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Weak signed Roman k-domination in graphs

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Abstract: Let $k \geq 1$ be an integer, and let G be a finite and simple graph with vertex set $V(G)$. A weak signed Roman k-dominating function (WSRkDF) on a graph G is a function $f : V(G) \to \{-1, 1, 2\}$ satisfying the conditions that $\sum_{x \in N[v]} f(x) \geq k$ for each vertex $v \in V(G)$, where $N[v]$ is the closed neighborhood of v. The weight of a WSRkDF f is $w(f) = \sum_{v \in V(G)} f(v)$. The weak signed Roman k-domination number $\gamma^k_{wsR}(G)$ of G is the minimum weight of a WSRkDF on G. In this paper we initiate the study of the weak signed Roman k-domination number of graphs, and we present different bounds on $\gamma^k_{wsR}(G)$. In addition, we determine the weak signed Roman k-domination number of some classes of graphs. Some of our results are extensions of well-known properties of the signed Roman k-domination number $\gamma^k_{sR}(G)$, introduced and investigated by Henning and Volkmann [5] as well as Ahangar, Henning, Zhao, Löwenstein and Samodivkin [1] for the case $k = 1$.

Keywords: Weak signed Roman k-dominating function, weak signed Roman k-domination number, Signed Roman k-dominating function, Signed Roman k-domination number

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1. Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [4]. Specifically, let G be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the *order* and the *size* of the graph G, respectively. The *open neighborhood* of a vertex v is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d_G(v) = d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. For a set $X \subseteq V(G)$, its *open neighborhood* is the set $N_G(X) = N(X) = \bigcup_{v\in X} N(v)$, and its *closed neighborhood* is the set $N_G[X] = N[X] = N(X) \cup X$. The *complement* of a graph G is denoted by $\overline{G}$. For sets $A, B \subseteq V(G)$, we say that A *dominates* B if $B \subseteq N[A]$. A *leaf* of a graph G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. An edge incident with a leaf is called a *pendant edge*. The *star* $K_{1,t}$ has on vertex of degree t and t leaves. A *spider* is the graph formed by subdividing all edges of a star $K_{1,t}$. Let P_n, C_n and K_n be the path, cycle and complete graph of order n, and let $K_{p,p}$ be the complete bipartite graph of order $2p$.

All along this paper we will assume that k is a positive integer. In 1985, Fink and Jacobson [3] introduced the concept of k-dominating sets. A subset $D \in V(G)$ is a k dominating set if every vertex in $V(D) - D$ has at least k neighbors in D. The minimum cardinality of a k-dominating set is the *k-domination number*, denoted by $\gamma_k(G)$.
In this paper we continue the study of Roman dominating functions in graphs and digraphs. For a subset $S \subseteq V(G)$ of vertices of a graph G and a function $f\colon V(G) \longrightarrow \mathbb{R}$, we define $f(S) = \sum_{x\in S} f(x)$. For a vertex v, we denote $f(N[v])$ by $f[v]$ for notational convenience.

If $k \geq 1$ is an integer, then Henning and Volkmann [5] defined the *signed Roman k-dominating function* (SRkDF) on a graph G as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $f[v] \geq k$ for every $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$. The weight of an SRkDF f on a graph G is $\omega(f) = \sum_{v\in V(G)} f(v)$. The *signed Roman k-domination number* $\gamma^k_{sR}(G)$ of G is the minimum weight of an SRkDF on G. The special case $k = 1$ was introduced and investigated by Ahangar, Henning, Zhao, Löwenstein and Samodivkin [1]. Sheikholeslami and Volkmann [6] studied the signed Roman domination number in digraphs. A $\gamma^k_{sR}(G)$-function is a signed Roman k-dominating function on G of weight $\gamma^k_{sR}(G)$.
A *weak signed Roman k-dominating function* (WSRkDF) on a graph G is defined as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ having the property $f[v] \geq k$ for every $v \in V(G)$. The weight of a WSRkDF f on a graph G is $\omega(f) = \sum_{v\in V(G)} f(v)$. The *weak signed Roman k-domination number* $\gamma^k_{wsR}(G)$ of G is the minimum weight of a WSRkDF on G. The special case $k = 1$ was introduced and investigated by Volkmann [7]. A $\gamma^k_{wsR}(G)$-function is a weak signed Roman k-dominating function on G of weight $\gamma^k_{wsR}(G)$. For a WSRkDF f on G, let $V_i = V_i(f) = \{v \in V(G) : f(v) = i\}$ for $i = -1, 1, 2$. A weak signed Roman k-dominating function $f : V(G) \longrightarrow \{-1, 1, 2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(G)$.
The weak signed Roman k-domination number exists when $\delta \geq \frac{k}{2} - 1$. Therefore we assume in this paper that $\delta \geq \frac{k}{2} - 1$. The definitions lead to $\gamma^k_{wsR}(G) \leq \gamma^k_{sR}(G)$. Therefore each lower bound of $\gamma^k_{wsR}(G)$ is also a lower bound of $\gamma^k_{sR}(G)$ and each upper bound of $\gamma^k_{sR}(G)$ is also an upper bound of $\gamma^k_{wsR}(G)$.

Our purpose in this work is to initiate the study of the weak signed Roman k-

domination number. We present basic properties and sharp bounds on $\gamma^k_{wsR}(G)$. In particular, we show that many lower bounds on $\gamma^k_{sR}(G)$ are also valid for $\gamma^k_{wsR}(G)$. Some of our results are extensions of well-known properties of the signed Roman k-domination number and the weak signed Roman domination number $\gamma_{wsR}(G) = \gamma^1_{wsR}(G)$, given by Ahangar, Henning, Zhao, Löwenstein and Samodivkin [1], Amjadi, Nazari-Moghaddam, Sheikholeslami and Volkmann [2], Henning and Volkmann [5] and Volkmann [7].

2. Preliminary results

In this section we present basic properties of the weak signed Roman k-dominating functions and the weak signed Roman k-domination numbers.

