Communications in Combinatorics and Optimization
Vol. 5 No. 2, 2020 pp.191-206
DOI: 10.22049/CCO.2020.26761.1140

Bounds on signed total double Roman domination

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Received: 7 February 2020; Accepted: 2 July 2020
Published Online: 4 July 2020

Abstract: A signed total double Roman dominating function (STDRDF) on an isolated-free graph $G = (V, E)$ is a function $f : V(G) \to \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ has at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ has at least one neighbor w with $f(w) \geq 2$ and (iii) $\sum_{u\in N(v)} f(u) \geq 1$ holds for any vertex v. The weight of an STDRDF is the value $f(V(G)) = \sum_{u\in V(G)} f(u)$. The signed total double Roman domination number $\gamma^t_{sdR}(G)$ is the minimum weight of an STDRDF on G. In this paper, we continue the study of the signed total double Roman domination in graphs and present some sharp bounds for this parameter.

Keywords: Roman domination; signed double Roman domination; signed total double Roman domination

AMS Subject classification: 05C69

1. Terminology and introduction

In this paper, G is a simple isolated-free graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed*

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neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We write P_n for the *path* of order n, C_n for the *cycle* of length n, K_n for the complete graph of order n and $K_{m,n}$ for the complete bipartite graph. We refer the reader to [20] for some basic terminology in graph theory.

A set $S \subseteq V$ in a graph G is called a *(total) dominating set* if every vertex of $V \setminus S$ (V) is adjacent to a vertex of S. The *(total) domination number* $\gamma(G)$ $(\gamma_t(G))$ equals the minimum cardinality of a (total) dominating set in G.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G. Roman domination was introduced by Cockayne et al. in [17] inspired by the work of ReVelle and Rosing [18], and Stewart [19]. Since 2004, so many papers have been published on this topic and its variations. The literature on Roman domination and its variations has been surveyed and detailed in two book chapters and three surveys [12–16].

In 2016, Beeler et al. [11] introduced the double Roman domination defined as follows. A function $f : V \to \{0, 1, 2, 3\}$ is a *double Roman dominating function* (DRDF) on a graph G if the following conditions hold.

(i) If $f(v) = 0$, then v must have either at least one neighbor in V_3 or at least two neighbors in V_2.

(ii) If $f(v) = 1$, then v must have at least one neighbor in $V_2 \cup V_3$.

The *double Roman domination number* $\gamma_{dR}(G)$ equals the minimum weight of an DRDF on G. For an SDRDF f, let $V_i(f) = \{v \in V : f(v) = i\}$. In the context of a fixed SDRDF, we suppress the argument and simply write V_{-1}, V_1, V_2 and V_3. Since this partition determines f, we can equivalently write $f = (V_{-1}, V_1, V_2, V_3)$. For further results on double Roman domination see [1, 2, 5].

In 2014 Abdollahzadeh Ahangar et al. [8] introduced the concept of *Signed Roman domination*. Also, the other variations of this concept have been introduced in [3, 4]. Abdollahzadeh Ahangar et al. [6], introduced the concept of a new variation of double Roman domination as signed double Roman domination number. A *signed double Roman dominating function* (SDRDF) on a graph $G = (V, E)$ is a function $f : V(G) \to \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ is adjacent to least two vertices assigned a 2 or to at least one vertex w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ is adjacent to at least one vertex w with $f(w) \geq 2$ and (iii) $f(v) = \sum_{u \in N[v]} f(u) \geq 1$ holds for any vertex v. The weight of an SDRDF f is the value $\omega(f) = \sum_{u \in V(G)} f(u)$. The *signed double Roman domination number* $\gamma_{sdR}(G)$ is the minimum weight of an SDRDF on G. For further results on signed double Roman domination number see [7, 10, 21].

Recently, Abdollahzadeh Ahangar et al. [9] introduced the concept of signed total double Roman domination number defined as follows. A *signed total double Roman dominating function* (STDRDF) on a graph $G = (V, E)$ is a function $f : V(G) \to \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ is adjacent to least two

vertices assigned a 2 or to at least one vertex w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ is adjacent to at least one vertex w with $f(w) \geq 2$ and (iii) $f(v) = \sum_{u\in N(v)} f(u) \geq 1$ holds for any vertex v. The weight of an STDRDF f is the value $\omega(f) = \sum_{u\in V(G)} f(u)$. The *signed total double Roman domination number* $\gamma^t_{sdR}(G)$ is the minimum weight of an STDRDF on G.
In this paper we study the signed total double Roman domination and present some sharp bounds for this parameter in general graphs. In addition, we determine the signed total double Roman domination number of some classes of graphs.

