Signed total Italian $k$-domination in graphs

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Abstract: Let $k \geq 1$ be an integer, and let $G$ be a finite and simple graph with vertex set $V(G)$. A signed total Italian $k$-dominating function (STIkDF) on a graph $G$ is a function $f: V(G) \to \{-1, 1, 2\}$ satisfying the conditions that $\sum_{x \in N(v)} f(x) \geq k$ for each vertex $v \in V(G)$, where $N(v)$ is the neighborhood of $v$, and each vertex $u$ with $f(u) = -1$ is adjacent to a vertex $v$ with $f(v) = 2$ or to two vertices $w$ and $z$ with $f(w) = f(z) = 1$. The weight of an STIkDF $f$ is $\omega(f) = \sum_{v \in V(G)} f(v)$. The signed total Italian $k$-domination number $\gamma_{stI}^k(G)$ of $G$ is the minimum weight of an STIkDF on $G$. In this paper we initiate the study of the signed total Italian $k$-domination number of graphs, and we present different bounds on $\gamma_{stI}^k(G)$. In addition, we determine the signed total Italian $k$-domination number of some classes of graphs. Some of our results are extensions of well-known properties of the signed total Roman $k$-domination number $\gamma_{stR}^k(G)$, introduced and investigated by Volkmann [9, 12].

Keywords: Signed total Italian $k$-dominating function, Signed total Italian $k$-domination number, Signed total Roman $k$-dominating function, Signed total Roman $k$-domination number.

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1. Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [4]. Specifically, let $G$ be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the order and the size of the graph $G$, respectively. The open neighborhood of a vertex $v$ is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the closed neighborhood of $v$ is $N_G[v] = N[v] = N(v) \cup \{v\}$. The degree of a vertex $v$ is $d_G(v) = d(v) = |N(v)|$. The minimum and maximum degree of a graph $G$ are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. For a set $X \subseteq V(G)$, its open neighborhood is the set $N_G(X) = N(X) = \{v \in V(G) | N(v) \cap X \neq \emptyset\}$.
$\bigcup_{v \in X} N(v)$, and its closed neighborhood is the set $N_G[X] = N[X] = N(X) \cup X$. The complement of a graph $G$ is denoted by $\overline{G}$. A leaf of a graph $G$ is a vertex of degree 1, while a support vertex of $G$ is a vertex adjacent to a leaf. Denote by $S(G)$ the set of support vertices of $G$. An edge incident with a leaf is called a pendant edge. Let $P_n$, $C_n$ and $K_n$ be the path, cycle and complete graph of order $n$. A complete bipartite graph with partite sets $X$ and $Y$, where $|X| = p$ and $|Y| = q$ is denoted by $K_{p,q}$. Specifically, $K_{1,q}$ is called a star. Let $S(r, s)$ be the double star with exactly two adjacent vertices $u$ and $v$ that are not leaves such that $u$ is adjacent to $r \geq 1$ leaves and $v$ is adjacent to $s \geq 1$ leaves. A spider graph is a tree with one vertex of degree at least 3 and all others of degree at most two.

All along this paper we will assume that $k$ is a positive integer. In this paper we continue the study of signed total Roman and signed total Italian dominating functions in graphs and digraphs. For a subset $S \subseteq V(G)$ of vertices of a graph $G$ and a function $f : V(G) \rightarrow \mathbb{R}$, we define $f(S) = \sum_{x \in S} f(x)$.

If $k \geq 1$ is an integer, then Volkmann [9] defined the signed total Roman $k$-dominating function (STR$k$DF) on a graph $G$ as a function $f : V(G) \rightarrow \{-1, 1, 2\}$ such that $f(N(v)) = \sum_{x \in N(v)} f(x) \geq k$ for every $v \in V(G)$, and every vertex $u$ for which $f(u) = -1$ is adjacent to a vertex $v$ for which $f(v) = 2$. The weight of an STR$k$DF $f$ on a graph $G$ is $\omega(f) = \sum_{v \in V(G)} f(v)$. The signed total Roman $k$-domination number $\gamma_{stR}^k(G)$ of $G$ is the minimum weight of an STR$k$DF on $G$. The special case $k = 1$ was introduced and investigated by Volkmann [12] and [7]. A $\gamma_{stR}^k(G)$-function is a signed total Roman $k$-dominating function on $G$ of weight $\gamma_{stR}^k(G)$. Amjadi and Soroudi [1], Dehgardi and Volkmann [3] and Volkmann [8] studied the signed total Roman domination number in digraphs.