Proposition 1. If $f = (V_{-1}, V_1, V_2)$ is a WSRkDF on a graph G of order n, then

(a) $|V_{-1}| + |V_1| + |V_2| = n$.

(b) $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$.

(c) $V_1 \cup V_2$ is a $\lceil \frac{k+1}{2} \rceil$-dominating set.

Proof. Since (a) and (b) are immediate, we only prove (c). If $|V_{-1}| = 0$ then $V_1 \cup V_2 = V(G)$ is a $\lceil \frac{k+1}{2} \rceil$-dominating set. Let now $|V_{-1}| \geq 1$ and let $v \in V_{-1}$ an arbitrary vertex. Assume that v has j neighbors in V_1 and q neighbors in V_2. The the condition $f[v] \geq k$ leads to $j + 2q - 1 \geq k$ and so $q \geq \frac{k+1-j}{2}$. This implies

$$j + q \geq j + \frac{k+1-j}{2} = \frac{k+1+j}{2} \geq \frac{k+1}{2}.$$

Therefore v has at least $j + q \geq \lceil \frac{k+1}{2} \rceil$ neighbors in $V_1 \cup V_2$. Since v was an arbitrary vertex in V_{-1}, we deduce that $V_1 \cup V_2$ is a $\lceil \frac{k+1}{2} \rceil$-dominating set. □

Corollary 1. If G is a graph of order n, then $\gamma^k_{wsR}(G) \geq 2\gamma_{\lceil \frac{k+1}{2} \rceil}(G) - n$.

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma^k_{wsR}(G)$-function. Then it follows from Proposition 1 that

$$\gamma^k_{wsR}(G) = |V_1| + 2|V_2| - |V_{-1}| = 2|V_1| + 3|V_2| - n \geq 2|V_1 \cup V_2| - n \geq 2\gamma_{\lceil \frac{k+1}{2} \rceil}(G) - n.$$

□

The graphs $\overline{K_n}$ and qK_2 show that Corollary 1 is sharp for $k = 1$ and $k = 2$. The proof of the next proposition is identically with the proof of Proposition 2 in [5] and is therefore omitted.

Proposition 2. Assume that $f = (V_{-1}, V_1, V_2)$ is a WSRkDF on a graph G of order n, $\Delta = \Delta(G)$ and $\delta = \delta(G)$. Then

(i) $(2\Delta + 2 - k)|V_2| + (\Delta + 1 - k)|V_1| \geq (\delta + k + 1)|V_{-1}|$.

(ii) $(2\Delta + \delta + 3)|V_2| + (\Delta + \delta + 2)|V_1| \geq (\delta + k + 1)n$.

(iii) $(\Delta + \delta + 2)\omega(f) \geq (\delta - \Delta + 2k)n + (\delta - \Delta)|V_2|$.

(iv) $\omega(f) \geq (\delta - 2\Delta + 2k - 1)n/(2\Delta + \delta + 3) + |V_2|$.

3. Bounds on the weak signed Roman k-domination number

We start with a general upper bound, and we characterize all extremal graphs.

Theorem 1. Let G be a graph of order n with $\delta(G) \geq \lceil \frac{k}{2} \rceil - 1$. Then $\gamma^k_{wsR}(G) \leq 2n$, with equality if and only if k is even, $\delta(G) = \frac{k}{2} - 1$, and each vertex of G is of minimum degree or adjacent to a vertex of minimum degeree.

Proof. Define the function $g : V(G) \longrightarrow \{-1, 1, 2\}$ by $g(x) = 2$ for each vertex $x \in V(G)$. Since $\delta(G) \geq \lceil \frac{k}{2} \rceil - 1$, the function g is a WSRkDF on G of weight $2n$ and thus $\gamma^k_{wsR}(G) \leq 2n$.
Now let k be even, $\delta(G) = \frac{k}{2} - 1$, and assume that each vertex of G is of minimum degree or adjacent to a vertex of minimum degeree. Let f be a WSRkDF on G, and let $x \in V(G)$ be an arbitray vertex. If $d(x) = \frac{k}{2} - 1$, then $f[x] \geq k$ implies that $f(x) = 2$. If x is not of minimum degree, then x is adjacent to a vertex w of minmum degree. The condition $f[w] \geq k$ implies $f(x) = 2$. Thus f is of weight $2n$, and we obtain $\gamma^k_{wsR}(G) = 2n$ in this case.
Conversely, assume that $\gamma^k_{wsR}(G) = 2n$. If $k = 2p + 1$ is odd, then $\delta(G) \geq p$. Define the function $h : V(G) \longrightarrow \{-1, 1, 2\}$ by $h(w) = 1$ for an arbitrary vertex w and $h(x) = 2$ for each vertex $x \in V(G) \setminus \{w\}$. Then

$$h[v] = \sum_{x \in N[v]} f(x) \geq 1 + 2\delta(G) \geq 1 + 2p = k$$

for each vertex $v \in V(G)$. Thus the function h is a WSRkDF on G of weight $2n - 1$, and we obtain the contradiction $\gamma^k_{wsR}(G) \leq 2n - 1$.
Let now k even, and assume that there exists a vertex w with $d(w) \geq \frac{k}{2}$ and $d(x) \geq \frac{k}{2}$ for each $x \in N(w)$. Define the function $h_1 : V(G) \longrightarrow \{-1, 1, 2\}$ by $h_1(w) = 1$ and $h_1(x) = 2$ for each vertex $x \in V(G) \setminus \{w\}$. Then $h_1[v] \geq k + 1$ for each $v \in N[w]$ and $h_1[x] \geq k$ for each $x \notin N[w]$. Hence the function h_1 is a WSRkDF on G of weight $2n - 1$, a contradiction to the assumption $\gamma^k_{wsR}(G) = 2n$. This completes the proof. □

The next result is an immediate corollary of Theorem 1.

