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Total Roman domination subdivision number in graphs

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Abstract: A *Roman dominating function* on a graph G is a function $f : V(G) \to \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. A *total Roman dominating function* is a Roman dominating function with the additional property that the subgraph of G induced by the set of all vertices of positive weight has no isolated vertices. The weight of a total Roman dominating function f is the value $\Sigma_{u\in V(G)} f(u)$. The *total Roman domination number* of G, $\gamma_{tR}(G)$, is the minimum weight of a total Roman dominating function on G. The *total Roman domination subdivision number* $\mathrm{sd}_{\gamma_{tR}}(G)$ of a graph G is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the total Roman domination number. In this paper, we initiate the study of total Roman domination subdivision number in graphs and we present sharp bounds for this parameter.

Keywords: Total Roman domination number, total Roman domination subdivision number

AMS Subject classification: 05C69

1. Introduction

Throughout this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V, E, respectively). For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. Similarly, the *open neighborhood* of a set $S \subseteq V(G)$ is the set $N(S) = \cup_{v\in S} N(v)$ and the *closed neighborhood* of S is $N[S] = N(S) \cup S$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *distance* between two vertices u and v is the length of a shortest path joining them. We denote by $N_2(v)$ the set of vertices at distance 2 from the vertex v and put $d_2(v) = |N_2(v)|$ and $\delta_2(G) = \min\{d_2(v) \mid v \in V(G)\}$. For a set S of vertices and a vertex $v \in S$, the *private neighborhood* of v with respect to S, $\text{pn}(v, S)$, is defined by $\text{pn}(v, S) = N[v] - N(S - \{v\})$.

A subset S of vertices of G is a *dominating (total dominating) set* if $N[S] = V$ ($N(S) = V$). The *domination (total domination) number* $\gamma(G)$ ($\gamma_t(G)$) is the minimum cardinality of a (total) dominating set of G. A (total) dominating set with cardinality $\gamma(G)$ ($\gamma_t(G)$) is called a $\gamma(G)$-set ($\gamma_t(G)$-set). The domination and its variations have been attracted considerable attention and surveyed in three books [19, 20, 23].

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an RDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G. Roman domination was introduced by Cockayne et al. in [9] and was inspired by the work of ReVelle and Rosing [26] and Stewart [27]. A Roman dominating function $f : V(G) \longrightarrow \{0, 1, 2\}$ may be represented by the ordered triple (V_0^f, V_1^f, V_2^f) of V where $V_i^f = \{v \in V(G) \mid f(v) = i\}$ for $i \in \{0, 1, 2\}$.

A *total Roman dominating function* of a graph G with no isolated vertex, abbreviated TRD-function (TRDF), is a Roman dominating function f on G with the additional property that the subgraph of G induced by the set of all vertices of positive weight under f has no isolated vertex. The *total Roman domination number* $\gamma_{tR}(G)$ is the minimum weight of a TRD-function on G. A TRD-function with weight $\gamma_{tR}(G)$ in G is called a $\gamma_{tR}(G)$-function. The concept of total Roman domination in graphs was introduced by Liu and Chang [25] and has been studied in [1–4].

In application, network design for example, if a parameter $\mu(G)$ is important to study, then it is important to know the effect that modifications of G have on $\mu(G)$. For example, vertices can be deleted and edges can be deleted, added or subdivided. In network design, deleting a vertex or an edge may represent components failure. From the other perspective, networks can be made fault tolerant by providing redundant communication link (adding edges). The effects on the domination number of a graph, when G is modified by deleting a vertex or deleting or adding an edge, have been investigated extensively (see chapter 7 of [20]).