A signed total Italian $k$-dominating function (STI$k$DF) on a graph $G$ is defined as a function $f : V(G) \rightarrow \{-1, 1, 2\}$ having the property $f(N(v)) \geq k$ for every $v \in V(G)$, and each vertex $u$ with $f(u) = -1$ is adjacent to a vertex $v$ with $f(v) = 2$ or to two vertices $w$ and $z$ with $f(w) = f(z) = 1$. Note that in the case $k \geq 2$, the second condition is superfluous. The weight of an STI$k$DF $f$ is $\omega(f) = \sum_{v \in V(G)} f(v)$. The signed total Italian $k$-domination number $\gamma_{stI}^k(G)$ of $G$ is the minimum weight of an STI$k$DF on $G$. The special case $k = 1$ was introduced and investigated by Volkmann [6]. A $\gamma_{stI}^k(G)$-function is a signed total Italian $k$-dominating function on $G$ of weight $\gamma_{stI}^k(G)$. For an STI$k$DF $f$ on $G$, let $V_i = V_i(f) = \{v \in V(G) : f(v) = i\}$ for $i = -1, 1, 2$. A signed total Italian $k$-dominating function $f : V(G) \rightarrow \{-1, 1, 2\}$ can be represented by the ordered partition $(V_{-1}, V_1, V_2)$ of $V(G)$.

The signed total Italian $k$-domination number exists when $\delta \geq \frac{k}{2}$. The definitions lead to $\gamma_{stI}^k(G) \leq \gamma_{stR}^k(G)$. Therefore each lower bound of $\gamma_{stI}^k(G)$ is also a lower bound of $\gamma_{stR}^k(G)$. Note that the signed total Italian $k$-domination number is the total version of the weak signed Roman $k$-domination number, see Volkmann [10, 11].

For an integer $q \geq 1$, Kulli [5] called a subset $D$ of vertices of a graph $G$ a total $q$-dominating set if every vertex $x \in V(G)$ has at least $q$ neighbors in $D$. The total $q$-domination number $\gamma_{tq}(G)$ is the minimum cardinality of a total $q$-dominating set.
of $G$. The special case $q = 1$ is the usual total domination number $\gamma_t(G)$, introduced by Cockayne, Dawes and Hedetniemi [2].

Our purpose in this work is to initiate the study of the signed total Italian $k$-domination number. We present basic properties and sharp bounds on $\gamma_{stI}^k(G)$. In particular, we show that many lower bounds on $\gamma_{stR}^k(G)$ are also valid for $\gamma_{stI}^k(G)$. Some of our results are extensions of well-known properties of the signed total Roman $k$-domination number and the signed total Italian domination number $\gamma_{stI}^1(G) = \gamma_{stI}^{1}(G)$, given by Volkmann [6, 9, 12].

2. Preliminary results

In this section we present basic properties of the signed total Italian $k$-dominating functions and the signed total Italian $k$-domination numbers.

**Proposition 1.** Let $k \geq 1$ be an integer, and let $G$ be a graph of order $n$ with $\delta(G) \geq \lceil k/2 \rceil$. If $f = (V_-1, V_1, V_2)$ is an STIkDF on $G$, then

(a) $|V_-1| + |V_1| + |V_2| = n$.

(b) $\omega(f) = |V_1| + 2|V_2| - |V_-1|$.

(c) If $\delta(G) \geq \lceil (k+s)/2 \rceil$ with $s \in \{0, 1\}$, then $V_1 \cup V_2$ is a total $\lceil (k+s)/2 \rceil$-dominating set of $G$.