2. A lower bounds

In this section we present some sharp bounds on the signed total double Roman domination number in graphs. First we present a lower bound in terms of the order and size. To do this, we introduce some notation for convenience.
Let $V'_{-1} = \{v \in V_{-1} \mid N(v) \cap V_3 \neq \emptyset\}$ and $V''_{-1} = V_{-1} - V'_{-1}$. For a subset $S \subseteq V$, we let $d_S(v)$ denote the number of vertices in S that are adjacent to v. In particular, $d_V(v) = \deg(v)$. For disjoint subsets U and W of vertices, we let $[U, W]$ denote the set of edges between U and W. For notational convenience, we let $V_{12} = V_1 \cup V_2, V_{13} = V_1 \cup V_3, V_{123} = V_1 \cup V_2 \cup V_3$ and let $|V_{12}| = n_{12}, |V_{13}| = n_{13}, |V_{123}| = n_{123}$, and let $|V_1| = n_1, |V_2| = n_2$ and $|V_3| = n_3$. Then, $n_{123} = n_1 + n_2 + n_3$. Further, we let $|V_{-1}| = n_{-1}$, and so $n_{-1} = n - n_{123}$. Let $G_{123} = G[V_{123}]$ be the subgraph induced by the set V_{123} and let G_{123} have size m_{123}. For $i = 1, 2, 3$, if $V_i \neq \emptyset$, let $G_i = G[V_i]$ be the subgraph induced by the set V_i and let G_i have size m_i. Hence, $m_{123} = m_1 + m_2 + m_3 + |[V_1, V_2]| + |[V_1, V_3]| + |[V_2, V_3]|$.

Theorem 1. Let G be a connected graph of order $n \geq 3$ and size m. Then

$$\gamma^t_{sdR}(G) \geq \frac{11n - 12m}{3}.$$

Proof. Let $L = \{v \in V(G) \mid \deg(v) = 1\}$ and let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma^t_{sdR}(G)$-function having the property that $|V_2 \cap L|$ is minimized. Let $V_{-1} = \emptyset$. Then $\gamma^t_{sdR}(G) \geq n + 1 \geq \frac{11n-12m}{3}$ since $m \geq n - 1$. So, from now on we assume that $V_{-1} \neq \emptyset$. We consider the following cases.

Case 1. $V_3 \neq \emptyset$.
We distinguish the following.

Subcase 1.1. $V_2 \neq \emptyset$.
By the definition of an STDRDF, each vertex in V_{-1} is adjacent to at least one vertex in V_3 or to at least two vertices in V_2, and so

$$|[V_{-1}, V_3]| + |[V_{-1}, V_2]| \geq |V'_{-1}| + 2|V''_{-1}| \geq n_{-1}.$$

Furthermore we have

$$2n_{-1} \leq 2|[V_{-1}, V_3]| + |[V_{-1}, V_2]| = 2\sum_{v\in V_3} d_{V_{-1}}(v) + \sum_{u\in V_2} d_{V_{-1}}(u).$$

For each vertex $v \in V_3$, we have that $3d_{V_3}(v)+2d_{V_2}(v)+d_{V_1}(v)-d_{V_{-1}}(v) = f(N(v)) \geq 1$, and so $d_{V_{-1}}(v) \leq 3d_{V_3}(v) + 2d_{V_2}(v) + d_{V_1}(v) - 1$. Similarly, for each vertex $u \in V_2$, we have that $d_{V_{-1}}(u) \leq 3d_{V_3}(u) + 2d_{V_2}(u) + d_{V_1}(u) - 1$. Now, we have

$$\begin{array}{rcl} 2n_{-1} & \leq & 2\sum_{v\in V_3} d_{V_{-1}}(v) + \sum_{u\in V_2} d_{V_{-1}}(u) \\ & \leq & 2\sum_{v\in V_3}(3d_{V_3}(v) + 2d_{V_2}(v) + d_{V_1}(v) - 1) \\ & + & \sum_{u\in V_2}(3d_{V_3}(u) + 2d_{V_2}(u) + d_{V_1}(u) - 1) \\ & = & (12m_3 + 4|[V_2, V_3]| + 2|[V_1, V_3]| - 2n_3) \\ & + & (3|[V_2, V_3]| + 4m_2 + |[V_1, V_2]| - n_2) \\ & = & 12m_3 + 4m_2 + 7|[V_2, V_3]| + 2|[V_1, V_3]| + |[V_1, V_2]| - 2n_3 - n_2 \\ & = & 12m_{123} - 12m_1 - 8m_2 - 5|[V_2, V_3]| - 10|[V_1, V_3]| - 11|[V_1, V_2]| - 2n_3 - n_2, \end{array}$$

which implies that

$$m_{123}=\frac{1}{12}(2n_{-1} + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]| + 2n_3 + n_2).$$

Hence,

$$\begin{aligned} m &= m_{123} + |[V_{-1}, V_{123}]| + m_{-1} \\ &\geq m_{123} + |[V_{-1}, V_{123}]| \\ &\geq \frac{1}{12}(2n_{-1} + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]| + 2n_3 + n_2) \\ &+ |[V_{-1}, V_1]| + n_{-1} \\ &= \frac{1}{12}(14n_{-1} + 2n_{123} - 2n_1 - n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| \\ &+ 11|[V_1, V_2]| + 12|[V_{-1}, V_1]|) \\ &= \frac{1}{12}(14n - 12n_{123} - 2n_1 - n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + \\ &11|[V_1, V_2]| + 12|[V_{-1}, V_1]|) \end{aligned}$$

and so

$$\begin{aligned} n_{123} \geq \frac{1}{12}(&-12m + 14n - 2n_1 - n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| \\ &+ 11|[V_1, V_2]| + 12|[V_{-1}, V_1]|). \end{aligned}$$

