ts$, $\delta(H) = s - 1$ and $\rho(H) = t$, Theorem 3 implies that $\gamma_{wsR}(H) \geq \rho(H)(\delta(H) + 2) - n(H) \geq t$. Thus $\gamma_{wsR}(H) \geq \rho(H)(\delta(H)+2)-n(H) = t$. Consequently, $\gamma_{wsR}(H) = \rho(H)(\delta(H)+2)-n(H) = t$.*

For a subset $S \subseteq V(G)$, we let $d_S(v)$ denote the number of vertices in S that are adjacent to the vertex v. For disjoint subsets U and W of vertices, we let $[U, W]$ denote the set of edges between U and W. Now let $f = (V_{-1}, V_1, V_2)$ be a WSRDF. For notational convenience, we let $V_{12} = V_1 \cup V_2$, $|V_{12}| = n_{12}$, $|V_1| = n_1$ and $|V_2| = n_2$. Furthermore, let $|V_{-1}| = n_{-1}$ and so $n_{-1} = n - n_{12}$. Let $G_{12} = G[V_{12}]$ be the subgraph induced by V_{12} and let G_{12} have size m_{12}. For $i = 1, 2$, if $V_i \neq \emptyset$, let $G_i = G[V_i]$ be the subgraph induced by V_i and let G_i have size m_i. Hence $m_{12} = m_1 + m_2 + |[V_1, V_2]|$.

For $k \geq 1$, let L_k be the graph obtained from a graph H of order k by adding $2d_H(v)+1$ pendant edges to each v of H. Note that $L_1 = K_2$. Let $\mathcal{H} = \{L_k \,|\, k \geq 1\}$. In [1], one can find the following lower bound on the signed Roman domination number.

Theorem 4. [1] If G is a graph of order n and size m without isolated vertices, then

$$\gamma_{sR}(G) \geq \frac{3n - 4m}{2}$$

with equality if and only if $G \in \mathcal{H}$.

Using the ideas in [1], we obtain the following improvement of Theorem 4.

Theorem 5. If G is a graph of order n and size m without isolated vertices, then

$$\gamma_{wsR}(G) \geq \frac{3n-4m}{2}$$

with equality if and only if $G \in \mathcal{H}$.

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{wsR}(G)$-function, and let $V_{-1}^2 \subseteq V_{-1}$ be the maximum set such that each vertex $v \in V_{-1}^2$ has at least one neighbor in V_2. In addition, let $V_{-1}^1 = V_{-1} \setminus V_{-1}^2$. Since each vertex of V_{-1}^1 has at least two neighbors in V_1, we observe that

$$2|V_{-1}^1| \leq |[V_{-1}^1, V_1]| = \sum_{v \in V_1} d_{V_{-1}^1}(v).$$

For each $v \in V_1$, we have $1 \leq f(N[v]) = f(v) + 2d_{V_2}(v) + d_{V_1}(v) - d_{V_{-1}}(v)$ and so $d_{V_{-1}}(v) \leq 2d_{V_2}(v) + d_{V_1}(v)$. Hence we obtain

$$\begin{aligned} 2|V_{-1}^1| &\leq \sum_{v \in V_1} d_{V_{-1}^1}(v) \leq \sum_{v \in V_1} d_{V_{-1}}(v) \\ &\leq \sum_{v \in V_1} (2d_{V_2}(v) + d_{V_1}(v)) = 2|[V_1, V_2]| + 2m_1. \end{aligned}$$

Since each vertex of V_{-1}^2 has at least one neighbor in V_2, we have

$$|V_{-1}^2| \leq |[V_{-1}^2, V_2]| = \sum_{v \in V_2} d_{V_{-1}^2}(v).$$

For each $v \in V_2$, we have $1 \leq f(N[v]) = f(v) + 2d_{V_2}(v) + d_{V_1}(v) - d_{V_{-1}}(v)$ and so $d_{V_{-1}}(v) \leq 2d_{V_2}(v) + d_{V_1}(v) + 1$. This leads to

$$\begin{aligned} |V_{-1}^2| &\leq \sum_{v \in V_2} d_{V_{-1}^2}(v) \leq \sum_{v \in V_2} d_{V_{-1}}(v) \\ &\leq \sum_{v \in V_2} (2d_{V_2}(v) + d_{V_1}(v) + 1) = 4m_2 + |[V_1, V_2]| + n_2 \\ &= 4m_{12} + n_2 - 4m_1 - 3|[V_1, V_2]|. \end{aligned}$$