**Proof.** Since (a) and (b) are immediate, we only prove (c). Suppose on the contrary, that there exists a vertex $v$ with at most $\lceil (k+s)/2 \rceil - 1$ neighbors in $V_1 \cup V_2$. Then $v$ has at least

$$\delta(G) - \lceil (k+s)/2 \rceil - 1 \geq \lceil (k+s)/2 \rceil - (\lceil (k+s)/2 \rceil - 1) = 1$$

neighbors in $V_-1$. Hence the definition implies the contradiction

$$k \leq f(N(v)) \leq 2(\lceil (k+s)/2 \rceil - 1) - 1 \leq \frac{2(k+s+1)}{2} - 3 = k + s - 2 \leq k - 1.$$  

Consequently, $V_1 \cup V_2$ is a total $\lceil (k+s)/2 \rceil$-dominating set of $G$. 

**Corollary 1.** If $G$ is a graph of order $n$ and minimum degree $\delta \geq \lceil \frac{k+s}{2} \rceil$ with $s \in \{0, 1\}$, then $\gamma_{stI}^k(G) \geq 2\gamma_t^{\lceil \frac{k+s}{2} \rceil} - n$.

**Proof.** Let $f = (V_-1, V_1, V_2)$ be a $\gamma_{stI}^k(G)$-function. Then it follows from Proposition 1 that

$$\gamma_{stI}^k(G) = |V_1| + 2|V_2| - |V_-1| = 2|V_1| + 3|V_2| - n \geq 2|V_1 \cup V_2| - n \geq 2\gamma_t^{\lceil \frac{k+s}{2} \rceil} - n.$$  

The graphs $qK_2$ and $qK_3$ show that Corollary 1 is sharp for $k = 1$ and $k = 2$. The proof of the next proposition is identically with the proof of Proposition 2 in [9] and is therefore omitted.

**Proposition 2.** Let $k \geq 1$ be an integer, and let $G$ be a graph of order $n$ with minimum degree $\delta \geq k/2$ and maximum degree $\Delta$. If $f = (V_1, V_2)$ is an STIkDF on $G$, then

(i) $(2\Delta - k)|V_2| + (\Delta - k)|V_1| \geq (\delta + k)|V_1|.$

(ii) $(2\Delta + \delta)|V_2| + (\Delta + \delta)|V_1| \geq (\delta + k)n.$

(iii) $(\Delta + \delta)\omega(f) \geq (\delta - \Delta + 2k)n + (\delta - \Delta)|V_2|.$

(iv) $\omega(f) \geq (\delta - 2\Delta + 2k)n/(2\Delta + \delta) + |V_2|.$

3. **Bounds on the signed total Italian $k$-domination number**

We start with a general upper bound, and we characterize all extremal graphs.

**Theorem 1.** Let $G$ be a graph of order $n$ with $\delta(G) \geq \lceil \frac{k}{2} \rceil$. Then $\gamma_{stI}^k(G) \leq 2n$, with equality if and only if $k$ is even, $\delta(G) = \frac{k}{2}$, and each vertex of $G$ is adjacent to a vertex of minimum degree.

**Proof.** Define the function $g : V(G) \rightarrow \{-1, 1, 2\}$ by $g(x) = 2$ for each vertex $x \in V(G)$. Since $\delta(G) \geq \lceil \frac{k}{2} \rceil$, the function $g$ is an STIkDF on $G$ of weight $2n$ and thus $\gamma_{stI}^k(G) \leq 2n$.

Now let $k$ be even, $\delta(G) = \frac{k}{2}$, and assume that each vertex of $G$ is adjacent to a vertex of minimum degree. Let $f$ be an STIkDF on $G$, and let $x \in V(G)$ be an arbitrary vertex. Then $x$ has a neighbor $v$ with $d(v) = \frac{k}{2}$. Therefore the condition $f(N(v)) \geq k$ implies $f(x) = 2$. Thus $f$ is of weight $2n$, and we obtain $\gamma_{stI}^k(G) = 2n$.

Conversely, assume that $\gamma_{stI}^k(G) = 2n$. If $k = 2p + 1$ is odd, then $\delta(G) \geq p + 1$. Define the function $h : V(G) \rightarrow \{-1, 1, 2\}$ by $h(w) = 1$ for an arbitrary vertex $w$ and $h(x) = 2$ for each vertex $x \in V(G) \setminus \{w\}$. Then

$$h(N(v)) = \sum_{x \in N(v)} f(x) \geq 2(p + 1) - 1 = 2p + 1 = k$$

for each vertex $v \in V(G)$. Thus the function $h$ is an STIkDF on $G$ of weight $2n - 1$, and we obtain the contradiction $\gamma_{stI}^k(G) \leq 2n - 1$.