Corollary 2. Let G be a graph of order n with $\delta(G) \geq \lceil \frac{k}{2} \rceil - 1$. Then $\gamma^k_{sR}(G) \leq 2n$, with equality if and only if k is even, $\delta(G) = \frac{k}{2} - 1$, and each vertex of G is of minimum degree or adjacent to a vertex of minimum degeree.

The next known corollary follows immediately from Corollary 2.

Corollary 3. ([2]) Let T be a tree of order n. Then $\gamma^4_{sR}(T) \leq 2n$, with equality if and only if every vertex of T is either a leaf or a support vetex.

Observation 2. *If G is a graph of order n with $\delta(G) \geq k-1$, then $\gamma^k_{wsR}(G) \leq \gamma^k_{sR}(G) \leq n$.*

Proof. Define the function $f : V(G) \longrightarrow \{-1, 1, 2\}$ by $f(x) = 1$ for each vertex $x \in V(G)$. Since $\delta(G) \geq k - 1$, the function f is an SRkDF on G of weight n and thus $\gamma^k_{wsR}(G) \leq \gamma^k_{sR}(G) \leq n$. □

As an application of Proposition 2 (iii), we obtain a lower bound on the weak signed Roman k-domination number for r-regular graphs.

Corollary 4. If G is an r-regular graph of order n with $r \geq \frac{k}{2} - 1$, then

$$\gamma^k_{wsR}(G) \geq \frac{kn}{r+1}.$$

Example 1. *If H is a $(k-1)$-regular graph of order n, then it follows from Corollary 4 that $\gamma^k_{wsR}(H) \geq n$ and thus $\gamma^k_{wsR}(H) = n$, according to Observation 2.*

Example 1 shows that Observation 2 and Corollary 4 are both sharp. The proof of the next observation is analogously to the proof of Proposition 3 in [7] and is therefore omitted.

Observation 3. *If G is a graph of order n with $\delta(G) \geq \frac{k}{2} - 1$, then*

$$\gamma^k_{wsR}(G) \geq k + 1 + \Delta(G) - n.$$

Let $n \geq k \geq 2$ be integers. Then it was shown in [5] that $\gamma^k_{sR}(K_n) = k$. This implies $\gamma^k_{wsR}(K_n) \leq \gamma^k_{sR}(K_n) = k$. According to Corollary 4, we deduce that $\gamma^k_{wsR}(K_n) \geq k$. Therefore we obtain $\gamma^k_{wsR}(K_n) = k$ for $n \geq k \geq 2$. This example shows that Observation 3 is sharp.

Corollary 5. Let G be a graph of order n, minimum degree $\delta \geq \frac{k}{2} - 1$ and maximum degree Δ. If $\delta < \Delta$, then

$$\gamma^k_{wsR}(G) \geq \frac{-2\Delta + 2\delta + 3k}{2\Delta + \delta + 3} n.$$

Proof. Multiplying both sides of the inequality in Proposition 2 (iv) by $\Delta - \delta$ and adding the resulting inequality to the inequality in Proposition 2 (iii), we obtain the desired lower bound. $\square$

Since $\gamma_{sR}^k(G) \geq \gamma_{wsR}^k(G)$, Corollary 5 leads immediately to the next lower bound, given by Henning and Volkmann [5].

Corollary 6. ([5]) Let G be a graph of order n, minimum degree $\delta \geq \frac{k}{2} - 1$ and maximum degree Δ. If $\delta < \Delta$, then

$$\gamma_{sR}^k(G) \geq \frac{-2\Delta + 2\delta + 3k}{2\Delta + \delta + 3} n.$$

Examples 9 and 10 in [5] demonstrate that Corollary 6 is sharp and therefore Corollary 5 too. The special case $k = 1$ of Corollary 6 can be found in [1].
A set $S \subseteq V(G)$ is a *2-packing* of the graph G if $N[u] \cap N[v] = \emptyset$ for any two distinct vertices $u, v \in S$. The *2-packing number* $\rho(G)$ of G is defined by

$$\rho(G) = \max\{|S| : S \text{ is a } 2-\text{packing of } G\}.$$

Theorem 4. If G is a graph of order n with $\delta(G) \geq \frac{k}{2} - 1$, then

$$\gamma_{wsR}^k(G) \geq \rho(G)(k + \delta(G) + 1) - n.$$

Proof. Let $\{v_1, v_2, \dots, v_{\rho(G)}\}$ be a 2-packing of G, and let f be a $\gamma_{wsR}^k(G)$-function. If we define the set $A = \bigcup_{i=1}^{\rho(G)} N[v_i]$, then since $\{v_1, v_2, \dots, v_{\rho(G)}\}$ is a 2-packing, we have that

$$|A| = \sum_{i=1}^{\rho(G)} (d(v_i) + 1) \geq \rho(G)(\delta(G) + 1).$$

It follows that

$$\begin{aligned}
\gamma_{wsR}^k(G) &= \sum_{x \in V(G)} f(x) = \sum_{i=1}^{\rho(G)} f[v_i] + \sum_{x \in V(G)-A} f(x) \\
&\geq k\rho(G) + \sum_{x \in V(G)-A} f(x) \geq k\rho(G) - n + |A| \\
&\geq k\rho(G) - n + \rho(G)(\delta(G) + 1) \\
&= \rho(G)(k + \delta(G) + 1) - n.
\end{aligned}$$

$\square$

Theorem 4 yields the next result immediately.

Corollary 7. ([5]) If G is a graph of order n with $\delta(G) \geq \frac{k}{2} - 1$, then

$$\gamma_{sR}^{k}(G) \geq \rho(G)(k + \delta(G) + 1) - n.$$

In [5], the authors presented an infinite family of graphs achieving equality in Corollary 7. Thus Corollary 7 and Theorem 4 are sharp. Using Corollary 4, one can prove the following Nordhaus-Gaddum type inequality analogously to Theorem 13 in [5].