Now, we have

$$\begin{aligned}
\gamma^t_{sdR}(G) &= 3n_3 + 2n_2 + n_1 - n_{-1}\\
&= 4n_3 + 3n_2 + 2n_1 - n\\
&= 4n_{123} - n - n_2 - 2n_1\\
&\geq \frac{1}{3}(-12m + 14n - 2n_1 - n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]|\\
&\quad + 11|[V_1, V_2]| + 12|[V_{-1}, V_1]|) - n - n_2 - 2n_1\\
&= \frac{1}{3}(11n - 12m) + \frac{1}{3}(-8n_1 - 4n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]|\\
&\quad + 10|[V_1, V_3]| + 11|[V_1, V_2]| + 12|[V_{-1}, V_1]|).
\end{aligned}$$

Let $\Theta = -8n_1 - 4n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]| + 12|[V_{-1}, V_1]|$. If $n_1 = 0$, then $\Theta = -4n_2 + 8m_2 + 5|[V_2, V_3]|$. By definition of an STDRDF of G, we have $d_{V_{23}}(v) \geq 1$ for each $v \in V_2$. Then

$$\begin{aligned}
\Theta &= -4n_2 + 8m_2 + 5|[V_2, V_3]|\\
&= 4\sum_{v\in V_2} d_{V_2}(v) + 4\sum_{v\in V_2} d_{V_3}(v) + (-4n_2 + |[V_2, V_3]|)\\
&= 4\sum_{v\in V_2} d_{V_{23}}(v) + (-4n_2 + |[V_2, V_3]|)\\
&\geq 4n_2 - 4n_2 + |[V_2, V_3]|\\
&= |[V_2, V_3]|\\
&> 0.
\end{aligned}$$

Therefore $\gamma^t_{sdR}(G) > \frac{11n-12m}{3}$.
Suppose now that $n_1 \geq 1$. Let $H_1, H_2, \ldots, H_t$ be the components of the induced subgraph $G[V_1]$ of order $h_1, h_2, \ldots, h_t$, respectively. Since G is connected, each component H_i contains a vertex adjacent to a vertex of $V_2 \cup V_3$ or to a vertex of V_{-1} for $1 \leq i \leq t$. This implies

$$\begin{aligned}
m_1 + |[V_1, V_{23}]| + |[V_1, V_{-1}]| &\geq (h_1 - 1) + (h_2 - 1) + \cdots + (h_t - 1) + t\\
&= h_1 + h_2 + \cdots + h_t = n_1.
\end{aligned}$$

By the definition of an STDRDF of G we have $d_{V_{23}}(v) \geq 1$ for each $v \in V_1$. Then

$$\begin{aligned}\Theta &= -4n_2 - 8n_1 + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]| \\ &\quad + 12|[V_{-1}, V_1]| \\ &\geq (8m_1 + 8|V_1, V_3| + 8|V_1, V_2| + 8|V_1, V_{-1}|) + (5|V_2, V_3| + 3|V_1, V_2| \\ &\quad + 4|V_2, V_2| + 4|[V_{-1}, V_1]|) - 4n_2 - 8n_1 \\ &\geq 4\sum_{v\in V_2}(d_{V_3}(v) + d_{V_2}(v)) + 3\sum_{v\in V_2} d_{V_1}(v) + 4\sum_{v\in V_1} d_{V_{-1}}(v) - 4n_2. \qquad (1)\end{aligned}$$

Note that if $d_{V_{23}}(v) \geq 1$ or $d_{V_1}(v) \geq 2$ for each $v \in V_2$, then $4\sum_{v\in V_2}(d_{V_3}(v)+d_{V_2}(v))+3\sum_{v\in V_2} d_{V_{-1}}(v) \geq 4n_2$. So we assume that there exists a vertex $v \in V_2$ for which $d_{V_{23}}(v) = 0$ and $d_{V_1}(v) = 1$. This implies that v is a leaf which is adjacent to a support vertex u with $f(u) = 1$. If $f(N(u)) \geq 2$, then by assigning 2 to u and 1 to v we obtain an STDRDF h of G which contradicts our choice of f. Therefore, $f(N(u)) = 1$. Since u is adjacent to v with $f(v) = 2$, it follows that u has a neighbor with weight -1 under f. So, the vertex v is counted at least four times in $4\sum_{v\in V_2} d_{V_{-1}}(v)$. All in all, we have shown that

$$4\sum_{v\in V_2}(d_{V_3}(v) + d_{V_2}(v)) + 3\sum_{v\in V_2} d_{V_1} + 4\sum_{v\in V_1} d_{V_{-1}}(v) \geq 4n_2.$$

Consequently, $\Theta \geq 0$ by (1). Therefore $\gamma_{sdR}(G) \geq \frac{11n-12m}{3}$.