Let now $k$ even, and assume that there exists a vertex $w$ such that $d(x) \geq \frac{k}{2} + 1$ for each $x \in N(w)$. Define the function $h_1 : V(G) \rightarrow \{-1, 1, 2\}$ by $h_1(w) = 1$ and $h_1(x) = 2$ for each vertex $x \in V(G) \setminus \{w\}$. Then $h_1(N(w)) \geq k$, $h_1(N(x)) \geq 2(\frac{k}{2} + 1) - 1 = k + 1$ for each $x \in N(w)$ and $h_1(N(y)) \geq k$ for each $y \notin N[w]$. Hence the function $h_1$ is an STIkDF on $G$ of weight $2n - 1$, a contradiction to the assumption $\gamma_{stI}^k(G) = 2n$. This completes the proof. □
The proof of Theorem 1 also leads to the next result.

**Theorem 2.** Let $G$ be a graph of order $n$ with $\delta(G) \geq \lceil \frac{k}{2} \rceil$. Then $\gamma_{\text{stR}}^k(G) \leq 2n$, with equality if and only if $k$ is even, $\delta(G) = \frac{k}{2}$, and each vertex of $G$ is adjacent to a vertex of minimum degree.

Let $k \geq 2$ be an even integer. If $H$ is a $\frac{k}{2}$-regular graph of order $n$, then Theorems 1 and 2 imply that $\gamma_{\text{stI}}^k(H) = \gamma_{\text{stR}}^k(H) = 2n$. The next corollaries follow from Theorems 1 and 2 immediately.

**Corollary 2.** Let $G$ be a connected graph of order $n \geq 2$ with $\delta(G) = 1$. Then $\gamma_{\text{stI}}^2(G) = 2n$ if and only if $n = 2$.

**Corollary 3.** Let $G$ be a connected graph of order $n \geq 2$ with $\delta(G) = 1$. Then $\gamma_{\text{stR}}^2(G) = 2n$ if and only if $n = 2$.

**Observation 3.** If $G$ is a graph of order $n$ with $\delta(G) \geq k$, then $\gamma_{\text{stI}}^k(G) \leq \gamma_{\text{stR}}^k(G) \leq n$.

*Proof.* Define the function $f : V(G) \rightarrow \{-1, 1, 2\}$ by $f(x) = 1$ for each vertex $x \in V(G)$. Since $\delta(G) \geq k$, the function $f$ is an STRkDF on $G$ of weight $n$ and thus $\gamma_{\text{stI}}^k(G) \leq \gamma_{\text{stR}}^k(G) \leq n$. \qed

As an application of Proposition 2 (iii), we obtain a lower bound on the signed total Italian $k$-domination number for $r$-regular graphs.

**Corollary 4.** If $G$ is an $r$-regular graph of order $n$ with $r \geq \frac{k}{2}$, then

$$\gamma_{\text{stI}}^k(G) \geq \frac{kn}{r}.$$  

**Example 1.** If $H$ is a $k$-regular graph of order $n$, then it follows from Corollary 4 that $\gamma_{\text{stI}}^k(H) \geq n$ and thus $\gamma_{\text{stI}}^k(H) = n$, according to Observation 3.

Example 1 shows that Observation 3 and Corollary 4 are both sharp.

**Theorem 4.** If $G$ is a graph of order $n$ with $\delta(G) \geq \frac{k}{2}$, then

$$\gamma_{\text{stI}}^k(G) \geq k + 2 + \delta(G) - n.$$  

If in addition $\delta(G) - k$ is odd, then $\gamma_{\text{stI}}^k(G) \geq k + 3 + \delta(G) - n$. 