Theorem 5. If G is an r-regular graph of order n such that $r \geq \frac{k}{2} - 1$ and $n - r - 1 \geq \frac{k}{2} - 1$, then

$$\gamma_{wsR}^{k}(G) + \gamma_{wsR}^{k}(\overline{G}) \geq \frac{4kn}{n+1}.$$

If n is even, then $\gamma_{wsR}^{k}(G) + \gamma_{wsR}^{k}(\overline{G}) \geq 4k(n+1)/(n+2)$.

Let $k \geq 1$ be an odd integer, and let H and $\overline{H}$ be $(k-1)$-regular graphs of order $n = 2k - 1$. By Example 1, we have $\gamma_{wsR}^{k}(H) = \gamma_{wsR}^{k}(\overline{H}) = n$. Consequently,

$$\gamma_{wsR}^{k}(H) + \gamma_{wsR}^{k}(\overline{H}) = 2n = \frac{4kn}{n+1}.$$

Thus the Nordhaus-Gaddum bound of Theorem 5 is sharp for odd k.

4. The weak signed Roman k-domination number of $K_{p,p}$

Example 2. *Let $k \geq 1$ and $p \geq k+1$ be integers.*

(1) If $p \geq k+3$, then $\gamma_{wsR}^{k}(K_{p,p}) = 2k + 2$.

(2) If $k + 1 \leq p \leq k + 2$, then $\gamma_{wsR}^{k}(K_{p,p}) = p + k - 1$.

(3) If $k \geq 2$, then $\gamma_{wsR}^{k}(K_{k,k}) = 2k$ and $\gamma_{wsR}^{1}(K_{1,1}) = 1$.

Proof. Let $X = \{x_1, x_2, \ldots, x_p\}$ and Y be a bipartition of the complete bipartite graph $K_{p,p}$.
(1) First we show that $\gamma_{wsR}^{k}(K_{p,p}) \geq 2k + 2$. Let $f : V(K_{p,p}) \longrightarrow \{-1, 1, 2\}$ be a WSRkDF. If $f(u) \geq 1$ for each $u \in V(K_{p,p})$, then $\omega(f) \geq 2p \geq 2k + 2$. Assume next, without loss of generality, that $f(x) = -1$ for at least one vertex $x \in X$ and $f(y) \geq 1$ for each $y \in Y$. If $w \in Y$, then it follows that

$$\omega(f) = f[w] + \sum_{y \in Y - \{w\}} f(y) \geq k + p - 1 \geq 2k + 2.$$

Finally, assume that $f(x) = -1$ for at least one vertex $x \in X$ and $f(y) = -1$ for at least one vertex $y \in Y$. We deduce that

$$\omega(f) = f[x] + f[y] - f(x) - f(y) \geq 2k + 2.$$

Since we have discussed all possible cases, we obtain $\gamma^k_{wsR}(K_{p,p}) \geq 2k+2$.
As $\gamma^k_{sR}(K_{p,p}) = 2k+2$ for $p \geq k+2$ (see Example 14 in [5]), we have the converse inequality $\gamma^k_{wsR}(K_{p,p}) \leq \gamma^k_{sR}(K_{p,p}) = 2k+2$ and so the desired result.
(2) First we show that $\gamma^k_{wsR}(K_{p,p}) \geq p+k-1$. Let $f : V(K_{p,p}) \longrightarrow \{-1,1,2\}$ be a WSRkDF. If $f(u) \geq 1$ for each $u \in V(K_{p,p})$, then $\omega(f) \geq 2p \geq p+k-1$. Assume next, without loss of generality, that $f(x) = -1$ for at least one vertex $x \in X$ and $f(y) \geq 1$ for each $y \in Y$. If $w \in Y$, then it follows that

$$\omega(f) = f[w] + \sum_{y \in Y - \{w\}} f(y) \geq k+p-1.$$

Finally, assume that $f(x) = -1$ for at least one vertex $x \in X$ and $f(y) = -1$ for at least one vertex $y \in Y$. We deduce that

$$\omega(f) = f[x] + f[y] - f(x) - f(y) \geq 2k+2 \geq k+p \geq p+k-1.$$

If $p = k+2$, then define the function $g : V(K_{p,p}) \longrightarrow \{-1,1,2\}$ by $g(x_1) = g(x_2) = -1$, $g(x_3) = 2$ and $g(x) = 1$ otherwise. Then g is a WSRkDF function on $K_{p,p}$ of weight $2k+1 = p+k-1$ and thus $\gamma^k_{wsR}(K_{p,p}) \leq p+k-1$ and so $\gamma^k_{wsR}(K_{p,p}) = p+k-1$ in this case.
If $p = k+1$, then define the function $h : V(K_{p,p}) \longrightarrow \{-1,1,2\}$ by $h(x_1) = -1$ and $h(x) = 1$ otherwise. Then h is a WSRkDF function on $K_{p,p}$ of weight $2k = p+k-1$ and thus $\gamma^k_{wsR}(K_{p,p}) \leq p+k-1$ and so $\gamma^k_{wsR}(K_{p,p}) = p+k-1$ also in this case.
(3) Clearly, $\gamma^1_{wsR}(K_{1,1}) = 1$. Let now $k \geq 2$.
First we show that $\gamma^k_{wsR}(K_{k,k}) \geq 2k$. Let $f : V(K_{k,k}) \longrightarrow \{-1,1,2\}$ be a WSRkDF. If $f(u) \geq 1$ for each $u \in V(K_{k,k})$, then $\omega(f) \geq 2k$. Assume next, without loss of generality, that $f(x) = -1$ for at least one vertex $x \in X$ and $f(y) \geq 1$ for each $y \in Y$. Then $f(u) = 2$ for at least one vertex $u \in Y$. If $w \in Y$ with $w \neq u$, then it follows that

$$\omega(f) = f[w] + \sum_{y \in Y - \{w\}} f(y) \geq k+k = 2k.$$

Finally, assume that $f(x) = -1$ for at least one vertex $x \in X$ and $f(y) = -1$ for at least one vertex $y \in Y$. We deduce that

$$\omega(f) = f[x] + f[y] - f(x) - f(y) \geq 2k+2.$$

Applying Observation 2, we obtain $\gamma^k_{wsR}(K_{k,k}) = 2k$. □

Example 1 implies $\gamma^k_{wsR}(K_{k-1,k-1}) = 2k-2$ for $k \geq 2$.