Subcase 1.2. $V_2 = \emptyset$.
By definition of an STDRDF, each vertex in V_{-1} is adjacent to at least one vertex in V_3, and so

$$|[V_{-1}, V_3]| \geq |V_{-1}| = n_{-1}.$$

Furthermore we have

$$n_{-1} \leq |[V_{-1}, V_3]| = \sum_{v\in V_3} d_{V_{-1}}(v).$$

For each vertex $v \in V_3$, we have that $3d_{V_3}(v) + d_{V_1}(v) - d_{V_{-1}}(v) = f(N(v)) \geq 1$, and so $d_{V_{-1}}(v) \leq 3d_{V_3}(v) + d_{V_1}(v) - 1$. Now, we have

$$\begin{aligned}n_{-1} &\leq \sum_{v\in V_3} d_{V_{-1}}(v) \\ &\leq \sum_{v\in V_3}(3d_{V_3}(v) + d_{V_1}(v) - 1) \\ &= 6m_3 + |[V_1, V_3]| - n_3 \\ &= 6m_{13} - 6m_1 - 5|[V_1, V_3]| - n_3,\end{aligned}$$

which implies that

$$m_{13} \geq \frac{1}{6}(n_{-1} + 6m_1 + 5|[V_1, V_3]| + n_3).$$

Hence,

$$\begin{aligned}
m &= m_{13} + |[V_{-1}, V_3]| + |[V_{-1}, V_1]| + m_{-1} \\
&\geq m_{13} + |[V_{-1}, V_3]| + |[V_{-1}, V_1]| \\
&\geq \frac{1}{6}(n_{-1} + 6m_1 + 5|[V_1, V_3]| + n_3) + n_{-1} + |[V_{-1}, V_1]| \\
&= \frac{1}{6}(7n_{-1} + n_3 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) \\
&= \frac{1}{6}(7n_{-1} + n_{13} - n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) \\
&= \frac{1}{6}(7n - 6n_{13} - n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|)
\end{aligned}$$

and so

$$n_{13} \geq \frac{1}{6}(-6m + 7n - n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|).$$

Now, we have

$$\begin{aligned}
\gamma^t_{sdR}(G) &= 3n_3 + n_1 - n_{-1} \\
&= 4n_3 + 2n_1 - n \\
&= 4n_{13} - n - 2n_1 \\
&\geq \frac{4}{6}(-6m + 7n - n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) - n - 2n_1 \\
&= \frac{2}{3}(-6m + 7n - \frac{3}{2}n) + \frac{2}{3}(-4n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) \\
&= \frac{1}{3}(11n - 12m) + \frac{2}{3}(-4n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1|]).
\end{aligned}$$

Let $\Theta = -4n_1 + 6m_1 + 5|[V_{-1}, V_3]| + 6|[V_{-1}, V_1]|$. We show that $\Theta \geq 0$. If $n_1 = 0$, then $\Theta = 0$. Suppose that $n_1 \geq 1$. As above, we let $H_1, H_2, \ldots, H_t$ be the components of the induced subgraph $G[V_1]$ of order $h_1, h_2, \ldots, h_t$, respectively. Since G is connected, each component H_i contains a vertex adjacent to a vertex of $V_2 \cup V_3$ or to a vertex of V_{-1} for $1 \leq i \leq t$. This implies $m_1 + |[V_1, V_{23}]| + |[V_1, V_{-1}]| \geq n_1$. It follows that

$$\begin{aligned}
\Theta &= -4n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]| \\
&\geq -4n_1 + 5m_1 + 5|[V_1, V_3]| + 5|[V_{-1}, V_1]| \\
&> 0.
\end{aligned}$$

Therefore $\gamma^t_{sdR}(G) > \frac{11n-12m}{3}$.

Case 2. $V_3 = \emptyset$.
Since $V_{-1} \neq \emptyset$, we conclude that $V_2 \neq \emptyset$. By definition of an STDRDF, each vertex in V_{-1} is adjacent to at least two vertices in V_2, and so

$$|[V_{-1}, V_{12}]| \geq |[V_{-1}, V_2]| \geq 2|V_{-1}| = 2n_{-1}.$$

Furthermore, we have

$$2n_{-1} \leq |[V_{-1}, V_2]| = \sum_{v \in V_2} d_{V_{-1}}(v).$$

For each vertex $v \in V_2$, we have that $2d_{V_2}(v) + d_{V_1}(v) - d_{V_{-1}}(v) = f(N(v)) \geq 1$, and so $d_{V_{-1}}(v) \leq 2d_{V_2}(v) + d_{V_1}(v) - 1$. Now, we have

$$\begin{aligned}
2n_{-1} &\leq \sum_{v \in V_2} d_{V_{-1}}(v) \\
&\leq \sum_{v \in V_2} (2d_{V_2}(v) + d_{V_1}(v) - 1) \\
&= 4m_2 + |[V_1, V_2]| - n_2 \\
&= 4m_{12} - 4m_1 - 3|[V_1, V_2]| - n_2,
\end{aligned}$$

