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Review Article

A survey of the studies on Gallai and anti-Gallai graphs

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Abstract: The Gallai graph and the anti-Gallai graph of a graph G are edge disjoint spanning subgraphs of the line graph $L(G)$. The vertices in the Gallai graph are adjacent if two of the end vertices of the corresponding edges in G coincide and the other two end vertices are nonadjacent in G. The anti-Gallai graph of G is the complement of its Gallai graph in $L(G)$. Attributed to Gallai (1967), the study of these graphs got prominence with the work of Sun (1991) and Le (1996). This is a survey of the studies conducted so far on Gallai and anti-Gallai of graphs and their associated properties.

Keywords: Line graphs, Gallai graphs, anti-Gallai graphs, irreducibility, cograph, total graph, simplicial complex, Gallai-mortal graph

AMS Subject classification: 05C76

1. Introduction

All the graphs under consideration are simple and undirected, but are not necessarily finite. The vertex set and the edge set of the graph G are denoted by $V(G)$ and $E(G)$ respectively. If H is a subgraph of G with $V(H) = V(G)$, then H is a *spanning subgraph* of G [21]. If H contains all the edges of G with end vertices in $V(H)$, the H is an *induced subgraph* of G; [7] we say that $V(H)$ induces or spans H in G. The *line graph* $L(G)$ of G is the graph on E in which $x, y \in E$ are adjacent as vertices if and only if they are adjacent as edges in G [7, 11]. The *Gallai graph* $\Gamma(G)$, of a

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graph G is a spanning subgraph of $L(G)$ in which two vertices are adjacent if the corresponding edges in G are adjacent and do not span a triangle in G [17]. The *anti-Gallai graph* $\Delta(G)$, of a graph G is a spanning subgraph of $L(G)$ in which two vertices are adjacent if the corresponding edges in G are adjacent and spans a triangle in G [21]. The basic properties of Gallai graphs that follow from the definition are; for any induced subgraph H of G, the Gallai graph of H is an induced subgraph of the Gallai graph of G and the Gallai graph of $G * K_1$ is isomorphic to the union of Gallai graph of G and its complement [20]. Although several authors have been finding many interesting results on Gallai and anti-Gallai graphs after the motivating work of Van Bang Le [21] in 1996, this construction was first used by Gallai [9] and then by Sun [30] in their investigations.
The smallest integer k such that G has a k-vertex coloring, is called the *chromatic number* of G and is denoted by $\chi(G)$. A graph G with $\chi(G) = k$ is called k-*chromatic*; if $\chi(G) \leq k$ [7]. The greatest integer r such that $K_r \subseteq G$ is the *clique number* $\omega(G)$ of G, and the greatest integer r such that $\overline{K_r} \subseteq G$ (induced) is the *independence number* [13] $\alpha(G)$ of G. Clearly, $\alpha(G) = \omega(\bar{G})$ and $\omega(G) = \alpha(\bar{G})$ [7].
The Gallai graph, $\Gamma(G)$ and the anti-Gallai graph, $\Delta(G)$ are edge disjoint subgraphs with $\Gamma(G) \cup \Delta(G) = L(G)$. Though the line graph $L(G)$ has forbidden sub graph characterization, none of the Gallai graphs or the anti-Gallai graphs has the vertex hereditary property. Hence, they cannot be characterized using forbidden subgraphs [21]. For a graph G, if some of its iterated Gallai graphs finally becomes the empty graph, the graph is said to be *Gallai-mortal* and the characterizations of these graphs are discussed in [20].
It is known that $K_{1,3}$ and K_3 are the only pair of non-isomorphic graphs having the same line graph. In [17], it is proved that there exists infinitely many pairs of non-isomorphic graphs of the same order having isomorphic Gallai and anti-Gallai graphs. Also, the existence of a finite family of forbidden subgraphs for the Gallai graphs and anti-Gallai graphs to be P_4-free and the characterization based on distance parameters are discussed in [17]. The theory of simplicial complexes are extensively studied for its wide range of applications namely in pattern generation and network analysis. Considerable amount of work on charaterizations of Gallai simplicial complexes and that of anti-Gallai simplicial complexes are discussed in [2] and [3]. The *triangular line graph* remarkably exhibits a similar behaviour as that of an anti-Gallai graph, which is also obtained by adding a restriction to the line graph. Hence, its results can also be inspected for further studies. The papers [5], [4] and [8] can be referred for the properties of triangular line graph and for another interesting area of study called triadic closure, one may refer to [14, 15].
It is observed that many meaningful results have been proved on some of the major parameters like chromatic number, clique number, radius and diameter of the graph with its Gallai and anti-Gallai graphs. However, no much extensive work has been done on other parameters like connectivity, covering, domination, etc. In this work the results are named as theorems, lemmas, propositions etc. as they were originally named by the respective authors.

2. Chromatic number and clique number

The problems of determining the clique number, and the chromatic number of a Gallai graph and anti-Gallai graph are NP-complete. However, a formula for the parameters $\omega(\Delta(G))$, $\chi(\Delta(G))$, $\omega(\Gamma(G))$ and $\chi(\Gamma(G))$ in terms of that of G is determined. Some of the major results are presented now.