Combining the corresponding inequalities, we obtain

$$\begin{aligned} n_{-1} &= |V_{-1}^1| + |V_{-1}^2| \leq |[V_1, V_2]| + m_1 + 4m_{12} + n_2 - 4m_1 - 3|[V_1, V_2]| \\ &= 4m_{12} + n_2 - 3m_1 - 2|[V_1, V_2]| \end{aligned}$$

and so $m_{12} \geq (n_{-1} - n_2 + 3m_1 + 2|[V_1, V_2]|)/4$. Hence we deduce that

$$\begin{aligned} m &\geq m_{12} + |[V_{-1}, V_{12}]| \\ &\geq \frac{1}{4}(n_{-1} - n_2 + 3m_1 + 2|[V_1, V_2]|) + n_{-1} \\ &= \frac{1}{4}(5n_{-1} - n_2 + 3m_1 + 2|[V_1, V_2]|) \\ &= \frac{1}{4}(5n - 6n_{12} + n_1 + 3m_1 + 2|[V_1, V_2]|) \end{aligned}$$

This yields

$$n_{12} \geq \frac{1}{6}(5n - 4m + n_1 + 3m_1 + 2|[V_1, V_2]|)$$

and thus

$$\begin{aligned} \gamma_{wsR}(G) &= 2n_2 + n_1 - n_{-1} = 3n_2 + 2n_1 - n = 3n_{12} - n - n_1 \\ &\geq \frac{1}{2}(5n - 4m + n_1 + 3m_1 + 2|[V_1, V_2]|) - n - n_1 \\ &= \frac{1}{2}(3n - 4m) + \frac{1}{2}(3m_1 + 2|[V_1, V_2]| - n_1). \end{aligned}$$

Let $\phi(n_1) = 3m_1 + 2|[V_1, V_2]| - n_1$. It suffices to show that $\phi(n_1) \geq 0$, since then $\gamma_{wsR}(G) \geq (3n - 4m)/2$, which is the desired bound. If $n_1 = 0$, then $\phi(n_1) = 0$, and we are done. Assume now that $n_1 \geq 1$. If $v \in V_1$ and $d_{V_{12}}(v) = 0$, then since by assumption there is no isolated vertex in G, we have $d_G(v) \geq 1$, and every neighbor of v belongs to V_{-1}. But then we obtain the contradiction $f(N[v]) \leq 0$. Hence $d_{V_{12}}(v) \geq 1$ for each $v \in V_1$. We deduce that

$$\begin{aligned} \phi(n_1) &= 3m_1 + 2|[V_1, V_2]| - n_1 > 2m_1 + |[V_1, V_2]| - n_1 \\ &= \sum_{v \in V_1} d_{V_1}(v) + \sum_{v \in V_1} d_{V_2}(v) - n_1 \\ &= \sum_{v \in V_1} d_{V_{12}}(v) - n_1 \geq n_1 - n_1 = 0 \end{aligned}$$

and so $\gamma_{wsR}(G) > (3n - 4m)/2$ when $n_1 \geq 1$.

Suppose that $\gamma_{wsR}(G) = (3n - 4m)/2$. Then all the inequalities above must be equalities. In particular, $V_{-1}^2 = V_{-1}$, $n_1 = 0$, $n_{12} = n_2$ and so $V_{12} = V_2$ and $V(G) = V_2 \cup V_{-1}$. Furthermore, $m_{12} = m_2$, $m = m_2 + |[V_{-1}, V_2]|$ and $|[V_{-1}, V_2]| = n_{-1}$. This implies that for each vertex $v \in V_{-1}$, we have $d_{V_2}(v) = 1$ and thus $d_{V_{-1}}(v) = 0$. Hence each vertex of V_{-1} is a leaf in G. Moreover, the identity $n_{-1} = |V_{-1}^2| = \sum_{v \in V_2}(2d_{V_2}(v) + 1)$ shows that $d_{V_{-1}}(v) = 2d_{V_2}(v) + 1$ for each $v \in V_2$ and therefore $G \in \mathcal{H}$.

Conversely, assume that $G \in \mathcal{H}$. Then it follows from Theorems 4 and 5 that

$$\frac{3n - 4m}{2} \leq \gamma_{wsR}(G) \leq \gamma_{sR}(G) = \frac{3n - 4m}{2}$$

and therefore $\gamma_{wsR}(G) = \frac{3n-4m}{2}$. □

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