Alternatively, one can consider how many modifications must take place before a parameter changes. Along these lines, Fink et al. [17], defined the bondage number of a graph to equal the minimum number of edges whose removal increases the domination number. On the other hand, Kok and Mynhardt [24] defined the reinforcement number of a graph to equal the minimum number of edges which must be added to a graph in order to decrease the domination number. Considering a different type of graph modification, Velammal [28] defined the *domination subdivision number* $sd_\gamma(G)$ to be the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the domination number. The domination subdivision number has been studied by several authors (see for instance [5, 18, 21]). A similar concepts related to connected domination were studied in [14], to total domination in [14–16, 22], to Roman domination in [7, 8], to rainbow domination in [10, 13], to 2-domination in [6], to weakly convex domination in [11] and to

convex domination in [12].
The *total Roman domination subdivision number* $\mathrm{sd}_{\gamma_{tR}}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the total Roman domination number of G. If $G_1, G_2, \ldots, G_s$ are the components of G, then $\gamma_{tR}(G) = \sum_{i=1}^{s} \gamma_{tR}(G_i)$ and $\mathrm{sd}_{\gamma_{tR}}(G) = \min\{\mathrm{sd}_{\gamma_{tR}}(G_i) \mid 1 \le i \le s\}$. Hence, it is sufficient to study $\mathrm{sd}_{\gamma_{tR}}(G)$ for connected graphs. Proposition 1 below shows that the total Roman domination number of graphs cannot decrease when an edge of graph is subdivided.
We make use of the following results in this paper.

Theorem A. *[2] For every graph G with no isolated vertex*

$$\gamma_t(G) \le \gamma_R(G) \le \gamma_{tR}(G) \le 2\gamma_t(G).$$

A graph G for which $\gamma_{tR}(G) = 2\gamma_t(G)$ is defined in [2] to be a *total Roman graph*. In their paper, they presented the following trivial necessary and sufficient condition for a graph to be a total Roman graph.

Theorem B. *[2] Let G be a graph without isolated vertices. Then, G is a total Roman graph if and only if there exists a $\gamma_{tR}(G)$-function $f = (V_0, V_1, V_2)$ such that $V_1 = \emptyset$.*

Theorem C. *[2] For any integer $n \ge 3$, $\gamma_{tR}(P_n) = \gamma_{tR}(C_n) = n$.*

As a consequence of Theorem C, we have:

Corollary 1. For any integer $n \ge 3$, $\mathrm{sd}_{\gamma_{tR}}(P_n) = \mathrm{sd}_{\gamma_{tR}}(C_n) = 1$.

Theorem D. *[1] For any graph G of order $n \ge 3$, $\gamma_{tR}(G) = 3$ if and only if $\Delta(G) = n-1$.*

Theorem E. *[1] For any graph G of order $n \ge 4$, $\gamma_{tR}(G) = 4$ if and only if $G = 2K_2$, $\Delta(G) = n-2$ or there are two adjacent vertices $u, v \in V(G)$ such that $N[u] \cup N[v] = V(G)$.*

Next result shows that subdividing an edge does not decrease the total Roman domination number.

Proposition 1. Let G be a simple connected graph of order $n \ge 3$ and $e = uv \in E(G)$. If G' is obtained from G by subdivision the edge e, then $\gamma_{tR}(G') \ge \gamma_{tR}(G)$.

Proof. Let x be the subdivision vertex and let f be a $\gamma_{tR}(G')$-function. Since f is a TRDF on G', we have that $f(u) + f(v) \ge 1$. With loss of generality, we suppose that v has positive weight under f. Let $g : V(G) \longrightarrow \{0, 1, 2\}$ be a function defined by $g(u) = \min\{2, f(u) + f(v)\}$ and $g(z) = f(z)$ whenever $z \in V(G) \setminus \{u\}$. Notice that

g is a TRDF on G and $\omega(g) \leq \omega(f)$. Hence $\gamma_{tR}(G') \geq \gamma_{tR}(G)$, which completes the proof. □

Observation 1. *[2] Let G be a connected graph of order at least three and let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_{tR}(G)$-function. Then the following assertions hold.*

1. *$|V_2^f| \leq |V_0^f|$.*
2. *If x is a leaf and y a support vertex in G, then $x \notin V_2^f$, $y \notin V_0^f$ and $f(x) + f(y) \geq 2$.*
3. *If z has at least three leaf-neighbors, then $f(z) = 2$ and at most one leaf-neighbor of z belongs to V_1^f.*

2. Sufficient conditions for small total Roman domination subdivision number

In this section we present some sufficient conditions for graphs to have small total Roman domination subdivision number.