5. Cycles

Let C_n be a cycle of length $n \geq 3$. In [1], the authors have shown that $\gamma_{sR}(C_n) = \lceil 2n/3 \rceil$. In addition, in [7] it is proved that $\gamma_{wsR}(C_n) = \lceil n/3 \rceil$ when $n \equiv 0, 1 \pmod 3$ and $\gamma_{wsR}(C_n) = \lceil n/3 \rceil + 1$ when $n \equiv 2 \pmod 3$. Now we determine $\gamma^k_{wsR}(C_n)$ as well as $\gamma_{sR}(C_n)$ for $2 \leq k \leq 6$.
Theorems 1 and 2 immediately lead to $\gamma^6_{wsR}(C_n) = \gamma^6_{sR}(C_n) = 2n$. In addition, according to Corollary 4 and Observation 2, we have $\gamma^3_{wsR}(C_n) = \gamma^3_{sR}(C_n) = n$.

Example 3. *For $n \geq 3$, we have $\gamma^5_{wsR}(C_n) = \gamma^5_{sR}(C_n) = \lceil \frac{5n}{3} \rceil$.*

Proof. Corollary 4 implies $\gamma^5_{sR}(C_n) \geq \gamma^5_{wsR}(C_n) \geq \lceil \frac{5n}{3} \rceil$. For the converse inequality $\gamma^5_{wsR}(C_n) \leq \gamma^5_{sR}(C_n) \leq \lceil \frac{5n}{3} \rceil$, we distinguish three cases.
Case 1. Assume that $n = 3t$ with an integer $t \geq 1$. Let $C_{3t} = v_0 v_1 \dots v_{3t-1} v_0$. Define the function $f : V(C_{3t}) \longrightarrow \{-1, 1, 2\}$ by $f(v_{3i}) = 1$ and $f(v_{3i+1}) = f(v_{3i+2}) = 2$ for $0 \leq i \leq t-1$. Then $f[v_j] = 5$ for each $0 \leq j \leq 3t-1$ and therefore f is an SR5DF on C_{3t} of weight $\omega(f) = 5t$. Thus $\gamma^5_{wsR}(C_{3t}) \leq \gamma^5_{sR}(C_{3t}) \leq 5t$. Consequently, $\gamma^5_{wsR}(C_n) = \gamma^5_{sR}(C_n) = 5t = \lceil \frac{5n}{3} \rceil$ in this case.
Case 2. Assume that $n = 3t+1$ with an integer $t \geq 1$. Let $C_{3t+1} = v_0 v_1 \dots v_{3t} v_0$. Define the function $f : V(C_{3t+1}) \longrightarrow \{-1, 1, 2\}$ by $f(v_{3i}) = 1$, $f(v_{3i+1}) = f(v_{3i+2}) = 2$ for $0 \leq i \leq t-1$ and $f(v_{3t}) = 2$. Then $f[v_j] \geq 5$ for each $0 \leq j \leq 3t$ and therefore f is an SR5DF on C_{3t+1} of weight $\omega(f) = 5t+2$. Thus $\gamma^5_{wsR}(C_{3t+1}) \leq \gamma^5_{sR}(C_{3t+1}) \leq 5t+2$. Consequently, $\gamma^5_{wsR}(C_n) = \gamma^5_{sR}(C_n) = 5t+2 = \lceil \frac{5n}{3} \rceil$ also in this case.
Case 3. Assume that $n = 3t+2$ with an integer $t \geq 1$. Let $C_{3t+2} = v_0 v_1 \dots v_{3t+1} v_0$. Define the function $f : V(C_{3t+2}) \longrightarrow \{-1, 1, 2\}$ by $f(v_{3i}) = 1$, $f(v_{3i+1}) = f(v_{3i+2}) = 2$ for $0 \leq i \leq t-1$ and $f(v_{3t}) = f(v_{3t+1}) = 2$. Then $f[v_j] \geq 5$ for each $0 \leq j \leq 3t+1$ and therefore f is an SR5DF on C_{3t+2} of weight $\omega(f) = 5t+4$. Thus $\gamma^5_{wsR}(C_{3t+2}) \leq \gamma^5_{sR}(C_{3t+2}) \leq 5t+4$. Consequently, $\gamma^5_{wsR}(C_n) = \gamma^5_{sR}(C_n) = 5t+4 = \lceil \frac{5n}{3} \rceil$ also in the last case. □

Example 4. *For $n \geq 3$, we have $\gamma^4_{wsR}(C_n) = \gamma^4_{sR}(C_n) = \lceil \frac{4n}{3} \rceil$.*