Proof. Let $f$ be a $\gamma^k_{stI}(G)$-function. Then there exists a vertex $w$ with $f(w) \geq 1$. It follows from the definitions that

$$
\gamma^k_{stI}(G) = \sum_{x \in V(G)} f(x) = f(w) + \sum_{x \in N(w)} f(x) + \sum_{x \in V(G) - N[w]} f(x)
\geq 1 + k - (n - d(w) - 1) = k + 2 + d(w) - n \geq k + 2 + \delta(G) - n.
$$

Now assume that $\delta(G) - k$ is odd. If there exists a vertex $w$ with $f(w) = 2$ or a vertex $v$ with $f(v) = 1$ and $d(v) \geq \delta(G) + 1$, then the inequality chain above leads to $\gamma^k_{stI}(G) \geq k + 3 + \delta(G) - n$. So assume that $f(x) \in \{-1, 1\}$ for each vertex $x \in V(G)$, and each vertex $y$ with $f(y) = 1$ has degree $d(y) = \delta(G)$. Let now $u$ be a vertex of minimum degree with $f(u) = 1$. Since $\delta(G) - k$ is odd, we observe that

$$
\sum_{x \in N(u)} f(x) \geq k + 1
$$

and thus

$$
\gamma^k_{stI}(G) = \sum_{x \in V(G)} f(x) = f(u) + \sum_{x \in N(u)} f(x) + \sum_{x \in V(G) - N[u]} f(x)
\geq 1 + k + 1 - (n - d(u) - 1) = k + 3 + d(u) - n = k + 3 + \delta(G) - n.
$$

This completes the proof. \qed

Example 2. If $k \geq 1$ and $n \geq 2$ are integers such that $2n - 2 \geq k$, then it holds:

(i) If $k \geq n$, then $\gamma^k_{stI}(K_n) = k + 2$.

(ii) If $k \leq n - 1$ and $n - k$ is odd, then $\gamma^k_{stI}(K_n) = k + 1$.

(iii) If $k \leq n - 1$ and $n - k$ is even, then $\gamma^k_{stI}(K_n) = k + 2$.

Proof. Let $f$ be a $\gamma^k_{stI}(K_n)$-function.

(i) Since $k \geq n$, there exists a vertex $w$ with $f(w) \geq 2$. This implies $\gamma^k_{stI}(K_n) = f(w) + f(N(w)) \geq 2 + k$.

For $\gamma^k_{stI}(K_n) \leq k + 2$, let the function $g : V(K_n) \rightarrow \{-1, 1, 2\}$ assign to $2n - k - 2$ vertices the value 1 and to the remaining $k + 2 - n$ vertices the value 2. Then $g$ is an STlkDF on $K_n$ of weight $\omega(g) = k + 2$ and so $\gamma^k_{stI}(K_n) \leq k + 2$. Thus $\gamma^k_{stI}(K_n) = k + 2$ in this case.

(ii) Since there exists a vertex $w$ with $f(w) \geq 1$, we note that $\gamma^k_{stI}(K_n) = f(w) + f(N(w)) \geq 1 + k$.

For $\gamma^k_{stI}(K_n) \leq k + 1$, let the function $g : V(K_n) \rightarrow \{-1, 1, 2\}$ assign to $\frac{n + k + 1}{2}$ vertices the value 1 and to the remaining $\frac{n - k - 1}{2}$ vertices the value -1. Then $g$ is an STlkDF on $K_n$ of weight $\omega(g) = k + 1$ and so $\gamma^k_{stI}(K_n) \leq k + 1$. Thus $\gamma^k_{stI}(K_n) = k + 1$ in the second case.
(iii) If there exists a vertex $w$ with $f(w) = 2$, then it follows that $\gamma^k_{stI}(K_n) = f(w) + f(N(w)) \geq 2 + k$. If $f(x) \in \{-1, 1\}$ for all $x \in V(K_n)$, then $f(N(x))$ is even when $n$ is odd and $f(N(x))$ is odd when $n$ is even. Since $f(N(x)) \geq k$, we observe that $f(N(x)) \geq k + 1$ when $n - k$ is even. If $w$ is a vertex with $f(w) = 1$, we therefore deduce that $\gamma^k_{stI}(K_n) \geq 1 + k + 1 = k + 2$.

For $\gamma^k_{stI}(K_n) \leq k + 2$, let the function $g : V(K_n) \rightarrow \{-1, 1, 2\}$ assign to $\frac{n + k + 2}{2}$ vertices the value 1 and to the remaining $\frac{n - k - 2}{2}$ vertices the value -1. Then $g$ is an STIkDF on $K_n$ of weight $\omega(g) = k + 2$ and so $\gamma^k_{stI}(K_n) \leq k + 2$. Thus $\gamma^k_{stI}(K_n) = k + 2$ in the last case.

