New results on upper domatic number of graphs

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Abstract: For a graph $G = (V,E)$, a partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of the vertex set $V$ is an upper domatic partition if $V_i$ dominates $V_j$ or $V_j$ dominates $V_i$ or both for every $V_i, V_j \in \pi$, whenever $i \neq j$. The upper domatic number $D(G)$ is the maximum order of an upper domatic partition of $G$. We study the properties of upper domatic number and propose an upper bound in terms of clique number. Further, we discuss the upper domatic number of certain graph classes including unicyclic graphs and power graphs of paths and cycles.

Keywords: Domination, upper domatic partition, upper domatic number, transitivity

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1. Introduction

Let $G = (V,E)$ be a graph of order $n = |V|$ and size $m = |E|$. Throughout this paper, we consider only finite, simple and undirected graphs. The degree of a vertex $v \in V$, denoted by $\text{deg}(v)$, is the number of vertices adjacent to $v$. The maximum (minimum) degree of a vertex in a graph $G$ is denoted by $\Delta(G)$ ($\delta(G)$). Complete graphs, cycles and paths of order $n$ are represented as $K_n$, $C_n$, and $P_n$ respectively. A maximal complete subgraph of a graph is called a clique. The maximum cardinality of a clique of $G$ is called the clique number $\omega(G)$.

A vertex $v \in V$ is a leaf vertex if $\text{deg}(v) = 1$. Given two disjoint subsets $A$ and $B$ of the vertex set of a graph, the set $A$ dominates $B$, denoted by $A \rightarrow B$, if every vertex of $B$ is adjacent to at least one vertex in $A$, else we write $A \not\rightarrow B$. The dominating set of a graph $G$ is a set $S \subset V$ such that $S \rightarrow V - S$. The domination number of a graph $G$ is the minimum cardinality of a dominating set of $G$. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be
a vertex partition of a graph \( G \). A set \( V_i \in \pi \) is a source set if \( V_i \) is a dominating set of \( G \) and a sink set if \( V_j \rightarrow V_i \), for all \( 1 \leq j \leq k \).
Partitioning the vertex set of a graph into dominating sets is a problem posed by Cockayne and Hedetniemi in 1977 [1]. This gave rise to the concept of *domatic number* \( d(G) \) of a graph \( G \) which is the maximum order \( k \) of a vertex partition \( \{V_1, V_2, \ldots, V_k\} \) such that each \( V_i \), \( 1 \leq i \leq k \), is a dominating set. This concept was further generalised by S. T. Hedetniemi and J. T. Hedetniemi taking into consideration the domination between the subsets of the vertex set [5]. Using this idea, they considered a vertex partition \( \{V_1, V_2, \ldots, V_k\} \) of a graph \( G \) such that \( V_i \rightarrow V_j \) for all \( i, j, 1 \leq i < j \leq k \) and defined *transitivity* \( Tr(G) \) of a graph as the maximum order of such a partition. Haynes et al. obtained another generalisation of domatic number by defining the *upper domatic number* \( D(G) \) which is the maximum order of a vertex partition \( \{V_1, V_2, \ldots, V_k\} \) of a graph \( G \) such that for each pair \( i \) and \( j \), \( 1 \leq i < j \leq k \), either \( V_i \rightarrow V_j \) or \( V_j \rightarrow V_i \) or both [4]. Any upper domatic partition of order \( D(G) \) is referred to as a *D*-partition of \( G \). It follows from these definitions that for any graph \( G \),

\[
1 \leq d(G) \leq Tr(G) \leq D(G) \leq n.
\]

In this paper, we continue the study of upper domatic number. The results include upper bounds of \( D(G) \) in terms of size and clique number of the graph \( G \), and a characterisation of graphs having transitivity at least four. It has been proved that for unicyclic graphs the upper domatic number is the same as its transitivity. We have also examined the upper domatic number of powers of graphs and have determined the upper domatic number of powers of paths and powers of cycles.

### 2. Preliminary Results

**Theorem 1.** [1] For any graph \( G \), \( d(G) \leq \delta(G) + 1 \).

Analogous to the result in Theorem 1, Haynes et al. obtained an upper bound for the upper domatic number in terms of maximum degree for any graph \( G \) [4].