Proof. Corollary 4 implies $\gamma^4_{sR}(C_n) \geq \gamma^4_{wsR}(C_n) \geq \lceil \frac{4n}{3} \rceil$. For the converse inequality $\gamma^4_{wsR}(C_n) \leq \gamma^4_{sR}(C_n) \leq \lceil \frac{4n}{3} \rceil$, we distinguish three cases.
Case 1. Assume that $n = 3t$ with an integer $t \geq 1$. Let $C_{3t} = v_0 v_1 \dots v_{3t-1} v_0$. Define the function $f : V(C_{3t}) \longrightarrow \{-1, 1, 2\}$ by $f(v_{3i}) = f(v_{3i+1}) = 1$ and $f(v_{3i+2}) = 2$ for $0 \leq i \leq t-1$. Then $f[v_j] = 4$ for each $0 \leq j \leq 3t-1$ and therefore f is an SR4DF on C_{3t} of weight $\omega(f) = 4t$. Thus $\gamma^4_{wsR}(C_{3t}) \leq \gamma^4_{sR}(C_{3t}) \leq 4t$. Consequently, $\gamma^4_{wsR}(C_n) = \gamma^4_{sR}(C_n) = 4t = \lceil \frac{4n}{3} \rceil$.
Case 2. Assume that $n = 3t+1$ with an integer $t \geq 1$. Let $C_{3t+1} = v_0 v_1 \dots v_{3t} v_0$. Define the function $f : V(C_{3t+1}) \longrightarrow \{-1, 1, 2\}$ by $f(v_{3i}) = f(v_{3i+1}) = 1$, $f(v_{3i+2}) = 2$ for $0 \leq i \leq t-1$ and $f(v_{3t}) = 2$. Then $f[v_j] \geq 4$ for each $0 \leq j \leq 3t$ and therefore f is

an SR4DF on C_{3t+1} of weight $\omega(f) = 4t+2$. Thus $\gamma^4_{wsR}(C_{3t+1}) \leq \gamma^4_{sR}(C_{3t+1}) \leq 4t+2$. Consequently, $\gamma^4_{wsR}(C_n) = \gamma^4_{sR}(C_n) = 4t+2 = \lceil \frac{4n}{3} \rceil$.
Case 3. Assume that $n = 3t+2$ with an integer $t \geq 1$. Let $C_{3t+2} = v_0 v_1 \ldots v_{3t+1} v_0$. Define the function $f : V(C_{3t+2}) \longrightarrow \{-1,1,2\}$ by $f(v_{3i}) = f(v_{3i+1}) = 1$, $f(v_{3i+2}) = 2$ for $0 \leq i \leq t-1$, $f(v_{3t}) = 1$ and $f(v_{3t+1}) = 2$. Then $f[v_j] \geq 4$ for each $0 \leq j \leq 3t+1$ and therefore f is an SR4DF on C_{3t+2} of weight $\omega(f) = 4t+3$. Thus $\gamma^4_{wsR}(C_{3t+2}) \leq \gamma^4_{sR}(C_{3t+2}) \leq 4t+3$. Consequently, $\gamma^4_{wsR}(C_n) = \gamma^4_{sR}(C_n) = 4t+3 = \lceil \frac{4n}{3} \rceil$. □

Examples 3 and 4 also show the sharpness of Corollary 4. In [5], we have determined $\gamma^2_{sR}(C_n)$.

Example 5. *([5]) For $n \geq 3$, we have* $\gamma^2_{sR}(C_n) = \left\lceil \frac{2n}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil - \left\lfloor \frac{n}{3} \right\rfloor$.

Analogously to Example 5, one can determine the weak signed Roman 2-domination number of a cycle.

Example 6. *For $n \geq 3$, we have* $\gamma^2_{wsR}(C_n) = \left\lceil \frac{2n}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil - \left\lfloor \frac{n}{3} \right\rfloor$.

6. Trees

Let P_n be a path of order n. Our aim in the section is to determine $\gamma^k_{wsR}(P_n)$, and to establish lower bounds on the weak signed Roman k-domination number of a tree for $2 \leq k \leq 4$.
We start with the path P_n. In [1] it is proved that $\gamma_{sR}(P_n) = \lfloor \frac{2n}{3} \rfloor$ and in [7], the author has shown that $\gamma_{wsR}(P_2) = 1$, $\gamma_{wsR}(P_n) = \lceil n/3 \rceil$ when $n \equiv 1 \pmod 3$ and $\gamma_{wsR}(P_n) = \lceil n/3 \rceil + 1$ when $n \equiv 0, 2 \pmod 3$ and $n \geq 3$. Now we determine $\gamma^k_{wsR}(P_n)$ for $2 \leq k \leq 4$. In [5], one can find the following result.

Example 7. *If $2 \leq n \leq 7$ then $\gamma^2_{sR}(P_n) = n$, and if $n \geq 8$, then $\gamma^2_{sR}(P_n) = \lceil \frac{2n+5}{3} \rceil$.*

If f is an SR2DF on P_n, then f is also a WSR2DF on P_n. Now assume that g is a WSR2DF on P_n. If $g(v) = -1$, then $g[v] \geq 2$ implies that v has a neighbor w with $g(w) = 2$. Therefore g is also an SR2Df on P_n. This observation and Example 7 lead to the next result immediately.

Example 8. *If $2 \leq n \leq 7$ then $\gamma^2_{wsR}(P_n) = n$, and if $n \geq 8$, then $\gamma^2_{wsR}(P_n) = \lceil \frac{2n+5}{3} \rceil$.*

In [2], the authors have shown that $\gamma^3_{sR}(P_n) = n+2$ when $n \geq 4$ and $\gamma^4_{sR}(P_n) = \lceil \frac{4n}{3} \rceil + 2$ when $n \geq 3$. The same arguments as above lead to $\gamma^3_{wsR}(P_2) = n+2$ when $n \geq 4$ and $\gamma^4_{wsR}(P_n) = \lceil \frac{4n}{3} \rceil + 2$ when $n \geq 3$.

Observation 6. *Let T be a tree of order n and let f be a WSR2DF on T. Then the following holds.*
(a) *If v is a leaf or a support vertex in T, then $f(v) \geq 1$.*
(b) *If $2 \leq n \leq 5$, then $\gamma_{sR}^2(T) = n$.*

The next result provided a lower bound on the weak signed Roman 2-domination number of a tree in terms of its order.