Theorem 1 ([17]). *Given two positive integers a, b, where $a > 1$, there exists a graph G such that $\chi(G) = a$ and $\chi(\Gamma(G)) = b$.*

Lemma 1 ([17]). *The anti-Gallai graph of a graph G cannot be bipartite except for the triangle free graphs.*

Theorem 2 ([17]). *Given two positive integers a, b, where $b < a$, $b \neq 2$, there exists a graph G such that $\chi(G) = a$ and $\chi(\Delta(G)) = b$. Further, for any odd number a, there exists a graph G such that $\chi(G) = \chi(\Delta(G)) = a$.*

Proposition 1 ([21]). *For every graph G with finite chromatic number, $\chi(\Delta(G)) \leq \chi(G)$. Moreover, if $\chi(G)$ is even, then $\chi(\Delta(G)) \leq \chi(G)$ - 1.*

Proposition 2 ([21]). *The following statements are equivalent.*
(i) For all planar graphs G, $\chi(G) \leq 4$ (the Four Colour Theorem).
(ii) For all planar graphs G, $\chi(\Delta(G)) = \omega(\Delta(G))$.

Proposition 3 ([21]). *For a graph G whose complement $\bar{G}$ has finite chromatic number, $\chi(\Gamma(G)) \leq \chi(\bar{G})$.*

Proposition 4. *For every graph G with $\omega(G) \geq 1$,*

$$\omega(\Delta(G)) = \begin{cases} \omega(G) - 1 & \text{if } \omega(G) \neq 3, \\ 3 & \text{if } \omega(G) = 3. \end{cases}$$

Proposition 5. *For a graph G whose complement $\bar{G}$ has finite chromatic number, $\chi(\Gamma(G)) \leq \chi(\bar{G})$.*

Let $\Delta^o(G)$ denote G and for every integer $t > 1$, $\Delta_t(G)$ denote $\Delta(\Delta_{t-1}(G))$, the iterated anti-Gallai graphs.

Corollary 1. *For every graph G with finite clique number, $\Delta^{\omega(G)-1}$ has no edges or is the union of vertex-disjoint triangles.*

3. Partial characterizations

3.1. Perfect graphs

The term "perfect graph" was introduced by Claude Berge in 1963. Berge proposed to call a graph *perfect* if, for each of its induced subgraphs F the chromatic number of F equals the largest number of pairwise adjacent vertices in F. This concept was in fact developed from a 1958 work of Tibor Gallai. It was conjectured that a graph is perfect if and only if none of its induced subgraphs is an odd hole or the complement of an odd hole. An *odd hole* is a chordless cycle whose number of vertices, $n \geq 5$ is odd. This is known as the *Strong Perfect Graph Conjecture*. However, this conjecture of Berge was proved in 2002 as the *Strong Perfect Graph Theorem*. Another special class of perfect graphs that emerged around 1975-1976 is known as "Raspail graph". A graph in which every cycle whose length is odd and at least five has at least one chord joining two vertices whose distance on the cycle is two is defined as *Raspail graph* [30]. All Raspail graphs are perfect is a corollary of the Strong Perfect Graph Theorem.

Liping Sun has presented two classes of perfect graphs, where one is the Gallai graph which was used by Gallai in his investigation and the other is the Berge graphs defined through a forbidden graph on five vertices. As seen in the definition of Gallai graphs, the two edges form a chordless path with three vertices in G, Liping Sun was keen on considering the Gallai graphs for the classification and defined a graph G to be Gallai-perfect if, and only if, the Gallai graph of G contains no odd hole [30]. However, his study on perfect graphs, led to the following partial-characterizations of Gallai graphs.

Theorem 3 ([30]). *Every Gallai-perfect graph is perfect.*

Theorem 4 ([30]). *If a Gallai-perfect graph G contains an induced dart (see Figure 1), then G contains a star-cutset or an even pair.*

Theorem 5 ([30]). *Every diamond-free Raspail graph is Gallai-perfect.*

Proposition 6 ([21]). *The Perfect Graph Conjecture is true for Gallai graphs of graphs having a 4-colorable complement.*

Theorem 6 ([21]). *Anti-Gallai graphs without odd holes are perfect.*

Proposition 7 ([21]). *The following statements are equivalent for every minimal imperfect graph G.*

1. *G or $\bar{G}$ is an odd hole.*
2. *$\Gamma(G)$ and $\Gamma(\bar{G})$ are triangle-free.*
3. *$\Gamma(G)$ or $\Gamma(\bar{G})$ is triangle-free.*

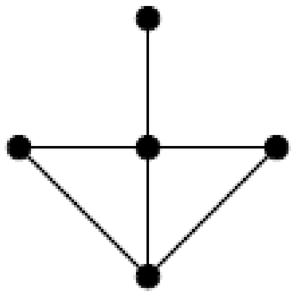

Figure 1. **Dart**

Figure 2. G_1**- A clique irreducible graph** G_2**- A clique reducible graph**

3.2. Irreducibility

A graph G is *clique irreducible* if every clique in G of size at least two, has an edge which does not lie in any other clique of G and it is clique reducible if it is not clique irreducible [32]. A graph G is *clique vertex irreducible* if every clique in G has a vertex which does not lie in any other clique of G and it is *clique vertex reducible* if it is not clique vertex irreducible.