which implies that

$$m_{12} \geq \frac{1}{4}(2n_{-1} + 4m_1 + 3|[V_1, V_2]| + n_2).$$

Hence,

$$\begin{aligned}
m =& m_{12} + |[V_{-1}, V_{12}]| + m_{-1} \\
&\geq m_{12} + |[V_{-1}, V_{12}]| \\
&\geq \frac{1}{4}(2n_{-1} + 4m_1 + 3|[V_1, V_2]| + n_2) + 2n_{-1} + |[V_1, V_{-1}]| \\
&= \frac{1}{4}(10n_{-1} + n_{12} - n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|) \\
&= \frac{1}{4}(10n - 9n_{12} - n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|)
\end{aligned}$$

and so

$$n_{12} \geq \frac{1}{9}(-4m + 10n - n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|).$$

Now, we have

$$\begin{aligned}
\gamma^t_{sdR}(G) &= 2n_2 + n_1 - n_{-1} \\
&= 3n_2 + 2n_1 - n \\
&= 3n_{12} - n - n_1 \\
&\geq \frac{1}{3}(-4m + 10n - n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|) - n - n_1 \\
&= \frac{1}{3}(-4m + 10n - 4n) + \frac{1}{3}(-4n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]| + n) \\
&\geq \frac{1}{3}(-4m + 6n) + \frac{1}{3}(-3n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|).
\end{aligned}$$

Since every vertex in V_1 has at least one neighbor in V_2, we have $3|[V_1, V_2]| \geq 3n_1$. Therefore, $\gamma^t_{sdR}(G) \geq \frac{1}{3}(6n - 4m) \geq \frac{1}{3}(11n - 12m)$. This completes the proof. □

In the next example, we present an infinite family of graphs that attain the bound of Theorem 1.

Example 1. For $t \geq 2$, let F_t be the graph obtained from a connected graph F of order t by adding $3d_F(v) - 1$ pendant edges to each vertex v of F. Then

$$n(F_t) = n(F) + \sum_{v \in V(F)} (3d_F(v) - 1) = 6m(F)$$

and

$$m(F_t) = m(F) + \sum_{v \in V(F)} (3d_F(v) - 1) = 7m(F) - n(F).$$

Assigning a 3 to every vertex in $V(F)$ and a -1 to every vertex in $V(F_t) - V(F)$ produces an STDRDF f of weight

$$\omega(f) = 3n(F) - \sum_{v \in V(F)} (3d_F(v) - 1) = 4n(F) - 6m(F) = \frac{11n(F_t) - 12m(F_t)}{3},$$

and so $\gamma^t_{tdR}(F_t) \leq \frac{11n(F_t)-12m(F_t)}{3}$. Using Theorem 1, we obtain $\gamma^t_{tdR}(F_t) = \frac{11n(F_t)-12m(F_t)}{3}$.

Next we establish a lower bound on the signed total double Roman domination number in terms of the order.

Theorem 2. Let G be a graph of order n. Then

$$\gamma^t_{sdR}(G) \geq \left\lceil 3\sqrt{\frac{n}{2}} + \frac{1}{2} \right\rceil - n + 1.$$

This bound is sharp for K_2, K_3.

Proof. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma^t_{sdR}(G)$-function. If $|V_{-1}| = 0$, then $\gamma_{sdR}(G) \geq n+1 \geq \lceil 3\sqrt{\frac{n}{2}} + \frac{1}{2} \rceil - n + 1$. Hence, let $|V_{-1}| \geq 1$. We consider the following cases.

Case 1. $|V_3| > 0$.
Since $V_3 \neq \emptyset$, we have $V'_{-1} \neq \emptyset$. Since each vertex in V'_{-1} is adjacent to at least one vertex in V_3, we conclude that at least one vertex v of V_3 is adjacent to at least $\frac{n'_{-1}}{n_3}$ vertices of V'_{-1}. Also, since each vertex in V''_{-1} (if any) is adjacent to at least two vertices in V_2, we conclude that at least one vertex u of V_2 is adjacent to at least $\frac{2n''_{-1}}{n_2}$ vertices of V''_{-1}. Then $1 \leq f(N(v)) \leq 3(n_3 - 1) + 2n_2 + n_1 - \frac{n'_{-1}}{n_3}$ which implies that $0 \leq 3n_3^2 + 2n_2n_3 + n_1n_3 - n'_{-1} - 4n_3$. Similarly, we have $0 \leq 3n_3n_2 + 2n_2^2 + n_1n_2 - 2n''_{-1} - 3n_2$ (note that this holds even if $n_2 = 0$). Then $0 \leq 3n_3^2 + 2n_2^2 + 5n_2n_3 + n_1n_3 + n_1n_2 - n'_{-1} - 2n''_{-1} - 3n_2 - 4n_3$. Since $n = n_3 + n_2 + n_1 + n_{-1}$, we have

$$\begin{aligned}
0 &\leq 3n_3^2 + 2n_2^2 + 5n_2n_3 + n_1n_3 + n_1n_2 + n_1 - 3n_3 - 2n_2 - n''_{-1} - n \\
&\leq \frac{32}{9}n_3^2 + 2n_2^2 + \frac{16}{3}n_2n_3 + \frac{8}{9}n_1^2 + \frac{32}{9}n_1n_3 + \frac{8}{3}n_1n_2 - \frac{8}{3}n_3 - \frac{4}{3}n_1 - 2n_2 - n''_{-1} - n.
\end{aligned}$$