Theorem 7. If T is a tree of order $n \geq 4$, then

$$\gamma_{wsR}^2(T) \geq \frac{n+4}{2}.$$

Proof. We proceed by induction on the order $n \geq 4$. If $n = 4$, then by Observation 6(b), $\gamma_{sR}^2(T) = n = (n+4)/2$. This establishes the base case when $n = 4$. Let $n \geq 5$ and suppose that if T' is a tree of order n' where $4 \leq n' < n$, then $\gamma_{sR}^2(T') \geq (n'+4)/2$. Let T be a tree of order n. Choose an optimal WSR2DF f on T, and so $\gamma_{sR}^2(T) = \omega(f)$. If $f(x) \geq 1$ for each vertex $x \in V(T)$, then $\omega(f) \geq n > (n+4)/2$. Now suppose that there is a vertex $v \in V(T)$ with $f(v) = -1$. Suppose that $T - v$ is the disjoint union of r trees $T_1, T_2, \ldots, T_r$. Let f_i be the restriction of f on T_i for $1 \leq i \leq r$. Clearly, f_i is a WSR2DF on T_i for $1 \leq i \leq r$. Since by Observation 6(a) a leaf and its only neighbor has a positive label, $r \geq 2$ and each T_i has $n_i \geq 2$ vertices. If $n_i = 2$, then in fact $\omega(f_i) \geq 3 = (n_i+4)/2$, and if $n_i = 3$, then $\omega(f_i) = 3$ or $\omega(f_i) \geq 4 > (n_i+4)/2$. If $n_i \geq 4$, then by the induction hypothesis $\omega(f_i) \geq (n_i+4)/2$. If $\omega(f_i) \geq (n_i+4)/2$ for all i, then since $r \geq 2$,

$$\omega(f) = -1 + \sum_{i=1}^{r} \omega(f_i) \geq -1 + \sum_{i=1}^{r} \frac{n_i+4}{2} = \frac{n+4r-3}{2} \geq \frac{n+5}{2}.$$

Hence we may assume that for some i, $n_i = 3$ and $\omega(f_i) = 3$, for otherwise the desired result follows. Assume that $T_1, T_2, \ldots, T_q$, $q \geq 1$, are exactly the trees with three vertices and with $\omega(f_i) = 3$, $1 \leq i \leq q$. We note that $f(w) = 1$ for each vertex w that belongs to such a tree T_i with $\omega(f_i) = 3$. If $r > q$, then

$$\begin{aligned}
\omega(f) &= -1 + \sum_{i=1}^{q} \omega(f_i) + \sum_{i=q+1}^{r} \omega(f_i) \\
&\geq -1 + 3q + \sum_{i=q+1}^{r} \frac{n_i+4}{2} \\
&= \frac{n+4(r-q)+3(q-1)}{2} \geq \frac{n+4}{2},
\end{aligned}$$

as desired. If $r = q$, then $q \geq 3$, and we obtain

$$\omega(f) = -1 + \sum_{i=1}^{q} \omega(f_i) = -1 + 3q = n - 2 \geq \frac{n+4}{2},$$

since $n \geq 10$ in this case. This completes the proof. $\square$

Corollary 8. ([5]) If T is a tree of order $n \geq 4$, then

$$\gamma_{sR}^{2}(T) \geq \frac{n+4}{2}.$$

In [2], one can find the following statement. If T is a tree of order $n \geq 2$, then $\gamma_{sR}^{3}(T) \geq \frac{4n+7}{5}$, with equality if and only if $T = P_2$. The next examples demonstrates that this statement is not valid.

Example 9. *Let $P = v_1 v_2 \ldots v_{2p+1}$ be a path of order $2p+1$ with $p \geq 1$. Now attach two pendant edges to v_1 and v_{2p+1} and three pendant edges to v_{2i+1} for $1 \leq i \leq p-1$. The resulting tree T_{5p+2} is of order $5p+2$. Define the function $f : V(T_{5p+2}) \longrightarrow \{-1, 1, 2\}$ by $f(v_{2i+1}) = 2$ for $0 \leq i \leq p$, $f(v_{2i}) = -1$ for $1 \leq i \leq p$ and $f(x) = 1$ otherwise. Then f is an SR3DF on T_{5p+2} of weight*

$$\omega(f) = 2(p+1) - p + 3p + 1 = 4p + 3 = \frac{4n(T_{5p+2}) + 7}{5}.$$

Therefore $\gamma_{sR}^{3}(T_{5p+2}) \leq \frac{4n(T_{5p+2})+7}{5}$. Since $f(u) + f(v) \geq 3$ if v is a leaf and u its support vertex, it easy to verify that $\gamma_{sR}^{3}(T_{5p+2}) = \frac{4n(T_{5p+2})+7}{5}$.

Example 10. *Let S_Δ be a spider and w be a vertex of maximum degree $\Delta \geq 1$. In addition, let $v_1, v_2 \ldots, v_\Delta$ be the neighbors of w and $u_i \neq w$ be the neighbor of v_i for $1 \leq i \leq \Delta$. Now attach $\Delta+1$ pendant edges to w and two pendant edges to u_i for $1 \leq i \leq \Delta$. The resulting tree H is of order $5\Delta + 2$. Define the function $f : V(H) \longrightarrow \{-1, 1, 2\}$ by $f(w) = 2$, $f(u_i) = 2$ for $1 \leq i \leq \Delta$, $f(v_i) = -1$ for $1 \leq i \leq \Delta$ and $f(x) = 1$ otherwise. Then f is an SR3DF on H of weight*

$$\omega(f) = 4\Delta + 3 = \frac{4n(H) + 7}{5}.$$

Therefore $\gamma_{sR}^{3}(H) = \frac{4n(H)+7}{5}$.

I conjecture that the bound $\gamma_{sR}^{3}(T) \geq \frac{4n+7}{5}$ is really valid for each tree T of order $n \geq 2$, however, I only can prove the following weaker bound.