**Theorem 2.** [4] For any graph \( G \), \( D(G) \leq \Delta(G) + 1 \).

The following results giving upper domatic number and transitivity of certain families of graphs are required for further discussion.

**Theorem 3.** [4]

1. For the path \( P_n \) with \( n \geq 4 \), \( Tr(P_n) = D(P_n) = 3 \).
2. For the cycle \( C_n \) with \( n \geq 3 \), \( Tr(C_n) = D(C_n) = 3 \).
3. For the complete graph \( K_n \), \( Tr(K_n) = D(K_n) = n \).
Given any $D$-partition of a graph $G$, we make the following observations.

**Observation 4.** If $\pi = \{V_1, V_2, \ldots, V_k\}$ is a $D$-partition of a graph $G$ such that the subset $V_i$ contains a leaf vertex, then $V_i$ is dominated by at most one set $V_j$, where $i \neq j$.

**Observation 5.** Every sink set in a $D$-partition is an independent set.

**Observation 6.** Let $\pi$ be a $D$-partition of a graph $G$. Then for any $V_i \in \pi$, $D(H) \geq D(G) - 1$, where $H = G[\pi - V_i]$.

The following theorem gives a property of graphs with $D(G) < 5$.

**Theorem 7.** [4] If $D(G) \leq 4$, then there exists a $D$-partition of graph $G$ that contains a sink set.

S. T. Hedetniemi and J. T. Hedetniemi noted that the transitivity of a graph is at least as large as the transitivity of any of its subgraphs [5].

**Proposition 1.** [5] If $H$ is a subgraph of a graph $G$, then $Tr(H) \leq Tr(G)$.

**Proposition 2.** [3] For any graph $G$ and a subgraph $H$, if $V(G) - V(H)$ dominates $H$, then $Tr(G) \geq Tr(H) + 1$.

Proposition 1 holds for the upper domatic number of a graph, provided the subgraph has a $D$-partition with a source set [4].

**Proposition 3.** [4] If $H$ is a subgraph of a graph $G$ and $H$ has a $D$-partition with source set, then $D(H) \leq D(G)$.

This proposition can be extended to disconnected graphs as shown in the following theorem.

**Theorem 8.** For a disconnected graph $G$ with finite number of components, $D(G) \leq \max\{D(G_i), \text{ where } G_i \text{ is a component of } G\}$.

**Proof.** Let $G$ be a disconnected graph with $r$ components such that $D(G) = k$ and $\pi$ be a $D$-partition of $G$. If possible, assume that $D(G) > \max\{D(G_i), \text{ where } G_i \text{ is a component of } G\}$. Then corresponding to each component $G_i$ of $G$, there exists an element say $V_{G_i} \in \pi$ such that none of the vertices of $G_i$ appears in $V_{G_i}$. Otherwise, the collection $\pi_i$ of order $k$ formed by taking the intersection of $V(G_i)$ with
the $k$ elements of $\pi$ will constitute a $D$-partition of $G$, contradicting the assumption. Arbitrarily choose one component $G_1$ of $G$ and let $V_{G_1} \in \pi$ be the corresponding set that contains no vertex of $G_1$. As the set $V_{G_1}$ is not empty, choose a component $G_2$ having at least one vertex in $V_{G_1}$ and let $V_{G_2} \in \pi$ be the set containing no vertex of $G_2$. Since $V_{G_2} \not\subseteq V_{G_1}$, the set $V_{G_1}$ dominates $V_{G_2}$ and $V_{G_2}$ does not contain any of the vertices of $G_1$ and $G_2$. Proceeding in this manner, choose the $r^{th}$ component $G_r$ and the set $V_{G_r}$ disjoint from the vertex set of $G_r$. Then, the set $V_{G_r}$ has empty intersection with the vertex set of components $G_i$, for $1 \leq i \leq r$ making $V_{G_r}$ an empty set, a contradiction. Hence, $D(G) \leq \max\{D(G_i), \text{ where } G_i \text{ is a component of } G\}$.