A graph G is *weakly clique irreducible* if every edge lies in an essential clique. Oth-

Figure 3. **A clique vertex irreducible graph** **A clique vertex reducible graph**

erwise, it is *weakly clique reducible* [31, 33]. For example, from the above figure we see that the graph G_2 is weakly clique irreducible and the graph G_1 is weakly clique reducible.

3.3. Gallai graphs and irreducibility

The Gallai graphs which are clique irreducible or weakly clique irreducible are characterized as follows.

Theorem 7 ([18]). *The Gallai graph $\Gamma(G)$ is clique vertex irreducible if and only if for every $v \in V(G)$, every maximal independent set I in $N(v)$ with $|I| > 2$ contains a vertex u such that $N(u)$-$\{v\} = N(v)$-I.*

Theorem 8. *If $\Gamma(G)$ is clique vertex reducible, then G contains one of the graphs in Figure 4 as an induced subgraph.*

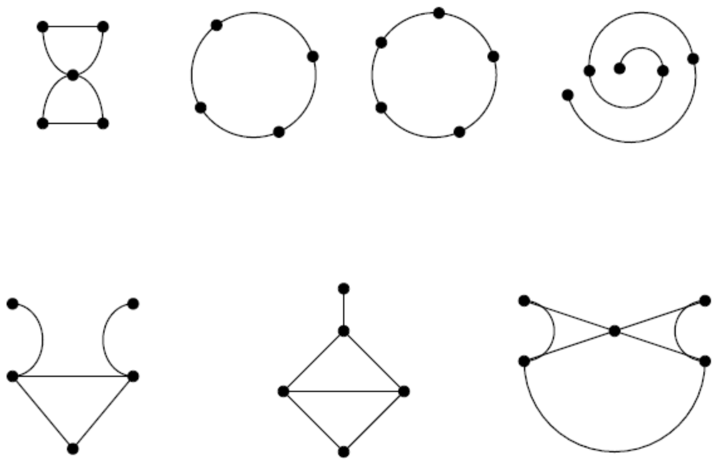

Figure 4. **Forbidden Graphs (Theorem 8)**

Theorem 9 ([18]). *The Gallai graph $\Gamma(G)$ is clique irreducible if and only if for every $v \in V(G)$, $< N(v) >^c$ is clique irreducible.*

Theorem 10 ([18]). *The second iterated Gallai graph $\Gamma^2(G)$ is clique irreducible if and only if for every $uv \in E(G)$, either $< N(u) - N(v) >$ and $< N(v) - N(u) >$ are clique vertex irreducible or one among them is a clique and the other is clique irreducible.*

Theorem 11 ([18]). *If $\Gamma(G)$ is clique reducible then G contains one of the graphs in Figure 5 as an induced subgraph.*

Figure 5. **Forbidden Graphs (Theorem 11)**

Theorem 12 ([18]). *The Gallai graph of a graph G, $\Gamma(G)$ is weakly clique irreducible if and only if for every vertex $u \in V(G)$, $< N(u) >^c$ is weakly clique irreducible.*

3.4. Anti-Gallai graphs and irreducibility

The anti-Gallai graphs which are clique irreducible or weakly clique irreducible are characterized as follows.

Theorem 13 ([18]). *The anti-Gallai graph $\Delta(G)$ is clique irreducible if and only if G neither contains K_4 nor one of the graphs in Figure 6 (Hajós' graphs) as an induced subgraph.*

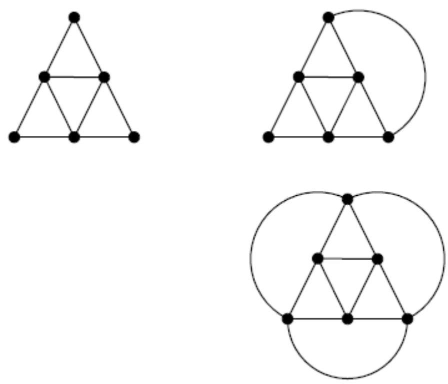

Figure 6. **Forbidden Graphs (Theorem 13)**

Theorem 14 ([18]). *The second iterated anti-Gallai graph $\Delta^2(G)$ is clique vertex irreducible if and only if G does not contain K_4 as an induced subgraph.*

Theorem 15 ([18]). *The n^{th} iterated anti-Gallai graph $\Delta^n(G)$ is clique vertex irreducible if and only if G does not contain K_{n+2} as an induced subgraph.*

Theorem 16 ([18]). *The anti-Gallai graph $\Delta(G)$ is clique irreducible if and only if G does not contain K_4 as an induced subgraph.*

Theorem 17 ([18]). *The n^{th} iterated anti-Galli graph $\Delta^n(G)$ is clique irreducible if and only if G does not contain an induced K_{n+3}.*

Theorem 18 ([19]). *The anti-Gallai graph of a graph G, $\Delta(G)$ is weakly clique irreducible if and only if G is K_4-free.*

Corollary 2 ([19]). *The anti-Gallai graph of a graph G, $\Delta(G)$ is weakly clique irreducible if and only if it is clique irreducible.*

3.5. Some miscellaneous results

Proposition 8 ([21]). *(i) Let H be a subgraph of the graph G. If H has no short chord, then $\Gamma(H)$ is an induced subgraph of $\Gamma(G)$ and $\Delta(H)$ is an induced subgraph of $\Delta(G)$.*
(ii) Every K_a in $\Gamma(G)$ stems from an induced star $K_{1,a}$ in G; every $K_a(a > 3)$ in $\Delta(G)$ stems from a K_{a+1} in G.
(iii) For all graphs G, $\Gamma(G * K_1) \approx \Gamma(G) \cup G^c$ and G are an induced subgraphs of $\Delta(G * K_1)$.