Proposition 2. If G contains a strong support vertex, then $\mathrm{sd}_{\gamma_{tR}}(G) = 1$.

Proof. Let w be a strong support vertex of G and let u, v be two leaves adjacent to w. Suppose G' is the graph obtained from G by subdivision the edge uw with vertex x and let f be a $\gamma_{tR}(G')$-function. By Observation 1 we have that $f(w) \neq 0, f(u)+f(x) \geq 2$ and $f(w) + f(v) \geq 2$. Define $g : V(G) \longrightarrow \{0, 1, 2\}$ by $g(w) = 2, g(u) = 1, g(v) = 0$ and $g(z) = f(z)$ for each $z \in V(G) \setminus \{u, v, w\}$. Observe that g is a TRDF on G with $\omega(g) < \omega(f)$. Hence $\mathrm{sd}_{\gamma_{tR}}(G) = 1$. □

Proposition 3. Let G be a connected graph of order $n \geq 3$. If $\gamma_{tR}(G) = 3$, then $\mathrm{sd}_{\gamma_{tR}}(G) = 1$.

Proof. Let $e = uv \in E(G)$ and let $G^{'}$ be the graph obtained from G by subdivision the edge uv with vertex x. Then $n(G') = n + 1$ and $\Delta(G') \leq n(G') - 2$. It follows from Theorem D that $\gamma_{tR}(G') \geq 4$ and so $\mathrm{sd}_{\gamma_{tR}}(G) = 1$. □

By Theorem D and Proposition 3 we have the next results.

Corollary 2. For any graph G of order $n \geq 3$ with $\Delta(G) = n - 1$, $\mathrm{sd}_{\gamma_{tR}}(G) = 1$.

Proposition 4. Let G be a connected graph of order $n \geq 5$. If $\gamma_{tR}(G) = 4$, then

$$\mathrm{sd}_{\gamma_{tR}}(G) \leq 2.$$

Furthermore, this bound is sharp for the graph illustrated in Figure 1.

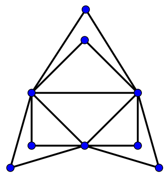

Figure 1. **Graph G with $\gamma_{tR}(G) = 4$ and $\mathrm{sd}_{\gamma_{tR}}(G) = 2$**

Proof. Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_{tR}(G)$-function such that $|V_2^f|$ is maximum. We claim that $V_1^f = \emptyset$. Suppose that $V_1^f \neq \emptyset$. Then we must have $|V_1^f| = 2$ and $|V_2^f| = 1$. Let $u, v \in V_1^f$ and $w \in V_2^f$. Since f is a TRDF of G, w is adjacent to any vertex in V_0 and to at least one vertex in $\{u, v\}$. We conclude from Theorem D and the assumption that w is adjacent to exactly one vertex in $\{u, v\}$. Suppose without loss of generality that $vw \in E(G)$. This implies that $uv \in E(G)$. Define $g : V(G) \longrightarrow \{0, 1, 2\}$ by $g(v) = 0, g(u) = 2$ and $g(z) = f(z)$ for each $z \in V(G) \setminus \{u, v\}$. Obviously, g is a $\gamma_{tR}(G)$-function contradicting the choice of f. Thus $V_1 = \emptyset$ and $|V_2| = 2$. Let $V_2^f = \{x, y\}$. By definition $xy \in E(G)$. Then each of x and y have at least one private neighbor in V_0^f, otherwise $\gamma_{tR}(G) = 3$, a contradiction.