**Corollary 1.** For a disconnected graph $G$ with finite number of components $G_1, G_2, \ldots, G_r$, if the component with maximum upper domatic number has a $D$-partition with source set, then $D(G) = \max\{D(G_i), \text{ where } G_i \text{ is a component of } G\}$.

![Disconnected graph $G$ with $D(G) < \max\{D(G_1), D(G_2)\}$.

**Remark 1.** The inequality in Theorem 8 is sharp. Figure 1 depicts a graph $G$ with components $G_1$ and $G_2$. The fact that $\pi_1 = \{\{v_1\}, \{v_2\}, \{v_3, v_4\}, \{v_5, v_6\}, \{v_7, v_8\}, \{v_9, v_{10}\}\}$ is an upper domatic partition of $G_1$ and $\Delta(G_1) = 5$ implies that $D(G_1) = 6$, while it can be verified that $D(G) = 5 < D(G_1)$.

An analogous result for the transitivity of a disconnected graph can be found in [5].

**Theorem 9.** [5] For a disconnected graph $G$ with finite number of components $G_1, G_2, \ldots, G_r$, $Tr(G) = \max\{Tr(G_i) \mid 1 \leq i \leq r\}$.

From Proposition 3, it is straightforward to see that the clique number of a graph is a lower bound for its upper domatic number [4]. The following theorem shows that the mean of order and clique number of a graph serves as an upper bound for upper domatic number of the graph.
**Theorem 10.** For any graph $G$ with clique number $\omega(G)$, $D(G) \leq \frac{n + \omega(G)}{2}$.

**Proof.** Assuming the contrary, let $G$ be a graph with $D(G) > \frac{n + \omega(G)}{2}$. As $\omega(G) > 0$, $\frac{n + \omega(G)}{2} > \frac{n}{2}$ and there are at least $\omega(G) + 1$ singleton sets in the $D$-partition of $G$. But the elements of the singleton sets in a $D$-partition induces a clique implying that $G$ contains a clique of order $\omega(G) + 1$ which contradicts the maximality of $\omega(G)$. Therefore, $D(G) \leq \frac{n + \omega(G)}{2}$.

**Remark 2.** There exist graphs for which the bound in Theorem 10 is attained. For example, consider the complete bipartite graph $K_{s,s}$ where $s \geq 1$. Then $D(K_{s,s}) = s + 1$ [4]. However, the difference between upper domatic number and the bound obtained in Theorem 10 can be arbitrarily large as one can see in the case of $K_{1,s}$, $s \geq 1$.

Further we observe that $D(G)$ has an obvious bound in terms of the size of $G$.

**Observation 11.** For any graph $G$ of size $m$, $D(G) \leq 1 + \frac{\sqrt{1 + 8m} - 1}{2}$. Moreover, $D(G) = 1 + \frac{\sqrt{1 + 8m}}{2}$ if and only if $G$ is a complete graph.

The graph families admitting small values of upper domatic number was characterized in [4]. A *star* on $n$ vertices is the complete bipartite graph $K_{1,n-1}$ and a *galaxy* is a disjoint union of stars.

**Theorem 12.** [4] For any graph $G$ of order $n$,

1. $D(G) = 1$ if and only if $G = K_n$,
2. $D(G) = 2$ if and only if $G$ is a galaxy with at least one edge,
3. $D(G) \geq 3$ if and only if $G$ contains a $K_3$ or a $P_3$.

Since transitivity serves as a lower bound for the upper domatic number of a graph, we provide a sufficient condition for $D(G) \geq 4$ by characterising the graphs with $Tr(G) \geq 4$. Let $G_i$, $1 \leq i \leq 12$ be the graphs shown in Figure 2 and $\mathcal{A} = \{G_i, 1 \leq i \leq 12\}$.