Theorem 8. If T is a tree of order $n \geq 2$, then

$$\gamma_{sR}^{3}(T) \geq \gamma_{wsR}^{3}(T) \geq \frac{3n+6}{4}.$$

Proof. Clearly, it is enogh to prove the right inequality. We proceed by induction on the order $n \geq 2$. If $n = 2$, then $\gamma_{wsR}^{3}(T) = 3 = (3n+6)/4$. If $n = 3$, then $\gamma_{wsR}^{3}(T) = 4 \geq (3n+6)/4$. Let now $n \geq 4$ and suppose that if T' is a tree of order n' where $2 \leq n' < n$, then $\gamma_{wsR}^{3}(T') \geq (3n'+6)/4$. Let T be a tree of order n. Choose

an optimal WSR3DF $f = (V_{-1}, V_1, V_2)$ on T, and so $\gamma^3_{wsR}(T) = \omega(f)$. If $f(x) \geq 1$ for each vertex $x \in V(T)$, then $\omega(f) \geq n + 1 > (3n + 6)/4$.
Now suppose that there is a vertex $v \in V(T)$ with $f(v) = -1$. If $|V_{-1}| \geq 2$, then choose $u, v \in V_{-1}$ such that $d(u, v)$ is as large as possible. Suppose that $T - v$ is the disjoint union of r trees $T_1, T_2, \ldots, T_r$. Let f_i be the restriction of f on T_i for $1 \leq i \leq r$. Clearly, f_i is a SWR3DF on T_i for $1 \leq i \leq r$. Since a leaf and its only neighbor has a positive label, $r \geq 2$ and each T_i has $n_i \geq 2$ vertices.
If $r \geq 3$, then we deduce from the induction hypothesis that

$$\begin{aligned}\omega(f) &= -1 + \sum_{i=1}^{r} \omega(f_i) \geq -1 + \sum_{i=1}^{r} \frac{3n_i + 6}{4} \\ &= -1 + \frac{3(n-1) + 6r}{4} \\ &= \frac{3n + 6}{4} + \frac{6r - 13}{4} > \frac{3n + 6}{4}.\end{aligned}$$

Let now $r = 2$. By the choice of u and v, we observe that $V_{-1} \cap V(T_1) = \emptyset$ or $V_{-1} \cap V(T_2) = \emptyset$, say $V_{-1} \cap V(T_2) = \emptyset$. Note that $\omega(f_i) = 4$ if $n_i = 2$. If $n_2 = 2$ then we deduce from the induction hypothesis that

$$\begin{aligned}\omega(f) &= \omega(f_1) + \omega(f_2) - 1 \geq \frac{3n_1 + 6}{4} + 4 - 1 \\ &= \frac{3(n-3) + 6 + 12}{4} > \frac{3n + 6}{4}.\end{aligned}$$

If $n_2 \geq 3$ then, $\omega(f_2) \geq n_2 + 1$, and we deduce from the induction hypothesis that

$$\begin{aligned}\omega(f) &= \omega(f_1) + \omega(f_2) - 1 \geq \frac{3n_1 + 6}{4} + n_2 + 1 - 1 \\ &= \frac{3n_1 + 6 + 4n_2}{4} = \frac{3(n_1 + n_2) + n_2 + 6}{4} \\ &\geq \frac{3(n_1 + n_2 + 1) + 6}{4} \geq \frac{3n + 6}{4},\end{aligned}$$

and the proof is complete. □

Observation 9. *Let T be a tree, and let f be a WSR4DF on T. If v is a leaf or a support vertex in T, then $f(v) = 2$.*

Theorem 10. If T is a tree of order $n \geq 4$, then $\gamma^4_{wsR}(T) \geq n + 4$.

Proof. We proceed by induction on the order $n \geq 4$. If $n = 4$, then by Observation 9, $\gamma^4_{sR}(T) = 8 = n + 4$. This establishes the base case when $n = 4$. Let $n \geq 5$ and suppose that if T' is a tree of order n' where $4 \leq n' < n$, then $\gamma^4_{sR}(T') \geq n' + 4$. Let T be a tree of order n. Choose an optimal WSR4DF f on T, and so $\gamma^4_{wsR}(T) = \omega(f)$. Assume first that $f(x) \geq 1$ for each vertex $x \in V(T)$. Since $n \geq 5$, the tree has at least 4 leaves or support vertices v_1, v_2, v_3, v_4. According to Observation 9, we note that $f(v_1) = f(v_2) = v(v_3) = f(v_4) = 2$ and hence $\omega(f) \geq 8 + n - 4 = n + 4$ in this case.

Now assume that there is a vertex $v \in V(T)$ with $f(v) = -1$. Suppose that $T - v$ is the disjoint union of r trees $T_1, T_2, \ldots, T_r$. Let f_i be the restriction of f on T_i for $1 \leq i \leq r$. Clearly, f_i is a WSR4DF on T_i for $1 \leq i \leq r$. Since $f[v] \geq 4$, we deduce that $r \geq 3$ and each T_i has $n_i \geq 3$ vertices. If $n_i = 3$, then in fact $\omega(f_i) = 6$, and if $n_i \geq 4$, then by the induction hypothesis $\omega(f_i) \geq n_i + 4$. Now assume that $n_1 = n_2 = \ldots = n_q = 3$ for $0 \leq q \leq r$ and $n_i \geq 4$ for $q + 1 \leq i \leq r$. We deduce from the induction hypothesis that

$$\begin{aligned}
\omega(f) &= -1 + \sum_{i=1}^{q} \omega(f_i) + \sum_{i=q+1}^{r} \omega(f_i) \\
&\geq -1 + 6q + \sum_{i=q+1}^{r} (n_i + 4) \\
&= -1 + 6q + (n - 3q - 1) + 4(r - q) \\
&= n + 3q - 2 + 4(r - q) \geq n + 4.
\end{aligned}$$

□

Corollary 9. If T is a tree of order $n \geq 4$, then $\gamma^4_{sR}(T) \geq n + 4$.

Note that if $H \in \{K_{1,3}, P_4, P_5, P_6\}$, then $\gamma^4_{wsR}(H) = \gamma^4_{sR}(H) = n(H) + 4$. Corollary 9 is an improvement of the bound $\gamma^4_{sR}(T) \geq n + 2$, given in [2].

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