Example 2 (ii) and (iii) show that Theorem 4 is sharp for $k \leq n - 1$. If $\Delta(G) \geq \delta(G) + 3$, then the next lower bound is an improvement of Theorem 4.

**Proposition 3.** If $G$ is a graph of order $n$ with $\delta(G) \geq \frac{k}{2}$, then

$$\gamma^k_{stI}(G) \geq k + \Delta(G) - n.$$  

**Proof.** Let $w \in V(G)$ be a vertex of maximum degree, and let $f$ be a $\gamma^k_{stI}(G)$-function. Then the definitions imply

$$\gamma^k_{stI}(G) = \sum_{x \in V(G)} f(x) = \sum_{x \in N(w)} f(x) + \sum_{x \in V(G) - N(w)} f(x) \geq k + \sum_{x \in V(G) - N(w)} f(x) \geq k - (n - \Delta(G)) = k + \Delta(G) - n,$$

and the proof of the desired lower bound is complete. 

**Corollary 5.** Let $G$ be a graph of order $n$, minimum degree $\delta \geq \frac{k}{2}$ and maximum degree $\Delta$. If $\delta < \Delta$, then

$$\gamma^k_{stI}(G) \geq \left\lceil \frac{2\delta + 3k - 2\Delta}{2\Delta + \delta} \right\rceil n.$$

**Proof.** Multiplying both sides of the inequality in Proposition 2 (iv) by $\Delta - \delta$ and adding the resulting inequality to the inequality in Proposition 2 (iii), we obtain the desired lower bound.

Since $\gamma^k_{stR}(G) \geq \gamma^k_{stI}(G)$, Corollary 5 leads to the next known lower bound immediately.

**Corollary 6.** ([9]) Let $G$ be a graph of order $n$, minimum degree $\delta \geq \frac{k}{2}$ and maximum degree $\Delta$. If $\delta < \Delta$, then

$$\gamma^k_{stR}(G) \geq \left\lceil \frac{2\delta + 3k - 2\Delta}{2\Delta + \delta} \right\rceil n.$$
Examples 12 and 13 in [9] demonstrate that Corollary 6 is sharp and therefore Corollary 5 too. The special case \( k = 1 \) of Corollaries 4 and 5 can be found in [6]. A set \( S \subseteq V(G) \) is a 2-packing of the graph \( G \) if \( N[u] \cap N[v] = \emptyset \) for any two distinct vertices \( u, v \in S \). The 2-packing number \( \rho(G) \) of \( G \) is defined by

\[
\rho(G) = \max \{|S| : S \text{ is a 2-packing of } G\}.
\]

Analogously to Theorem 14 in [9], one can prove the next lower bound on the signed total Italian \( k \)-domination number.

**Theorem 5.** If \( G \) is a graph of order \( n \) with \( \delta(G) \geq \frac{k}{2} \), then

\[
\gamma_{stI}^k(G) \geq \rho(G)(k + \delta(G)) - n.
\]

**Corollary 7.** ([9]) If \( G \) is a graph of order \( n \) with \( \delta(G) \geq \frac{k}{2} \), then

\[
\gamma_{stR}^k(G) \geq \rho(G)(k + \delta(G)) - n.
\]

In [9], the author presented an infinite family of graphs achieving equality in Corollary 7. Thus Corollary 7 and Theorem 5 are sharp. Using Corollary 4, one can prove the following Nordhaus-Gaddum type inequality analogously to Theorem 15 in [9].

**Theorem 6.** If \( G \) is an \( r \)-regular graph of order \( n \) such that \( r \geq \frac{k}{2} \) and \( n - r - 1 \geq \frac{k}{2} - 1 \), then

\[
\gamma_{stI}^k(G) + \gamma_{stI}^k(G) \geq \frac{4kn}{n-1}.
\]

If \( n \) is even, then

\[
\gamma_{stI}^k(G) + \gamma_{stI}^k(G) \geq 4k(n-1)/(n-2).
\]

Let \( k \geq 2 \) be an even integer, and let \( H \) and \( \overline{H} \) be \( k \)-regular graphs of order \( n = 2k+1 \). By Example 1, we have \( \gamma_{stI}^k(H) = \gamma_{stI}^k(\overline{H}) = n \). Consequently,

\[
\gamma_{stI}^k(H) + \gamma_{stI}^k(\overline{H}) = 2n = \frac{4kn}{n-1}.
\]

Thus the Nordhaus-Gaddum bound of Theorem 6 is sharp for even \( k \).