**Theorem 13.** For any graph $G$, $Tr(G) \geq 4$ if and only if $G$ contains at least one of the graphs in class $\mathcal{A}$. 
Proof. Let $G$ be a graph containing one of the graphs $G_i$; $1 \leq i \leq 12$ in class $\mathcal{A}$ as its subgraph. Then by Proposition 1, $\text{Tr}(G) \leq \text{Tr}(G_i)$. Since $\Delta(G_i) = 3$ for each $G_i \in \mathcal{A}$, $\text{Tr}(G_i) \leq 4$ by Theorem 2. On the other hand every $G_i \in \mathcal{A}$ contains a subgraph $H \in \{P_4, C_3\}$, such that $[V(G_i) - V(H)] \rightarrow H$, hence by Proposition 2 and Theorem 3, $\text{Tr}(G) \geq 4$, thus proving the sufficiency part.

Conversely assume that $G$ is a graph with $\text{Tr}(G) \geq 4$. Then any $\text{Tr}$-partition of $G$ will have at least four elements. Consider a transitive partition $\pi = \{V_1, V_2, V_3, V_4\}$ where $V_i \subset V(G)$ of $G$. Then $V_i \rightarrow V_j$ for $i > j$, $i = 1, 2, 3$; $j = 2, 3, 4$. Each $V_i$; $1 \leq i \leq 4$ being non-empty, we can find a vertex, say $v_4 \in V_4$. Then there exist vertices $v_1 \in V_1$; $v_2 \in V_2$ and $v_3 \in V_3$ that dominate $v_4$, as $\pi$ is a transitive partition. Again corresponding to $v_3 \in V_3$, there exist vertices say $v'_2 \in V_2$ and $v'_1 \in V_1$ that dominate $v_3$. It is to be noted that $v'_1$ and $v'_2$ need not be necessarily different from $v_1 \in V_1$ and $v_2 \in V_2$. In the same way let $v''_1$ and $v''_2$ be the vertices in $V_1$ that dominates $v'_2$ and $v_2$ respectively. Then the following cases arise.

**Case 1:** All the vertices $v_1, v_2, v_3, v_4, v'_1, v''_1, v''_2$ and $v'_2$ are distinct.

In this case, the graph induced by these vertices is isomorphic to $G_1$ as depicted in Figure 3.

**Case 2:** The vertices $v_2$ and $v'_2$ are distinct.

In this case there arise eight possibilities depending upon the nature of vertices $v_1$, $v'_1$, $v''_1$, $v''_2$. 

Figure 2. Class $\mathcal{A}$. 

![Figure 2: Class A.](image-url)
Figure 3. The graph $G_1$.

$v_1', v_1'', v_1'''$ belonging to $V_1$.  
(i) $v_1 = v_1'''$; $v_1' \neq v_1''$ or $v_1 \neq v_1'''$; $v_1' = v_1''$, in which case the resulting graph induced by these vertices is $G_2$.  
(ii) $v_1 = v_1'$ and $v_1'' \neq v_1'''$, then the graph induced by these vertices is $G_3$.  
(iii) $v_1'' = v_1'''$ and $v_1 \neq v_1'$, where these vertices induces the graph $G_4$.  
(iv) $v_1 = v_1'$ and $v_1'' \neq v_1'''$ or $v_1 \neq v_1''$ and $v_1' = v_1'''$, then the resulting graph induced by these vertices is $G_5$.  
(v) $v_1 = v_1'''$ and $v_1' = v_1''$, in which case these vertices induces the graph $G_6$.  
(vi) $v_1 = v_1'$ and $v_1'' = v_1'''$, then these vertices induces the graph $G_7$.  
(vii) $v_1 = v_1''$ and $v_1' = v_1'''$, while these vertices induces the graph $G_8$.  
(viii) $v_1 = v_1'' = v_1''' \neq v_1'$ or $v_1' = v_1'' = v_1''' \neq v_1$, where the graph induced by these vertices is $G_9$.  

Case 3: The vertices $v_2$ and $v_2'$ are the same.  
In this case, it suffices to consider only three possibilities for the vertices in $V_1$.  
(i) $v_1 = v_1'' = v_1''' \neq v_1'$ or $v_1 = v_1'' \neq v_1'$ or $v_1' = v_1'' \neq v_1$, in which case these vertices induces the graph $G_{10}$.  
(ii) $v_1 = v_1' = v_1''$, then the resulting graph induced is $G_{11}$.  
(iii) $v_1, v_1', v_1''$ are all distinct, where these vertices induces the graph $G_{12}$.  