Lemma 2 ([18]). *If G is K_4-free then $\Gamma(G)$ is diamond-free.*

Theorem 19 ([26]). *There does not exist a graph G such that $\Gamma(G) \cong \Delta(G) \cong nK_1 \cup mK_3 \cup pH$, where H is a bow and $p \neq 0$.*

Theorem 20 ([26]). *There does not exists a graph G for which $\Gamma(G) \cong \Delta(G) \cong nK_1 \cup mK_3 \cup pD$, where D is a diamond and $p \neq 0$.*

Theorem 21 ([26]). *If $\Gamma(G) \cong \Delta(G) \cong H$, then H cannot have a component with exactly 4 vertices.*

Lemma 3 ([26]). *There does not exist a graph G for which $\Gamma(\Delta(G)) \cong \Delta(\Gamma(G)) \cong nK_1$ and $n \neq 0$.*

Theorem 22 ([26]). *There does not exist a graph G for which $\Gamma(\Delta(G)) \cong \Delta(\Gamma(G)) \cong H$, where H is diamond free.*

Let r and d denote the radius and diameter of a graph, respectively.

Theorem 23 ([17]). *Let G be a graph such that $\Gamma(G)$ is connected. Then $r(\Gamma(G)) > r(G) - 1$ and $d(\Gamma(G)) > d(G) - 1$.*

Theorem 24 ([17]). *If G is a graph such that $\Delta(G)$ is connected and $r(G) > 1$, $r(\Delta(G)) > 2(r(G) - 1)$ and $d(\Delta(G)) > 2(d(G) - 1)$.*

There are no forbidden characterization for the Gallai graphs. However, Gallai graphs with certain properties can be characterized. The Gallai graphs that are trivially perfect graphs, 3-sun-free graphs, interval graphs etc. are characterized in [28]. In a recent work [27], Gallai and anti-Gallai operators of graphs are considered and their commutativity property is dealt with.

3.6. Anti-Gallai triangle

If the edges $\bar{u}$, $\bar{v}$ and $\bar{w}$ induce a triangle in G then the triangle uvw in $L(G)$ is referred to as an anti-Gallai triangle. All the triangles in $\Delta(G)$ need not be an anti-Gallai triangle and the number of anti-Gallai triangles in $L(G)$ is equal to the number of triangles in G. Since each edge of an anti-Gallai graph belongs to some anti-Gallai triangle, the set of all anti-Gallai triangles in $L(G)$ induces $\Delta(G)$.

Lemma 4 ([29]). *Consider a line graph $H \cong K_3$. If a triangle uvw in H is an anti-Gallai triangle, then for all $x \in V(H)\backslash\{u, v, w\}$, one of the following holds.*

1. $< u, v, w, x > \cong K_4 - e$
2. $< u, v, w, x >$ *is disconnected.*

Lemma 5 ([29]). *If a triangle uvw is not an anti-Gallai triangle in a line graph $H \cong L(G)$, then there is at most one common neighbor z for an edge of uvw in H such that $< u, v, w, z > \cong K_4 - e$.*

Theorem 25 ([29]). *Consider a line graph $H \cong K_3$, $K_4 - e$, $C_4 \vee K_1$ and $C_4 \vee 2K_1$. A triangle uvw in H is an anti-Gallai triangle if and only if $< u, v, w, x > \cong K_4 - e$ or disconnected for all $x \in V(H)\backslash\{u,\ v,\ w\}$.*

Lemma 6 ([29]). *An anti-Gallai triangle in a DHL (Distance Hereditary Line) graph has a vertex of degree two.*

3.7. Bipartite Gallai graph

In 1999, Maffray and Reed had characterized graphs whose Gallai graphs are bipartite. According to them, "a graph G is elementary if and only if Gallai graph of G is bipartite" [24]. They have also depicted the *five wonders*, which are the five graphs-*pyramid, lighthouse, mausoleum, garden, colossus* in their work. With these they have formulated one of the main results that the Gallai graph of G is bipartite if and only if G is claw-free, Berge, and contains none of the five wonders; and if and only if G is an augmentation of a line-graph of bipartite multigraph.
Later on Joos, Le and Rautenbach extensively characterized certain graphs whose Gallai graphs are forests or trees, the main results of which are mentioned below.