Let x', y' be two private neighbors of x, y in V_0^f, respectively and let G' be a graph obtained from G by subdividing the edges xx' and yy' with vertices x_1, y_1, respectively. We claim that $\gamma_{tR}(G') \geq 5$. Suppose, to the contrary, that $\gamma_{tR}(G') = 4$. Assume that $g = (V_0^g, V_1^g, V_2^g)$ is a $\gamma_{tR}(G')$-function such that $|V_2^g|$ is as large as possible. As above we may assume that $|V_2^g| = 2$. To total Roman dominate the vertices x_1, y_1, we must have $g(x) + g(x') + g(x_1) \geq 2$ and $g(y) + g(y') + g(y_1) \geq 2$. It follows from $\gamma_{tR}(G') = 4$ that $g(x) + g(x') + g(x_1) = 2$ and $g(y) + g(y') + g(y_1) = 2$. This implies that $g(x_1) = g(y_1) = 0$.

Now to Roman dominate x_1 and using the fact $g(x) + g(x') + g(x_1) = 2$, we must have $g(x) = 2$ and $g(x') = 0$ or $g(x) = 0$ and $g(x') = 2$. Similarly, $g(y) = 2$ and $g(y') = 0$ or $g(y) = 0$ and $g(y') = 2$. Since g is a TRDF of G', we must have $g(x) = g(y) = 2$ or $g(x') = g(y') = 2$. If $g(x) = g(y) = 2$, then x' is not Roman dominated and if $g(x') = g(y') = 2$, then x is not Roman dominated which is a contradiction. Thus $\gamma_{tR}(G') \geq 5$, and this implies that $\mathrm{sd}_{\gamma_{tR}}(G) \leq 2$. □

Next result is an immediate consequence of Theorem E and Proposition 4.

Corollary 3. If G is a connected graph of order $n \geq 4$ with $\Delta(G) = n - 2$, then $\mathrm{sd}_{\gamma_{tR}}(G) \leq 2$. Furthermore, this bound is sharp for the graph illustrated in Figure 2.

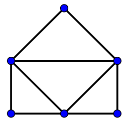

Figure 2. **Graph G with $\Delta(G) = n(G) - 2$ and $\mathrm{sd}_{\gamma_{tR}}(G) = 2$**

Proposition 5. Let G be a simple connected graph of order $n \geq 3$. If G has a vertex $v \in V(G)$ which is contained in a triangle uvw such that $N(u) \cup N(w) \subseteq N[v]$, then $\mathrm{sd}_{\gamma_{tR}}(G) \leq 3$.

Proof. Let G' be obtained from G by subdividing the edges vu, vw, uw with vertices x, y, z, respectively, and let f be a $\gamma_{tR}(G')$-function. It is easy to see that $f(x) + f(y) + f(z) + f(u) + f(v) + f(w) \geq 4$. Define $g : V(G) \longrightarrow \{0, 1, 2\}$ by $g(v) = 2, g(u) = 1, g(w) = 0$ and $g(a) = f(a)$ for each $a \in V(G) - \{u, v, w\}$. Obviously, g is a TRDF of G with $\omega(g) < \omega(f)$ and the proof is complete. $\square$

3. Bounds

In this section we present some sharp upper bounds on $\mathrm{sd}_{\gamma_{tR}}(G)$.

Theorem 2. Let G be a connected graph. If $v \in V(G)$ has degree at least two, then

$$\mathrm{sd}_{\gamma_{tR}}(G) \leq \deg(v).$$

Furthermore, this bound is sharp for the graphs illustrated in Figures 1 and 2.

Proof. Let $N(v) = \{v_1, v_2, \ldots, v_d\}$ where $d = \deg(v)$. Let G' be the graph obtained from G by subdividing the edges $vv_1, vv_2, \ldots, vv_d$ with vertices $x_1, x_2, \ldots, x_d$, respectively, and let f be a $\gamma_{tR}(G')$-function. Then $f(x_i) \geq 1$ for some $1 \leq i \leq d$, say $i = 1$, and so $f(v) \geq 1$ or $f(v_1) \geq 1$. Suppose G'' is obtained from G by subdividing the edges $vv_2, vv_3, \ldots, vv_d$ with vertices $x_2, x_3, \ldots, x_d$. If $f(v_1) \geq 1$ and $f(v) \geq 1$, then the function f, restricted to G'', is a TRDF of G'' which implies that $\gamma_{tR}(G) \leq \omega(f|_{G''}) < \omega(f)$ by Proposition 1. We now consider the cases depending on the value $f(v)$.