Hence we have proved that every graph $G$, with $Tr(G) \geq 4$ will have one of the members of class $A$ as its subgraph.  

**Corollary 2.** For any graph $G$, if $G$ contains at least one of the graphs in class $A$, then $D(G) \geq 4$.  

**Corollary 3.** For any graph $G$, $Tr(G) = 3$, if and only if $G$ contains $P_4$ or $C_3$ and $G$ is $H$–free, for all $H \in A$.  

\[ \square \]
3. Unicyclic Graphs

We now explore the family of unicyclic graphs, obtained from trees by adding a single edge between two non-adjacent vertices of the tree. A unicyclic graph can also be understood as a connected graph containing exactly one cycle [2]. Primarily, in this section, we show that a unicyclic graph is another class of graph having equal upper domatic number and transitivity. By the definition of a unicyclic graph, it is immediate from Theorem 3 and Proposition 3 that the upper domatic number of a unicyclic graph is at least three.

**Proposition 4.** For any unicyclic graph $G$, $D(G) \geq 3$.

The bound in Proposition 4 is obviously sharp as seen in the case of $C_n$. Tadpole graph form another family of unicyclic graphs with upper domatic number three. The tadpole graph $T_{s,t}$ is obtained by joining any vertex of a cycle $C_s$ and an end vertex of a path $P_t$ with an edge. We establish a result necessary to determine the upper domatic number of a tadpole graph.

**Lemma 1.** For any path $P_n$, $n \geq 4$, there exists no $D$-partition without a sink set.

**Proof.** For $n \geq 4$, the upper domatic number of a path $P_n$ being three, a $D$-partition $\pi' = \{V_1, V_2, V_3, V_4\}$ can be either a transitive triple, such that $V'_1 \rightarrow V'_2$, $V'_1 \rightarrow V'_3$ and $V'_2 \rightarrow V'_3$ or a cyclic triple, where $V'_1 \rightarrow V'_2$, $V'_2 \rightarrow V'_3$ and $V'_3 \rightarrow V'_1$. We consider the $D$-partition to be a cyclic triple otherwise, $\pi'$ has a sink set. To construct such a partition from a path with vertex set $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, we assign $v_1$ to $V'_1$, $v_2$ to $V'_3$ as $V'_3 \rightarrow V'_1$, and $v_3$ to $V'_2$. Assigning the vertices $v_4, v_5, \ldots, v_n$ to the sets $V'_1$, $V'_2$ and $V'_3$ sequentially, let $v_i$ be assigned to the set $V'_i$, for some $i$, $1 \leq i \leq 3$, there is no vertex in $V'_{(i-1)\mod 3}$ to dominate $v_n$ as $v_{n-1}$ is in $V'_{(i+1)\mod 3}$.

**Remark 3.** Let $G$ be a disjoint union of path, then there exists no $D$-partition of $G$ without a sink set.

**Theorem 14.** For a tadpole graph $T_{s,t}$, where $s \geq 3$ and $t \geq 1$, $D(T_{s,t}) = 3$.

**Proof.** Let $T_{s,t}$, where $s \geq 3$ and $t \geq 1$, be a tadpole graph. By Theorem 2 and Proposition 4, it is evident that $3 \leq D(T_{s,t}) \leq 4$. If possible, let $D(T_{s,t}) = 4$. Then by Theorem 7, $T_{s,t}$ has a $D$-Partition $\pi = \{V_1, V_2, V_3, V_4\}$ with a sink set. Without loss of generality, let $V_4$ be the sink set in $\pi$, then vertices belonging to $V_4$ has degree at least three. Since there exists only one vertex of degree three in $T_{s,t}$, $V_4$ is a singleton set. It can be observed that the graph induced by $T_{s,t} - V_4$ is a disconnected graph having two components which are paths
For any graph $G$, we know that $Tr(G) \leq D(G)$. It remains to show that for any unicyclic graph $G$, $D(G) = Tr(G)$. Assuming that $G$ is a unicyclic graph for which $Tr(G) < D(G)$, let $G_0$ be the unicyclic graph with least number of vertices such that $Tr(G_0) < D(G_0) = k$. The graph $G_0$ is not a cycle since the upper domatic number of a cycle coincides with its transitivity. Therefore, $G_0$ has at least one leaf vertex. Let $\pi = \{V_1, V_2, \ldots , V_k\}$ be a $D$-partition of $G_0$. Without loss of generality, assume that $V_1$ contains a leaf vertex.