Theorem 26 **([12]).** *The Gallai graph $\Gamma(G)$ of a graph G is a forest if and only if G is a $\{F_1, \ldots, F_9\}$-free chordal graph.*

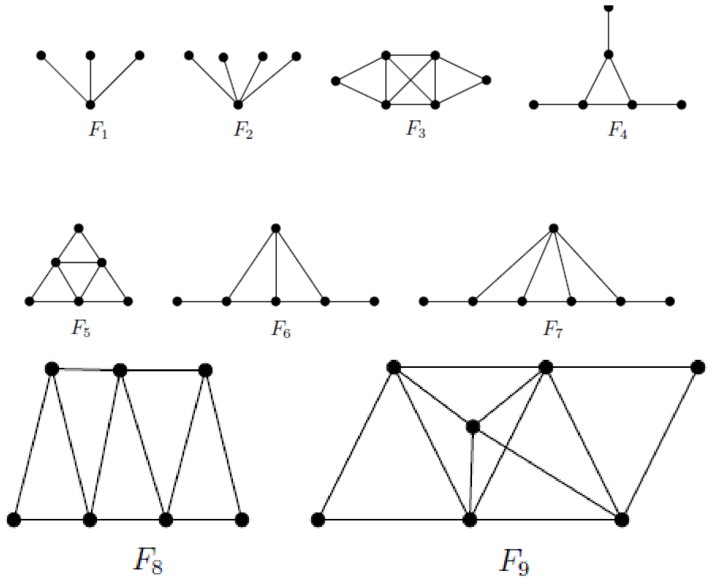

Figure 7. **The graphs $F_1, \ldots, F_9$ (Theorem 26)**

Theorem 27 **([12]).** *For a graph G without isolated vertices, the following statements are equivalent:*

1. *The Gallai graph $\Gamma(G)$ of G is a tree.*
2. *Every non-trivial homogeneous set in G is independent, and G is an $\{F_1, \ldots, F_9\}$-free chordal graph.*
3. *G is either the graph F_8^- or G is connected and satisfies the following conditions:*
   - *Every block of G is isomorphic to K_2,K_3, or a gem.*
   - *Every cut-vertex of G lies in at most two blocks and has degree atmost 3 in G.*
   - *Every block of G that is isomorphic to K_3 has exactly two cut-vertices.*
   - *Every block of G that is isomorphic to a gem has exactly one cut-vertex.*

4. Semi-characterizations based on graph operations

Graph operations are performed to create new graphs from the original one. Some unary and binary operations and their effects on Gallai and anti-Gallai graph are discussed in brief here.

4.1. Cograph

A graph is cograph (or compliment reducible graph or P_4-free graph) if it can be reduced to an empty graph by successively taking compliments within components.

Theorem 28 ([17]). *If G is a connected cograph without a vertex of full degree then $\Gamma(G)$ is a cograph if and only if $G = ({}^{p}K_2)^c$, the complement of p copies of K_2.*

Theorem 29 ([17]). *Let G be a graph. Then, $\Gamma(G)$ is a cograph if and only if G does not have the graphs in Figure 8 as induced subgraphs.*

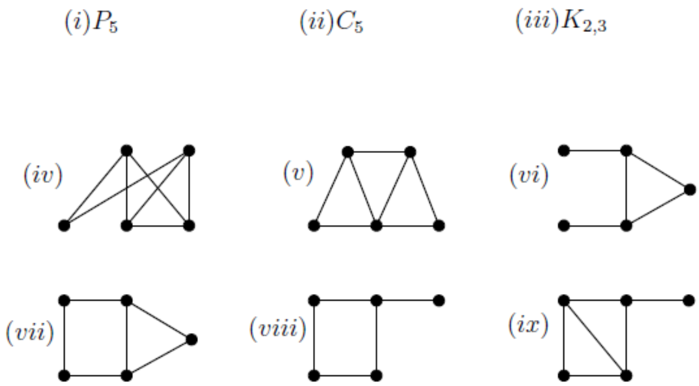

Figure 8. **Forbidden Graphs (Theorem 29)**

Theorem 30 ([17]). *Let G be a graph. Then, $\Delta(G)$ is a cograph if and only if G does not have the graphs in Figure 9 as induced subgraphs.*

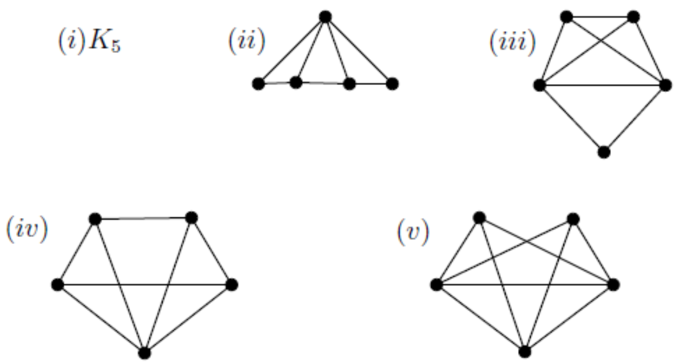

Figure 9. **Forbidden Graphs (Theorem 30)**

Theorem 31 ([17]). *For a connected cograph G, $\Gamma(G)$ is a cograph if and only if $G \in \mathcal{H}$.*

The class $\mathcal{H}$ of graphs is defined as follows:

- $\mathcal{H}_1 = \{G : G = ({}^pK_2)^c \vee (K_q)$, where $p, q \geq 0\}$
- $\mathcal{H}_i = \{G : G = (\bigcup H_{i-1}) \vee K_r,$ where $H_{i-1} \in \mathcal{H}_{i-1}$ and $r > 0\}$ for $i > 1$
- $\mathcal{H} = \bigcup \mathcal{H}_i$

Theorem 32. *For a connected cograph G, $\Delta(G)$ is a cograph if and only if*

1. *$G = G_1 \vee G_2$, where G_1 is edgeless and G_2 does not contain P_4 as a subgraph (which need not be induced) or*
2. *G is C_4.*

4.2. Triangular line graph

The triangular line graph $T(G)$ of a graph G is the graph with vertex set $E(G)$, with two distinct vertices e and f adjacent in $T(G)$ if and only if there exists a subgraph H of G with $e, f \in E(H)$ and $H \cong K_3$. Any such subgraph H is called a triangle of G. Some of its properties are mentioned below.