**Theorem 7.** Let \( G \) be connected graph of order \( n \geq 3 \) and let \( S(G) \) be the set of its support vertices. Then \( \gamma_{stI}^2(G) \leq \gamma_{stR}^2(G) \leq n + |S(G)| \).

**Proof.** Define the function \( f : V(G) \to \{-1, 1, 2\} \) by \( f(x) = 2 \) when \( x \) is a support vertex and \( f(x) = 1 \) otherwise. If \( v \) is a leaf, then \( v \) is adjacent to a support vertex and so \( f(N(v)) = 2 \). If \( v \) is not a leaf, then \( d(v) \geq 2 \) and thus \( f(N(v)) \geq 2 \). Therefore \( f \) is an STR2DF on \( G \) of weight \( n + |S(G)| \) and hence \( \gamma_{stI}^2(G) \leq \gamma_{stR}^2(G) \leq n + |S(G)| \).
Example 3. For $n \geq 4$, we have $\gamma_{stR}^2(P_n) = \gamma_{stI}^2(P_n) = n + 2$.

Proof. Let $P_n = v_1v_2\ldots v_n$, and let $f$ be a $\gamma_{stI}^2(P_n)$-function. According to Theorem 7, we have $\gamma_{stI}^2(P_n) \leq \gamma_{stR}^2(P_n) \leq n + 2$. Conversely, it is easy to see that $f(v_i) \geq 1$ for $1 \leq i \leq n$ and $f(v_2) = f(v_{n-1}) = 2$. Therefore $\gamma_{stR}^2(P_n) \geq \gamma_{stI}^2(P_n) \geq n + 2$ and thus $\gamma_{stR}^2(P_n) = \gamma_{stI}^2(P_n) = n + 2$.

Since $\gamma_{stR}^2(P_3) = \gamma_{stI}^2(P_3) = 4$, the condition $n \geq 4$ in Example 3 is necessary.

Example 4. Let $H$ be a spider graph, and let $w \in V(H)$ with $d(w) \geq 3$. If every leaf of $H$ has distance at least two from $w$, then $\gamma_{stR}^2(H) = \gamma_{stI}^2(H) = n + |S(H)|$.

Proof. According to Theorem 7, we have $\gamma_{stI}^2(H) \leq \gamma_{stR}^2(H) \leq n + |S(H)|$. Conversely, let $f$ be a $\gamma_{stI}^2(H)$-function. Since every leaf of $H$ has distance at least two from $w$, we note that $f(x) \geq 1$ for $x \in V(H)$. In addition, we have $f(x) = 2$ for each $x \in S(H)$. Therefore $\gamma_{stR}^2(H) \geq \gamma_{stI}^2(H) \geq n + |S(H)|$ and thus $\gamma_{stR}^2(H) = \gamma_{stI}^2(H) = n + |S(H)|$.

Examples 3 and 4 demonstrate that Theorem 7 is sharp.

Example 5. If $n \geq 2$, then $\gamma_{stI}^2(K_{1,n-1}) = \gamma_{stR}^2(K_{1,n-1}) = 4$.

Proof. Let $G = K_{1,n-1}$, and let $f$ be a $\gamma_{stI}^2(G)$-function. If $n = 2$, then the result is obviously. Let now $n \geq 3$, and let $w$ be the central vertex of the star $G$. Clearly, $f(w) = 2$. Hence it follows that