Claim 1: $V_1$ is a source set.

If $V_1$ is not a source set, then by Observation 4 there exists a set $V_j \in \pi$ that dominates $V_1$. Let $v \in V_1$ be the leaf vertex. Since $V_j$ dominates $V_1$, the graph obtained from $G_0$ by the removal of $v$ is still a unicyclic graph with the same upper domatic number as $G_0$, which contradicts the assumption that $G_0$ is a unicyclic graph with the least number of vertices such that $Tr(G_0) < D(G_0)$.

Now consider the graph $G_0'$ obtained from $G_0$ by removing the source set $V_1$. Then $G_0'$ is either a connected graph that is acyclic or unicyclic, or disconnected graph where the components are trees or trees and a unicyclic graph.

Claim 2: The subgraph $G_0'$ has equal transitivity and upper domatic number.

The inequality $Tr(G_0') \leq D(G_0')$ follows immediately. If $G_0'$ is acyclic, then by Theorem 3, $D(G_0') = Tr(G_0')$. On the other hand, if $G_0'$ is unicyclic, then it follows by the assumption that $D(G_0') = Tr(G_0')$. If $G_0'$ is a disconnected graph with unicyclic and acyclic components say $G_1, G_2, \ldots G_r$, by Theorem 8, $D(G_0') \leq \max\{D(G_i) \mid 1 \leq i \leq r\}$. But for $1 \leq i \leq r$, $D(G_i) = Tr(G_i)$ and by Theorem 9, $Tr(G_0') = \max\{Tr(G_i) \mid 1 \leq i \leq r\}$. Therefore, $D(G_0') \leq \max\{D(G_i) \mid 1 \leq i \leq r\} = \max\{Tr(G_i) \mid 1 \leq i \leq r\} = Tr(G_0')$.

Thus proving the claim.

By Observation 6, $D(G_0') \geq D(G_0) - 1$ and $V_1$ being a source set in the upper domatic partition $\pi$, we can conclude that,

$$Tr(G_0) \geq Tr(G_0') + 1 = D(G_0') + 1 \geq D(G_0).$$
Thus, $D(G_0) = Tr(G_0)$, which is a contradiction to the assumption that $Tr(G_0) < D(G_0)$. 

A graph with at most one cycle in each of its components is called a pseudoforest. Theorem 15 can be extended for pseudoforests as well.

**Theorem 16.** If $G$ is a pseudoforest, then $D(G) = Tr(G)$.

**Proof.** Consider a pseudoforest $G$ having finite number of components, say $r$. Since each component of $G$ is either acyclic or unicyclic, by Theorem 3 and Theorem 15, $D(G_i) = Tr(G_i)$ for each component $G_i$ of $G$. Hence, $D(G) = \max\{D(G_i)\} = Tr(G)$, but $D(G) \geq Tr(G)$. Therefore, $D(G) = Tr(G)$.

We close this section by characterising the unicyclic graphs with equal upper domatic number and domatic number.

**Theorem 17.** For any unicyclic graph $G$, $D(G) = d(G)$ if and only if $G$ is $C_{3k}$, where $k$ is a positive integer.

**Proof.** If $G = C_{3k}$, then $D(G) = d(G) = 3$. On the other hand, let $G$ be a unicyclic graph with $D(G) = d(G)$. Note that by Proposition 4, $D(G) \geq 3$ for any unicyclic graph $G$. But, by Theorem 1, $d(G) \leq 2$ for any graph $G$ with a leaf vertex. A unicyclic graph $G$ with $d(G) \geq 3$, does not have a leaf vertex, which implies that $G$ is a cycle. It is well known that $d(C_{3k+1}) = d(C_{3k+2}) = 2$ while $d(C_{3k}) = 3$ [1], where $k$ is a positive integer. Hence, $G$ is $C_{3k}$, if $D(G) = d(G)$.