Hence

$$\begin{aligned}
0 &\leq 16n_3^2 + 9n_2^2 + 24n_2n_3 + 4n_1^2 + 16n_1n_3 + 12n_1n_2 - 6n_1 - 12n_3 - 9n_2 + \frac{9}{4} - \frac{9}{2}n \\
&= (4n_3 + 3n_2 + 2n_1 - \frac{3}{2})^2 - \frac{9}{2}n
\end{aligned}$$

which implies that $3\sqrt{\frac{n}{2}} \leq 4n_3 + 3n_2 + 2n_1 - \frac{3}{2}$. Therefore

$$\begin{aligned}
\gamma_{sdR}(G) &= 3n_3 + 2n_2 + n_1 - n_{-1} \\
&= 4n_3 + 3n_2 + 2n_1 - n \\
&\geq 3\sqrt{\frac{n}{2}} + \frac{3}{2} - n.
\end{aligned}$$

Case 2. $V_3 = \emptyset$.
Since $|V_{-1}| > 0$, we conclude that $V_2 \neq \emptyset$. As in Case 1, at least one vertex u of V_2 is adjacent to at least $\frac{2n_{-1}}{n_2}$ vertices of V_{-1}. Then $0 \leq 2n_2^2 + n_1n_2 - 2n_{-1} - 3n_2$. Since $n = n_2 + n_1 + n_{-1}$, we have

$$\begin{aligned}
0 &\leq 2n_2^2 + n_1n_2 + n_1 - 2n_2 - n \\
&\leq 2n_2^2 + \frac{8}{9}n_1^2 + \frac{8}{3}n_1n_2 - 2n_2 - \frac{4}{3}n_1 - n.
\end{aligned}$$

Hence

$$0 \leq 9n_2^2 + 4n_1^2 + 12n_1n_2 - 6n_1 - 9n_2 + \frac{9}{4} - \frac{9}{2}n = (3n_2 + 2n_1 - \frac{3}{2})^2 - \frac{9}{2}n$$

which implies that $3\sqrt{\frac{n}{2}} \leq 3n_2 + 2n_1 - \frac{3}{2}$. Therefore

$$\begin{aligned}\gamma_{sdR}(G) &= 2n_2 + n_1 - n_{-1}\\ &= 3n_2 + 2n_1 - n\\ &\geq 3\sqrt{\frac{n}{2}} + \frac{3}{2} - n.\end{aligned}$$

This leads to the desired bound since $\gamma_{sdR}(G)$ is an integer. This completes the proof. □

Theorem 3. If G is a graph of order $n \geq 3$ with $\delta \geq 1$, then

$$\gamma^t_{sdR}(G) \geq max\{\Delta - n + 1, \delta - n + 4\}.$$

This bound is sharp for K_n.

Proof. Let f be a $\gamma^t_{sdR}(G)$-function. If $f(x) \geq 1$ for all $x \in V(G)$, then by definition $f(y) \geq 2$ for some $y \in V(G)$ and so $\gamma^t_{sdR}(G) = n + 1 \geq max\{\Delta - n + 1, \delta - n + 4\}$. Now assume that there exists a vertex u with $f(u) = -1$. Then u has a neighbor w with $f(w) \geq 2$ and so

$$\begin{aligned}\gamma^t_{sdR}(G) &= f(w) + f(N(w)) + \textstyle\sum_{x\in V(G)-N[w]} f(x) \geq 2 + 1 - (n - d(w) - 1)\\ &= 4 - n + d(w)\\ &\geq 4 - n + \delta.\end{aligned}$$

On the other hand, for any vertex v with maximum degree Δ, we have

$$\gamma^t_{sdR}(G)=f(v) + f(N(v)) + \sum_{x\in V(G)-N[v]} f(x) \geq -1 + 1 - (n - \Delta - 1) = 1 - n + \Delta,$$

and the proof is complete. □

3. Trees

In this section we present bounds on the signed total double Roman domination number in trees. A *double star* is a tree containing exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at each support vertex is denoted by $DS_{p,q}$.