Case 1. $f(v) = 0$.
Then $f(v_1) \geq 1$ and $f(x_i) = 2$ for some $i \in \{1, \ldots, d\}$. (Note that if $f(x_i) \geq 1$,

then $f(v_i) \geq 1$ for each $i \in \{2, \dots, d\}$). Define the function $g : V(G) \longrightarrow \{0, 1, 2\}$ by $g(v) = 1$ and $g(x) = f(x)$ for each $x \in V(G) - \{v\}$. Clearly, g is a TRDF of G with $\omega(g) < \omega(f)$ and so $\gamma_{tR}(G) \leq \omega(g) < \omega(f)$.

Case 2. $f(v) \geq 1$.
If $f(v_1) \geq 1$, then the function f, restricted to G'', is a TRDF of G'' which implies that $\gamma_{tR}(G) \leq \omega(f|_{G''}) < \omega(f)$ by Proposition 1. Assume that $f(v_1) = 0$. If $f(x_1) = 2$, then define $g : V(G'') \longrightarrow \{0, 1, 2\}$ by $g(v_1) = 1$, and $g(z) = f(z)$ for $z \in V(G'') \setminus \{v_1\}$. Clearly, g is a TRDF of G'' with $\omega(g) < \omega(f)$ and hence $\gamma_{tR}(G) \leq \omega(g) < \omega(f)$. Assume $f(x_1) = 1$. If $f(x_i) = 2$ for some $i \in \{2, \dots, d\}$, then the function $g : V(G) \longrightarrow \{0, 1, 2\}$ define by $g(v) = 2, g(v_1) = 1$, and $g(z) = f(z)$ otherwise, is a TRDF of G of weight less than $\omega(f)$ and so $\gamma_{tR}(G) < \omega(f)$. Let $f(x_i) \leq 1$ for each $i \in \{2, 3, \dots, d\}$. If $f(x_i) = 1$ for some $i \in \{2, \dots, d\}$, then the function $g : V(G) \longrightarrow \{0, 1, 2\}$ define by $g(v_1) = 1$, and $g(z) = f(z)$ for $z \in V(G) \setminus \{v_1\}$, is a TRDF of G with $\omega(g) < \omega(f)$. Now we assume that $f(x_i) = 0$ for $i \in \{2, 3, \dots, d\}$. If $f(v) = 2$, then the function $g : V(G) \longrightarrow \{0, 1, 2\}$ define by $g(v) = g(v_1) = 1$ and $g(z) = f(z)$ for $z \in V(G) \setminus \{v, v_1\}$, is a TRDF of G with $\omega(g) < \omega(f)$ yielding $\gamma_{tR}(G) < \omega(f)$. Let $f(v) = 1$. Then $f(v_i) = 2$ for each $i \in \{2, \dots, d\}$ and clearly $f|_G$ is a TRDF of G of weight less than $\omega(f)$ implying that $\gamma_{tR}(G) < \omega(f)$. Thus $\mathrm{sd}_{\gamma_{tR}}(G) \leq \deg(v)$ and the proof is complete. $\square$

Next result is an immediate result of Proposition 2 and Theorem 2.

Corollary 4. For every tree T of order at least 3,

$$\mathrm{sd}_{\gamma_{tR}}(T) \leq 2.$$

Furthermore, this bound is sharp for the trees illustrated in Figure 3.

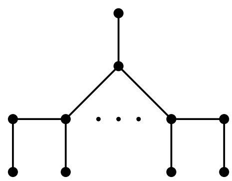

Figure 3. **Trees with total Roman domination subdivision number two**

Another consequence of Theorem 2 is that $\mathrm{sd}_{\gamma_{tR}}(G)$ is defined for every connected graph G of order $n \geq 3$. In addition:

Corollary 5. For every connected graph G with $\delta \geq 2$, $\mathrm{sd}_{\gamma_{tR}}(G) \leq \delta$.