4. Powers of graphs

The $k^{th}$ power of a graph $G$ is the graph $G^k$, with vertex set $V(G^k) = V(G)$ and two vertices $u, v \in V(G^k)$ are adjacent if and only if the distance between $u$ and $v$ is at most $k$ in $G$. It is to be noted that for the graph $G^k$, the value of $k$ is at most the diameter of $G$. Moreover, for any graph $G$ of order $n$, $G^k = K_n$, when $k$ coincides with the diameter of $G$. In this section, we discuss the upper domatic number of powers of paths and cycles. The $k^{th}$ powers of a path on $n$ vertices and a cycle on $n$ vertices are denoted as $P^k_n$ and $C^k_n$ respectively. The following properties which follow from the definition are required for further discussion.

**Proposition 5.** For the graphs $P^k_n$ and $C^k_n$,

1. $P^k_n$ is an induced subgraph of $P^k_{n+1}$,
2. $P^k_n$ is a subgraph (not induced) of $C^k_n$.

3. $\Delta(P^k_n) = \Delta(C^k_n) = 2k$,

4. $\omega(P^k_n) = k + 1$,

5. $\omega(C^k_n) = k + 1$, $k < \lfloor \frac{n}{2} \rfloor$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{graphs.png}
\caption{Graphs $C^3_8$ and $P^3_8$.}
\end{figure}

**Theorem 18.** The upper domatic number of the $k^{th}$ power of path $P_n$ is

$$D(P^k_n) = \begin{cases} 
  k + 1, & \text{if } n = k + 1, \\
  l + k + 1, & \text{if } k + 2 \leq n \leq 3k \text{ where } l = \left\lfloor \frac{n-k-1}{2} \right\rfloor, \\
  2k + 1, & \text{if } n \geq 3k + 1.
\end{cases}$$

**Proof.** Consider the $k^{th}$ power of a path $P_n$, $k \leq n-1$ with $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, where $v_iv_{i+1} \in E(P_n)$, for $i = 1, 2, \ldots, n-1$. For different values of $n$, we consider the following cases.

**Case 1:** $n = k + 1$

The $k^{th}$ power of path on $k + 1$ vertices is a complete graph of order $k + 1$. Thus, Theorem 3 implies $D(P^k_{k+1}) = k + 1$.

**Case 2:** $k + 2 \leq n \leq 3k$

When $k + 2 \leq n \leq 3k$, the vertices $v_{l+1}, v_{l+2}, \ldots, v_{l+k+1}$ where $l = \left\lfloor \frac{n-k-1}{2} \right\rfloor$ of $P^k_n$ induces a clique of order $k + 1$. Now partition the vertex set of $P^k_n$ into $l + k + 1$ sets in the following manner.

$$V_i = \begin{cases} 
  \{v_{l+i}\}, & \text{if } 1 \leq i \leq k + 1, \\
  \{v_{i-k-1}, v_{i+k+1}\}, & \text{if } k + 2 \leq i \leq l + k, \\
  \{v_l, v_{2l+k+1}\}, & \text{if } i = l + k + 1 \text{ and } n - k - 1 \text{ is even}, \\
  \{v_l, v_{2l+k+1}, v_n\}, & \text{if } i = l + k + 1 \text{ and } n - k - 1 \text{ is odd}.
\end{cases}$$
We claim that the partition $\pi = \{V_1, V_2, \ldots, V_{l+k+1}\}$ is an upper domatic partition. Note that the vertices $v_1, v_2, \ldots, v_l$ induces $K_l$, the vertices $v_{l+1}, v_{l+2}, \ldots, v_{l+k+1}$ induces $K_{k+1}$ and the vertices $v_{l+k+2}, v_{l+k+3}, \ldots, v_n$ induces $K_l$ or $K_{l+1}$ depending on the parity of $n - k - 1$. Since the sets $V_1, V_2, \ldots, V_{l+k+1}$ contains exactly one vertex from $\{v_1, v_2, \ldots, v_{l+k+1}\}$, the sets $V_1, V_2, \ldots, V_{l+k+1}$ dominate mutually. Further the sets $V_{l+k+2}, V_{l+k+3}, \ldots, V_{l+k+1}$ also dominate each other and each vertex of $V_1 \cup V_2 \cup \ldots \cup V_{l+k+1}$ is adjacent to at least one vertex from each of the sets $V_{l+k+2}, V_{l+k+3}, \ldots, V_{l+k+1}$. Therefore, the sets $V_{l+k+2}, V_{l+k+3}, \ldots, V_{l+k+1}$ are dominating sets and the partition $\pi$ is an upper domatic partition. Thus, $D(P_n^k) \geq l + k + 1$.