Proposition 9 ([5]). *If G is a graph, then $T(G)$ contains a copy of $K_4 - e$ if and only if G contains a copy of K_4.*

Theorem 33 ([5]). *For a graph G, the triangular line graph $T(G)$ is Eulerian if and only if $T(G)$ is connected.*

Lemma 7 ([16]). *Assume that F is a connected non-trivial graph and F is the anti-Gallai graph of F'. Then F' is K_4-free if and only if every edge of F is contained in exactly one triangle, or equivalently, every maximal clique of F is a triangle and any two triangles share at most one vertex.*

4.3. Gallai and anti-Gallai total graph

From the notion of total graphs, Gallai graphs and anti-Gallai graphs, two new operators Gallai total graph and anti-Gallai total graph were introduced by Garg et al. [10]

Let $G = (V, E)$ be a graph. The *Gallai total graph*, $\Gamma_T(G)$ of G is the graph, where $V(\Gamma_T(G)) = V \cup E$ and $uv \in E(\Gamma_T(G))$ if and only if

1. u and v are adjacent vertices in G, or
2. u is incident to v or v is incident to u in G, or
3. u and v are adjacent edges in G which do not span a triangle in G.

The *anti-Gallai total graph*, $\Delta_T(G)$ of G is the graph, where $V(\Delta_T(G)) = V \cup E$ and $uv \in E(\Delta_T(G))$ if and only if

1. u and v are adjacent vertices in G, or
2. u is incident to v or v is incident to u in G, or
3. u and v are adjacent edges in G and lie on a same triangle in G.

Proposition 10 ([10]). *Let $\Gamma_T(G)$ be the Gallai total graph of a graph $G = (V, E)$, then the degree of $v \in V(\Gamma_T(G))$ is given by,*

$$d_\Gamma(v) = \begin{cases} 2d(v) & \text{if } v \in V(G) \\ d(v_1) + d(v_2) - 2t & \text{if } v = v_1v_2 \in E(G) \end{cases}$$

where t denotes the number of triangles containing v in G.

Proposition 11 ([10]). *Let $G = (V, E)$ be a (p, q) graph, then $|E(\Gamma_T(G))| = 2q + \frac{1}{2} \sum_{i=1}^{p} (d(v_i))^2$ - $3(\frac{tr(A^3)}{6})$, where A is the adjacency matrix of G.*

Proposition 12 ([10]). *Let $\Delta_T(G)$ be the anti-Gallai total graph of a graph $G = (V, E)$, then*

$$d_\Delta(v) = \begin{cases} (2d(v)) & \text{if } v \in V(G), \\ 2t + 2 & \text{if } v = v_1v_2 \in E(G) \end{cases}$$

where t denotes the number of triangles containing v in G.

Proposition 13 ([10]). *Let $G = (V, E)$ be a (p, q) graph, then $|E(\Delta_T(G))| = 3q + 3(\frac{tr(A^3)}{6})$, where A is the adjacency matrix of G.*

Some Eulerian and Hamiltonian properties of Gallai and anti-Gallai total graph are also discussed in [10].

4.4. Eulerian property

Proposition 14 ([10]). *The Gallai total graph $\Gamma_T(G)$ of a graph G is regular if and only if G is regular and triangle free.*

Proposition 15 ([10]). *For a connected graph G, the Gallai total graph $\Gamma_T(G)$ has a spanning Eulerian subgraph.*

Proposition 16 ([10]). *The Gallai total graph $\Gamma_T(G)$ of G is connected if and only if G is connected.*

Corollary 3. *The $\Gamma_T^n(G)$ of G is connected if and only if G is connected for all $n \geq 1$.*

Theorem 34 ([10]). *Let G be a connected graph, then the Gallai total graph $\Gamma_T(G)$ is Eulerian if and only if all the vertices of G are of the same parity; where parity of a vertex means parity of its degree.*

Corollary 4. *If G is Eulerian, then $\Gamma_T^n(G)$ is Eulerian for all $n \geq 1$, but converse is not true.*

Proposition 17 ([10]). *The anti-Gallai total graph $\Delta_T(G)$ of G is regular if and only if G is l-triangular and $(l+1)$-regular; where a graph is called l-triangular if each edge of G lies on l number of triangles in G.*

Proposition 18 ([10]). *For a connected graph G, the anti-Gallai total graph $\Delta_T(G)$ has a spanning Eulerian subgraph.*

Proposition 19 ([10]). *For a connected graph G, $\Delta_T^n(G)$ has a spanning Eulerian subgraph for all $n \geq 1$ [10].*