The graphs in Figures 1 and 2 shows that the bound of Corollary 5 is sharp. It is well known that every planar graph contains at least one vertex of degree at most five. Thus, the following result is an immediate consequence of Corollary 5.

Corollary 6. For every planar graph G, $\mathrm{sd}_{\gamma_{tR}}(G) \leq 5$.

Corollary 7. For any connected graph G with adjacent vertices u and v, each of degree at least two,

$$\mathrm{sd}_{\gamma_{tR}}(G) \leq\mid N(u) \cup N(v) \mid -1.$$

In the above results, we essentially encountered graphs with small total Roman domination subdivision numbers. Next we show that the total Roman domination subdivision number of a graph can be arbitrarily large. The following graph was introduced by Haynes et al. in [22] to prove a similar result on $\mathrm{sd}_{\gamma_t}(G)$.

Theorem 3. For any integer $k \geq 4$, there exists a connected graph G with $\mathrm{sd}_{\gamma_{tR}}(G) = k$.

Proof. Let $X = \{1, 2, \ldots, 3(k-1)\}$ and let $\mathcal{Y} = \{Y \subset X \mid |Y| = k\}$. Thus $\mathcal{Y}$ consists of all k-elements of X, and so $\mid \mathcal{Y} \mid = \binom{3(k-1)}{k}$. Let G be the graph with vertex set $X \cup \mathcal{Y}$ and with edge set constructed as follows: add an edge joining every two distinct vertices of X and for each $x \in X$ and $Y \in \mathcal{Y}$, add an edge joining x and Y if and only if $x \in Y$. Then, G is a connected graph of order $n = \binom{3(k-1)}{k} + 3(k-1)$. The set X induces a clique in G, while the set $\mathcal{Y}$ is an independent set and each vertex of $\mathcal{Y}$ has degree k in G. By the proof of Theorems 13 in [22] and 17 in [7], we have $\gamma_t(G) = 2k-2$ and $\gamma_R(G) = 4k-5$, respectively. It follows from Theorem A that $4k-5 \leq \gamma_{tR}(G) \leq 4(k-1)$.
We claim that $\gamma_{tR}(G) = 4(k-1)$. Suppose, to the contrary, that $\gamma_{tR}(G) = 4k-5$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_{tR}(G)$-function. Since $\gamma_{tR}(G) = \gamma_R(G)$, f is a $\gamma_R(G)$-function and this implies that $V_2 \neq \emptyset$ and no edge of G joins a vertex in V_1 to a vertex in V_2. Moreover, $\gamma_{tR}(G) = 4k-5$ implies that $V_1 \neq \emptyset$. If $|X \cap V_1| \geq 1$, then we must have $V_2 \cap X = \emptyset$ because $G[X]$ is a clique. Hence $V_2 \subseteq \mathcal{Y}$. Since $\mathcal{Y}$ is an independent set, to totaly Roman dominate any vertex in $V_2 \cap \mathcal{Y}$ we must have an edge between $V_2 \cap \mathcal{Y}$ and $V_1 \cap X$ which is a contradiction. Thus $X \cap V_1 = \emptyset$. This implies that $V_1 \subseteq \mathcal{Y}$. Let $z \in V_1$. Then to totaly Roman dominate the vertex z, z must be adjacent to a vertex in $V_2 \cap X$ because $\mathcal{Y}$ is independent and $X \cap V_1 = \emptyset$, and this leads to a contradiction. Therefore $\gamma_{tR}(G) = 4(k-1)$.
Let $F = \{e_1, e_2, \ldots, e_{k-1}\}$ be an arbitrary subset of $k-1$ edges of G. Assume H is obtained from G by subdividing each edge in F. We show that $\gamma_{tR}(H) = \gamma_{tR}(G)$. By the proof of Theorem 13 [22], $\gamma_t(G) = \gamma_t(H) = 2(k-1)$. We deduce from Theorem A and Proposition 1 that $\gamma_{tR}(G) \leq \gamma_{tR}(H) \leq 4(k-1)$. Consequently, $\gamma_{tR}(H) = 4(k-1)$, whence $\mathrm{sd}_{\gamma_{tR}}(G) \geq k$.
By Theorem 2, we have $\mathrm{sd}_{\gamma_{tR}}(G) = k$. Note that since $\delta = k$, G is an example of equality in Corollary 5. $\square$

In the next result, we establish an upper bound on the total Roman domination number of a connected graph in terms of its order and minimum degree.