If $D(P_n^k) > l + k + 1$, there exist an upper domatic partition of $l + k + 2$ sets. By pigeonhole principle, in such a partition there should be at least $k + 2$ singleton sets, which contradicts the fact that clique number of $P_n^k$ is $k + 1$. Thus, $D(P_n^k) = l + k + 1$.

**Case 3:** $n = 3k + 1$

For $P_n^{3k+1}$, the partition considered in Case 2 provides an upper domatic partition of order $l + k + 1$. Hence, $D(P_n^{3k+1}) \geq l + k + 1$ and by substituting for $l = \left\lfloor \frac{n-k-1}{2} \right\rfloor$ and $n = 3k + 1$, we get $D(P_n^{3k+1}) \geq 2k + 1$. Since $\Delta(P_n^k) = 2k$, by Theorem 2 $D(P_n^k) \leq 2k + 1$, thus proving $D(P_n^k) = 2k + 1$.

**Case 4:** $n > 3k + 1$

For $n > 3k + 1$, $P_n^{3k+1}$ is a subgraph of $P_n^k$ having a $D$-partition of order $2k+1$ with a source set. Therefore, by Proposition 3 and Theorem 2, $D(P_n^k) = 2k + 1$.

**Theorem 19.** For the graph $C_n^k$,

$$D(C_n^k) = \begin{cases} n, & \text{if } 2k \leq n \leq 2k + 1, \\ l + k + 1, & \text{if } 2k + 1 < n \leq 3k \quad \text{where } l = \left\lfloor \frac{n-k-1}{2} \right\rfloor, \\ 2k + 1, & \text{if } n \geq 3k + 1. \end{cases}$$

**Proof.** We consider the following cases for different values of $n$.

**Case 1:** $2k \leq n \leq 2k + 1$

Since $C_{2k}^k$ is an upper domatic partition of $C_{2k}^k$, for $2k \leq n \leq 2k + 1$, by Theorem 3, $D(C_n^k) = n$.

**Case 2:** $2k + 1 < n \leq 3k$

Since $P_n^k$ is a subgraph of $C_n^k$, for $2k + 1 < n \leq 3k$, by Proposition 3 and Theorem 18, $D(C_n^k) \geq l + k + 1$. But by pigeonhole principle, if $D(C_n^k) > l + k + 1$, such an upper domatic partition will contain at least $k + 2$ singleton sets which contradicts the fact that the clique number of $C_n^k$ is $k + 1$.

**Case 3:** $n \geq 3k + 1$

By Theorem 18, $D(P_n^k) = 2k + 1$, for $n \geq 3k + 1$ and $P_n^k$ is a subgraph of $C_n^k$, therefore $D(C_n^k) \geq 2k + 1$. However, Theorem 2 implies that $D(C_n^k) \leq 2k + 1$. Thus, $D(C_n^k) = 2k + 1$.

A graph is said to be upper domatically full if $D(G) = \Delta(G) + 1$.

**Corollary 4.** For $n \geq 3k + 1$, the power graphs $P_n^k$ and $C_n^k$ are upper domatically full.
Proof. Since $\Delta(P^k_n) = \Delta(C^k_n) = 2k$, it follows from Theorems 18 and 19 that both $P^k_n$ and $C^k_n$ are upper domatically full whenever $n \geq 3k + 1$. 

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