Proposition 20 ([10]). *The anti-Gallai total graph $\Delta_T(G)$ of G is connected if and only if G is connected.*

Corollary 5. *The $\Delta_T^n(G)$ of G is connected if and only if G is connected for all $n \geq 1$.*

Theorem 35 ([10]). *Let G be a connected graph, then $\Delta_T(G)$ of G is Eulerian.*

Corollary 6. *Let G be a connected graph, then $\Delta_T^n(G)$ of G is Eulerian.*

Corollary 7. *If G is Eulerian, then $\Delta_T^n(G)$ of G is Eulerian, but converse is not true.*

4.5. Hamiltonian property

Theorem 36 ([10]). *The Gallai total graph $\Gamma_T(G)$ of a non-trivial graph G is Hamiltonian if and only if the set of all elements of G can be ordered in such a way that consecutive elements are neighbors as are the first and last elements, but the two edges are not consecutive elements, if both the edges are of same triangle.*

It is seen in [10] that if G is Hamiltonian, then its total graph is also Hamiltonian. However, only if G is Hamiltonian and triangle-free, then $\Gamma_T(G)$ is Hamiltonian.

Theorem 37 ([10]). *The anti-Gallai total graph $\Delta_T(G)$ of a non-trivial graph G is Hamiltonian if and only if the set of all elements of G can be ordered in such a way that consecutive elements are neighbors as are the first and last elements, but the two edges are not consecutive elements, if both the edges are not of same triangle.*

4.6. Simplicial complex

The line simplicial complex containing Gallai and anti-Gallai simplicial complexes as spanning subcomplexes is defined and a relation between Euler characteristics of line and Gallai simplicial complexes is established in [2]. Also in [1–3, 23, 25], connectedness of simplicial complexes based on various combinatorial and topological aspects and the shellability [6] of line and anti-Gallai simplicial complexes associated to various classes of graphs are studied. Some basic definitions mentioned are as follows. Let G be a finite simple graph on the vertex set $V(G) = [n]$. Let $E(G) = \{e_{i,j} = \{i, j\} \mid i, j \in V(G)\}$ be the edge set of G.
The set of *line indices* $\gamma(G)$ associated to the graph G is defined as the collection of subsets of $V(G)$ such that if $e_{i,j}$ and $e_{j,k}$ are adjacent in $L(G)$, then $F_{i,j,k} = \{i, j, k\} \in \gamma(G)$ or if $e_{i,j}$ is an isolated vertex in $L(G)$ then $F_{i,j} = \{i, j\} \in \gamma(G)$. The line index is said to be a *Gallai index* if the incident edges $e_{i,j}$ and $e_{j,k}$ of G do not span a triangle in G. The set of Gallai indices are denoted by $\Omega_\Gamma(G)$. The line index is said to be an *anti-Gallai index* if the incident edges $e_{i,j}$ and $e_{j,k}$ of G lie on a triangle in G. The set of anti-Gallai indices are denoted by $\Omega'_\Gamma(G)$. The set of line indices $\gamma(G)$ contains $\Omega_\Gamma(G)$ and $\Omega'_\Gamma(G)$ as spanning subsets.

Theorem 38 ([2]). *Let $\Delta_L(G)$ and $\Delta_\Gamma(G)$ be line and Gallai simplicial complexes of a finite simple graph G. Then, the Euler characteristic of the line simplicial complex $\Delta_L(G)$ is given by $\chi(\Delta_L(G)) = \chi(\Delta_\Gamma(G)) + |\Omega_{\Gamma'}(G)|$, where $|\Omega_{\Gamma'}(G)|$ is the number of anti-Gallai indices associated to G.*

Theorem 39 ([2]). *The anti-Gallai simplicial complex $\Delta_{\Gamma'}(W_{n+1})$ associated to wheel graph W_{n+1} is shellable.*

Proposition 21 ([2]). *Let G be a finite simple connected graph. Then the Gallai simplicial complex $\Delta_\Gamma(G)$ of G is one dimensional if and only if $\Delta_\Gamma(G) = G$.*

Lemma 8 ([2]). *Let G be a simple connected graph, then its Gallai simplicial complex $\Delta_\Gamma(G)$ will be connected.*

Theorem 40 ([2]). *The face ring of Gallai simplicial complex $\Delta_\Gamma(T)$ associated to a tree T is Cohen-Macaulay.*

Theorem 41 ([3]). *Let $\Delta_\Gamma(L_n^*)$ be the Gallai simplicial complex of triangular ladder graph L_n^* with $2n$ vertices for $n \geq 3$. Then, the Euler characteristic of $\Delta_\Gamma(L_n^*)$ is $\chi(\Delta_\Gamma(L_n^*)) = \sum_{k=0}^{N} = (-1)^k f_k = 0$.*

Theorem 42 ([3]). *Let $\Delta_\Gamma(Y_{3,n})$ be the Gallai-simplicial complex of prism graph $Y_{3,n}$, with $3n$ vertices for $n \geq 3$. Then, the Euler characteristic of $\Delta_\Gamma(Y_3, n)$ is $\chi(\Delta_\Gamma(Y_3, n)) = \sum_{k=0}^{n} = (-1)^k f_k = 3(n-1)$.*

4.7. Gallai-Mortal graph

A graph G is said to be Gallai-mortal if $\Gamma^t(G)$ is empty for some $t \geq 0$; where $\Gamma^t(G)$ is the iterated Gallai graph of G. Some of its characterizations can be seen in [20]. Also, its main result is as follows.