Proposition 6. For any connected graph G of order $n \geq 3$,

$$\gamma_{tR}(G) \leq n - \delta + 2.$$

Furthermore, this bound is sharp for complete graphs.

Proof. Suppose v is a vertex with minimum degree $\delta(G)$ and define $f : V(G) \longrightarrow \{0,1,2\}$ by $f(v) = 2, f(w) = 1$ for some $w \in N(v), f(u) = 0$ for each $u \in N(v) \setminus \{w\}$ and $f(u) = 1$ for each $u \in V(G) \setminus N[v]$. Since $N(u) \cap ((V(G) \setminus N(v)) \cup \{w\}) \neq \emptyset$ for any $u \in V(G) \setminus N[v]$, f is a TRDF of G. Hence $\gamma_{tR}(G) \leq n - \delta + 2$ and the proof is complete. □

The following corollary is an immediate consequence of Corollary 5 and Proposition 6.

Corollary 8. For any connected graph G with $\delta \geq 2$,

$$\mathrm{sd}_{\gamma_{tR}}(G) \leq n - \gamma_{tR}(G) + 2.$$

Ahangar et al. in [1] proved that for any connected graph G of order $n \geq 3$, $\gamma_{tR}(G) \geq \lceil \frac{2n}{\Delta} \rceil$. Using this bound and Corollary 8 we obtain the next result.

Corollary 9. For any connected graph G with $n \geq 3$,

$$\mathrm{sd}_{\gamma_{tR}}(G) \leq n - \lceil \frac{2n}{\Delta} \rceil + 2.$$

In the sequel, we present an upper bounds on the total Roman domination subdivision number in terms of d_2 for certain graphs.

Proposition 7. Let G be a connected graph of order $n \geq 3$. If $v \in V(G)$ is a support vertex and has a neighbor u with $N(u) \setminus N[v] \neq \emptyset$, then

$$\mathrm{sd}_{\gamma_{tR}}(G) \leq 2 + |N(u) - N[v]|.$$

In particular, $\mathrm{sd}_{\gamma_{tR}}(G) \leq 2 + d_2(v)$.

Proof. Assume $N(v) = \{u = v_1, v_2, \ldots, v_{\deg(v)}\}$ where $\deg(v_2) = 1$, and $N(u) \setminus N[v] = \{y_1, y_2, \ldots, y_k\}$. Let G_1 be the graph obtained from G by subdividing the edge vv_i with a vertex x_i for $i = 1, 2$, and the edge uy_j with a vertex z_j for $1 \leq j \leq k$. Let f be a $\gamma_{tR}(G')$-function. Then $f(v_2) + f(x_2) \geq 2$ and $f(v) + f(x_1) + f(u) \geq 2$. If $\sum_{j=1}^{k} f(z_j) + f(x_1) + f(v) + f(u) \geq 3$, then the function $g : V(G) \longrightarrow \{0, 1, 2\}$ defined by $g(v) = 2, g(u) = 2, g(v_2) = 0$ and $g(x) = f(x)$ for each $x \in V(G) \setminus \{v, v_2, u\}$ is a TRDF of G of weight less than $\omega(f) = \gamma_{tR}(G)$. Let $\sum_{j=1}^{k} f(z_j) + f(x_1) + f(v) + f(u) = 2$. Then $\sum_{j=1}^{k} f(z_j) = 0$. It is easy see that the function $g : V(G) \longrightarrow \{0, 1, 2\}$ defined by $g(v) = 2, g(u) = 1, g(v_2) = 0$ and $g(x) = f(x)$ for each $x \in V(G) \setminus \{v, v_2, u\}$ is a TRDF of G with $\omega(g) < \omega(f)$. Thus $\mathrm{sd}_{\gamma_{tR}}(G) \leq 2 + |N(u) - N[v]|$ and the proof is complete. □