Observation 4. *For $q \geq p \geq 1$,*

$$\gamma^t_{sdR}(DS_{p,q}) < \frac{4(p+q+2)}{3}.$$

Proof. Let u and v be the support vertices of the double star $DS_{p,q}$ which are adjacent to leaves $u_1, \ldots, u_s$, and $v_1, \ldots, v_s$, respectively. Define $f : V(DS_{p,q}) \to \{-1, 1, 2, 3\}$ by $f(u) = f(v) = 3$ and $f(u_i) = f(v_j) = -1$ for $i, j \leq 2$, $f(u_i) = (-1)^{i+1}$ for $i \geq 3$ and $f(v_i) = (-1)^{i+1}$ for $i \geq 3$. Clearly, f is an STDRDF of $DS_{p,q}$ of weight at most 4 and $\gamma^t_{sdR}(DS_{p,q}) \leq 4 < \frac{4(p+q+2)}{3}$. □

Proposition 1. ([9]) For $n \geq m \geq 1$,

$$\gamma^t_{sdR}(K_{m,n}) = \begin{cases} 4, & (m = n = 2, 4),\ \ (m = 2,\ \ n = 4), \text{ or } (m = 1,\ \ n \geq 2) \\ 2, & (m = 3,\ \ n \neq 4) \text{ or } m \geq 5 \\ 3 & \text{otherwise.} \end{cases}$$

Theorem 5. Let T be a tree of order $n \geq 3$. Then

$$\gamma^t_{sdR}(T) \leq \frac{4n}{3}.$$

This bound is sharp for P_3.

Proof. The proof is by induction on n. If $n = 3$, then clearly $\gamma^t_{sdR}(T) = 4$. Let $n \geq 4$ and let the statement hold for all trees of order $3 \leq n' < n$. Assume T a tree of order n. If $diam(T) = 2$, then T is a star and by Proposition 1, we have $\gamma^t_{sdR}(T) = 4 < \frac{4n}{3}$. If $diam(T) = 3$, then T is a double star $DS_{p,q}$ with $q \geq p \geq 1$ and by Proposition 4, we have $\gamma^t_{sdR}(T) < \frac{4n}{3}$. Therefore, we assume that $diam(T) \geq 4$. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma^t_{sdR}(T)$-function. Let $v_1 v_2 \ldots v_k$ ($k \geq 5$) be a diametral path in T such that $d_T(v_2)$ is as large as possible and root T at v_k. Let $d_T(v_2) \geq 3$. Consider $T' = T - v_1$ and let f' be a $\gamma^t_{sdR}(T')$-function. We have $f'(v_2) \geq 1$ since f' is an STDRDF and v_2 is adjacent to at least one leaf of T'. Suppose first that $f'(v_2) = 1$. This shows that $f'(x) \geq 2$ for each leaf adjacent to v_2. Suppose that $y \neq v_1$ is a leaf adjacent to v_2. It is easy to see that the function g defined by $g(y) = f'(y) - 1, g(v_2) = 2, g(v_1) = 1$ and $g(v) = f'(v)$ for the other vertices v, is an STDRDF of T with weight $\omega(f')+1$. So, $\gamma^t_{sdR}(T) \leq \omega(f')+1 \leq 4(n-1)/3+1 < 4n/3$. Suppose now that $f'(v_2) \geq 2$. It is then easily observed that f' can be extended to an STDRDF of T by assigning 1 to v_1. So, we deduce again that $\gamma^t_{sdR}(T) < \frac{4n}{3}$.
Let $d_T(v_2) = 2$. Suppose first that $d_T(v_3) = 2$. Let $T' = T - \{v_1, v_2, v_3\}$. If $n(T') = 2$, then $T = P_5$ with $\gamma^t_{sdR}(P_5) = 6 < \frac{4n}{3}$. So, we let $n(T') \geq 3$. By the induction hypothesis, $\gamma^t_{sdR}(T') \leq 4(n-3)/3$. It is easily seen that any $\gamma^t_{sdR}(T')$-function can be extended to an STDRDF of T by assigning 2, 3 and -1 to v_3, v_2 and v_1, respectively, with weight $\gamma^t_{sdR}(T') + 4$. So, $\gamma^t_{sdR}(T) \leq \gamma^t_{sdR}(T') + 4 \leq 4n/3$.

Now let $d_T(v_3) \geq 3$. Note that all children of v_3 are leaves or support vertices which are adjacent to only one leaf, by our choice of the diametral path. We distinguish two cases depending on the behavior of children of v_3.

Case 1. Let v_3 have a child leaf w.
Let f'' be a $\gamma^t_{sdR}(T'')$-function in which $T'' = T - v_1 - v_2$. Since v_3 is adjacent to a leaf, it follows that $f''(v_3) \geq 1$. We need to consider two more possibilities depending on $f''(v_3)$.

Subcase 1.1. $f''(v_3) \geq 2$.
This shows that the function g'' defined by $(g''(v_1), g''(v_2)) = (-1, 3)$ and $g''(v) = f''(v)$ for the other vertices v gives us an STDRDF of T with weight $\gamma^t_{sdR}(T'') + 2$. Therefore, $\gamma^t_{sdR}(T) \leq \gamma^t_{sdR}(T'') + 2 \leq 4(n-2)/3 + 2 < 4n/3$.

Subcase 1.2. $f''(v_3) = 1$.
This implies that $f''(w) \geq 2$. Now, the function h'' defined by $(h''(w), h''(v_3), h''(v_2), h''(v_1)) = (f''(w)-1, 2, 3, -1)$ and $h''(v) = f''(v)$ for the other vertices v, is an STDRDF of T with weight $\gamma^t_{sdR}(T'') + 2$. Therefore, $\gamma^t_{sdR}(T) < 4n/3$ by a similar fashion.