Theorem 43. *A finite graph G is Gallai-mortal if and only if*

1. *for any integer $t \geq 0$, $\Gamma^t(G)$ is chordal.*
2. *G contains an induced path of maximum length.*
3. *G contains a neighborhood chain of maximum length.*

5. Krausz-Type characterizations

The notion of simplices, generalised line graphs, Gallai graphs and anti-Gallai graphs give rise to k- line graphs, k- Gallai graph and the k-anti-Gallai graph of a graph G having the k- simplices of G as vertices [21]. For an integer $k > 1$, the vertices of the k-Gallai graph $\Gamma_k(G)$ represent the k-cliques of G, moreover its two vertices are adjacent if and only if the corresponding k-cliques of G share a $(k-1)$-clique but they are not contained in a common $(k+1)$-clique. For an integer $k > 1$, the k-anti-Gallai graph $\Delta_k(G)$ of a graph G has one vertex for each k-clique of G, and two vertices in $\Delta_k(G)$ are adjacent if and only if the union of the two k-cliques represented by them span a $(k+1)$-clique in G [16]. The following are the results obtained with respect to the k- Gallai graphs and the k-anti-Gallai graphs.

Theorem 44 ([21]). *Let $k > 1$ be an integer. A graph G is a k-Gallai graph if and only if there is a family $G_i : i \in I$ of induced subgraphs of G satisfying the following properties.*

1. *Every vertex of G lies in exactly k members of the family.*
2. *Every edge of G lies in exactly $k-1$ members of the family.*
3. *$|\bigcap_{i\in J} G_i| \leq 1$ for any k-element subset J of I.*
4. *The family has the k-Helly property.*
5. *For any $k-1$-element subset A of I and any $i \neq j \in I\backslash A$ and vertices $x \neq y$ of G with $\{x\} = G_i \cap \bigcap_{a\in A} G_a$ and $\{y\} = G_j \cap \bigcap_{a\in A} G_a$, $xy \in E(G)$ if and only if $G_i \cap G_j \neq \phi$ holds true.*

The following result is an immediate application of Theorem 44.

Remark 1 ([21]). The following statements are equivalent for every graph G.

1. Every induced subgraph of G is a Gallai graph.
2. G has no $K_{1,3}$ and no K_4^-; $K_4 - e$.

3. G is a Gallai graph of a triangle-free graph (and thus, G is a line graph).

Lemma 9 ([16]). *For every $k > 2$, every graph G and its k-anti-Gallai graph $\Delta_k(G)$ satisfy the following relations:*

1. *G is K_{k+2}-free if and only if $\Delta_k(G)$ is diamond-free.*
2. *G is K_{k+2}-free if and only if each maximal clique of $\Delta_k(G)$ is either an isolated vertex or a $(k+1)$-clique. Moreover, any two maximal cliques intersect in at most one vertex.*

6. NP-Complete problems

Theorem 45 ([16]). *The following problems are NP-complete for every $k > 3$.*

1. *Recognizing k-anti-Gallai graphs.*
2. *Recognizing k-anti-Gallai graphs on the class of connected and diamond-free graphs.*
3. *Deciding whether a given connected graph is a k-anti-Gallai graph of a K_{k+2}-free graph.*

Corollary 8 ([21]). *The problems of determining the clique number and the chromatic number of a Gallai graphs are NP-complete.*

7. Conjectures and open problems

We list some of the conjectures and open problems for further research.

7.1. Conjectures

Conjecture 1 ([22]). Gallai graphs without induced odd holes are perfect.

If the Conjecture is true, it implies Sun's Theorem [21]. Also, it is easy to see that the Conjecture implies a theorem of König, which tells that the line graphs of bipartite graphs are perfect. In [21], the Conjecture is proved for Gallai graphs $\Gamma(G)$ of graphs G with $\chi(G) \geq 4$.

Conjecture 2. Weakly Gallai perfect graphs are perfect.

Conjecture 3 ([22]). A graph G is perfect if and only if, for all induced subgraphs $G^{'}$ of G, $\chi(\Gamma(G^{'})) \leq \omega(G^{'c})$ and $\chi(\Gamma(G^{'c})) \leq \omega(G^{'})$.

7.2. Open problems

Problem 1. Characterize Gallai graphs.

Problem 2. Characterize anti-Gallai graphs.

Problem 3. Characterize graphs for which $\Gamma(G) \cong \Delta(G)$ other than those mentioned in [26].

Problem 4. Characterize graphs for which $\Gamma(\Delta(G)) \cong \Delta(\Gamma(G))$.

Problem 5. Characterize Gallai and anti-Gallai mortal graphs.

Problem 6. Study Gallai and anti-Gallai periodic graphs.

Problem 7 ([16]). Determine the time complexity of the recognition problem of k-Gallai graphs for $k > 2$.

8. Conclusion

Research on Gallai and anti-Gallai graphs has been escalated after Le's [20–22] studies. We hope that further studies on Gallai and anti-Gallai graphs can still pave way for the substantial growth in its research by working on various other parameters and characterizations.

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