Proposition 8. Let G be a connected graph of order $n \geq 3$. If $v \in V(G)$ is a support vertex, then

$$\mathrm{sd}_{\gamma_{tR}}(G) \leq 2 + d_2(v).$$

Proof. Assume that $N(v) = \{v_1, v_2, \ldots, v_{\deg(v)}\}$ where $\deg(v_1) = 1$. If v has a neighbor u such that $N(u) \setminus N[v] \neq \emptyset$, the it follows from Proposition 8 that $\mathrm{sd}_{\gamma_{tR}}(G) \leq 2 + |N(u) - N[v]| \leq 2 + d_2(v)$. Let $N(u) \subseteq N[v]$ for each $u \in N(v)$. If G is a star, then clearly $\mathrm{sd}_{\gamma_{tR}}(G) = 1$. Thus we may assume that $v_2v_3 \in E(G)$. Let G_1 be obtained from G by subdividing the edges vv_1 and v_2v_3, with vertices x, y respectively. Let f be a $\gamma_{tR}(G)$- function. Then $f(v_1) + f(x) \geq 2$ and $f(v_2) + f(y) + f(v_3) \geq 2$. Define $g : V(G) \longrightarrow \{0, 1, 2\}$ by $g(v) = 2, g(v_2) = 1, g(v_3) = g(v_1) = 0$ and $g(x) = f(x)$ for each $x \in V(G) \setminus \{v, v_1, v_2, v_3\}$. Since $N(v_3) \subseteq N(v)$, g is a TRDF of G with $\omega(g) < \omega(f)$ and hence $\mathrm{sd}_{\gamma_{tR}}(G) \leq 2$. This completes the proof. □

Proposition 9. Let G be a simple connected graph of order $n \geq 3$. If G has a vertex $v \in V(G)$ which is contained in a triangle vuw such that $N(u) \subseteq N[v]$ and $N(w) \setminus N[v] \neq \emptyset$, then

$$\mathrm{sd}_{\gamma_{tR}}(G) \leq 3 + |N(w) \setminus N[v]|.$$

In particular, $\mathrm{sd}_{\gamma_{tR}}(G) \leq 3 + d_2(v)$.

Proof. Let $N(w) \setminus N[v] = \{w_1, w_2, \ldots, w_k\}$ and let G' be a graph obtained from G by subdividing the edges vu, vw, uw with vertices x, y, z, respectively, and, the edge ww_i with the vertex z_i for each $1 \leq i \leq k$. Suppose $T = \{z_1, z_2, \ldots, z_k\}$.
Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_{tR}(G')$-function. Then $f(x) + f(y) + f(z) + f(u) + f(v) + f(w) \geq 4$. Define $g : V(G) \longrightarrow \{0, 1, 2\}$ by $g(v) = 2, g(u) = 0$,

$$g(w) = \begin{cases} 2 & \text{if } \sum_{z \in T} f(z) \geq 2 \\ 1 & \text{if } \sum_{z \in T} f(z) \leq 1 \end{cases}$$

and $g(a) = f(a)$ for each $a \in V(G) \setminus \{v, u, w\}$. It is easy to see that g is a TRDF of G with $\omega(g) < \omega(f)$ and so $\mathrm{sd}_{\gamma_{tR}}(G) \leq 3 + |N(w) \setminus N[v]|$. □

We conclude this section with two open problem.

Problem 1. Characterize all trees T attaining the bound of Corollary 4.

Problem 2. Is it true that, for any connected graph G with $\delta(G) \geq 2$, $\mathrm{sd}_{\gamma_{tR}(G)} \leq \delta_2(G) + 3$.

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