Case 2. Suppose that all children of v_3 are support vertices and $k \geq 2$ is the number of them.
We let L_{v_3} be the subtree of T induced by the vertices v_3 and its descendants. Let $T''' = T - L_{v_3}$. If $V(T''') = \{v_4, v_5\}$, we assign 3 to the children of v_3, -1 to the grandchildren of v_3, 2 to v_3 and v_4, and 1 to v_5. It is then easily checked that $\gamma^t_{sdR}(T) \leq 2k + 5 < 4(2k+3)/3 = 4n/3$. So, we may assume that $n(T''') \geq 3$. Let f''' be a $\gamma^t_{sdR}(T''')$-function. Then, $n(T''') = n(T) - 2k - 1$ and therefore $\gamma^t_{sdR}(T''') \leq 4(n-2k-1)/3$ by the induction hypothesis. It is a routine matter to see that the function g''' defined by $g'''(v_3) = 2$, $g'''(v) = 3$ for all removed support vertices v, $g'''(u) = -1$ for all removed leaves u, and $g'''(x) = f'''(x)$ for the other vertices x, is an STDRDF of T with weight $\gamma^t_{sdR}(T''') + 2k + 2$. Therefore, we end up in

$$\gamma^t_{sdR}(T) \leq \omega(g''') \leq \frac{4(n-2k-1)}{3} + 2k + 2 < \frac{4n}{3}.$$

This completes the proof.

$\square$

Theorem 6. If T is a tree of order n and maximum degree $\Delta(T) \geq 3$, then

$$\gamma^t_{sdR}(T) \geq \Delta(T) + 5 - n.$$

Proof. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma^t_{sdR}(T)$-function, v a vertex of maximum degree $\Delta(T)$ and $\Delta_i = |N(v) \cap V_i|$ for $i \in \{-1, 1, 2, 3\}$. If $f(v) = 1$, then by definition $\Delta_2 + \Delta_3 \geq 1$ and each vertex x in $V_{-1} \cap N(v)$ must have a neighbor x' with label at least two. Note that $x' \neq y'$ when $x \neq y$. Thus we have

$$\begin{array}{rcl}\gamma^t_{sdR}(T) & \geq & f(v) + 3\Delta_3 + 2\Delta_2 + \Delta_1 \\ & + & \sum_{x\in N(v)\cap V_{-1}}(f(x) + f(x')) - (n - \Delta - \Delta_{-1} - 1) \\ & \geq & 2 + \Delta + 3\Delta_3 + 2\Delta_2 + \Delta_1 + 2\Delta_{-1} - n \\ & = & 2 + 2\Delta + 2\Delta_3 + \Delta_2 + \Delta_{-1} - n \\ & > & 5 + \Delta - n.\end{array}$$

If $f(v) = 2$, then each vertex x in $V_{-1} \cap N(v)$ must have a neighbor x' with label at least two. Note that $x' \neq y'$ when $x \neq y$. As above we can see that $\gamma^t_{sdR}(T) > 5 + \Delta - n$. If $f(v) = 3$, then we have

$$\begin{array}{rcl}\gamma^t_{sdR}(T) & \geq & f(v) + 3\Delta_3 + 2\Delta_2 + \Delta_1 - \Delta_{-1} - (n - \Delta - 1) \\ & = & 4 + \Delta + 3\Delta_3 + 2\Delta_2 + \Delta_1 - \Delta_{-1} - n \\ & = & 4 + \Delta - n + f(N(v)) \\ & \geq & 5 + \Delta - n.\end{array}$$

If $f(v) = -1$, then by definition $\Delta_2 \geq 2$ or $\Delta_3 \geq 1$ and each vertex x in $V_{-1} \cap N(v)$ must have a neighbor x' with label at least two.

$$\begin{array}{rcl}\gamma^t_{sdR}(T) & \geq & f(v) + 3\Delta_3 + 2\Delta_2 + \Delta_1 \\ & + & \sum_{x\in N(v)\cap V_{-1}}(f(x) + f(x')) - (n - \Delta - \Delta_{-1} - 1) \\ & \geq & \Delta + 3\Delta_3 + 2\Delta_2 + \Delta_1 + 2\Delta_{-1} - n \\ & = & 2\Delta + 2\Delta_3 + \Delta_2 + \Delta_{-1} - n \\ & \geq & 5 + \Delta - n\end{array}$$

as desired. □

Example 2. Let $t \geq 1$ be an integer, and let T be the tree formed by subdividing exactly t edges of the star $K_{1,4t-1}$. Assign -1 to all leaves of T and to the remaining vertices the weight 3. This is an STDRDF on T of weight

$$3(t + 1) - (4t - 1) = -t + 4 = \Delta(T) + 5 - n.$$

Acknowledgements

The authors are grateful to anonymous referees for their remarks and suggestions that helped improve the manuscript. H. Abdollahzadeh Ahangar was supported by the Babol Noshirvani University of Technology under research grant number BNUT/385001/99.

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