$$\gamma_{stR}^2(G) \geq \gamma_{stI}^2(G) = f(w) + f(N(w)) \geq 2 + 2 = 4.$$  

For the converse inequality $\gamma_{stI}^2(G) \leq \gamma_{stR}^2(G) \leq 4$, let $v_1,v_2,\ldots,v_{n-1}$ be the leaves of $G$. If $n - 1 = 2p$ is even, define $g : V(G) \rightarrow \{-1,1,2\}$ by $g(w) = 2$, $g(v_i) = 1$ for $1 \leq i \leq p + 1$ and $g(v_i) = -1$ for $p + 2 \leq i \leq 2p$. Then $g$ is an STR2DF on $G$ of weight 4 and thus $\gamma_{stI}^2(G) \leq \gamma_{stR}^2(G) \leq 4$ in this case. If $n = 2p + 1$ is odd, then define $h : V(G) \rightarrow \{-1,1,2\}$ by $h(w) = h(v_{2p+1}) = 2$, $h(v_i) = 1$ for $1 \leq i \leq p$ and $h(v_i) = -1$ for $p + 1 \leq i \leq 2p$. Then $h$ is an STR2DF on $G$ of weight 4 and thus $\gamma_{stI}^2(G) \leq \gamma_{stR}^2(G) \leq 4$ also in this case.

Example 5 shows that $\gamma_{stR}^2(H) = \gamma_{stI}^2(H) = n + |S(H)|$ is not valid for each spider graph $H$. Therefore the condition that every leaf of $H$ has distance at least two from $w$ is important in Example 4.
4. Special classes of graphs

Let $C_n$ be a cycle of length $n \geq 3$. In [6], the author has shown that $\gamma_{stI}(C_n) = n/2$ when $n \equiv 0 \pmod{4}$, $\gamma_{stI}(C_n) = (n + 3)/2$ when $n \equiv 1, 3 \pmod{4}$ and $\gamma_{stI}(C_n) = (n + 6)/2$ when $n \equiv 2 \pmod{4}$. Now we determine $\gamma_{stI}^k(C_n)$ as well as $\gamma_{stR}^k(C_n)$ for $2 \leq k \leq 4$.

Corollary 4 and Observation 3 immediately lead to $\gamma_{stI}^2(C_n) = \gamma_{stI}^2(C_n) = n$. In addition, Theorems 1 and 2 imply $\gamma_{stI}^4(C_n) = \gamma_{stR}^4(C_n) = 2n$.

**Example 6.** For $n \geq 3$, we have $\gamma_{stR}^3(C_n) = \gamma_{stI}^3(C_n) = \lceil \frac{3n}{2} \rceil + 1$ when $n \equiv 2 \pmod{4}$ and $\gamma_{stR}^3(C_n) = \gamma_{stI}^3(C_n) = \lceil \frac{3n}{2} \rceil$ otherwise.

**Proof.** Let $C_n = v_1v_2 \ldots v_nv_1$. Assume first that $n \equiv 0, 1, 3 \pmod{4}$. Applying Corollary 4, we obtain $\gamma_{stR}^3(C_n) \geq \gamma_{stI}^3(C_n) \geq \lceil \frac{3n}{2} \rceil$.

If $n = 4t$ for an integer $t \geq 1$, then define $g : V(C_n) \rightarrow \{-1, 1, 2\}$ by $g(v_{4i+1}) = g(v_{4i+2}) = 2$ and $g(v_{4i+3}) = g(v_{4i+4}) = 1$ for $0 \leq i \leq t - 1$. Then $g$ is an STR3DF on $C_n$ of weight $\omega(g) = 6t = \frac{3n}{2}$ and so $\gamma_{stI}^3(C_n) \leq \gamma_{stR}^3(C_n) \leq \lceil \frac{3n}{2} \rceil$ and thus $\gamma_{stI}^3(C_n) = \gamma_{stR}^3(C_n) = \lceil \frac{3n}{2} \rceil$ in this case.

If $n = 4t + 1$ for an integer $t \geq 1$, then define $g : V(C_n) \rightarrow \{-1, 1, 2\}$ by $g(v_{4i+1}) = g(v_{4i+2}) = 2$ and $g(v_{4i+3}) = g(v_{4i+4}) = 1$ for $0 \leq i \leq t - 1$ and $g(v_{4t+1}) = 2$. Then $g$ is an STR3DF on $C_n$ of weight $\omega(g) = 6t + 2 = \frac{3n}{2}$ and thus $\gamma_{stR}^3(C_n) = \gamma_{stI}^3(C_n) = \lceil \frac{3n}{2} \rceil$ in this case too.

If $n = 4t + 3$ for an integer $t \geq 1$, then define $g : V(C_n) \rightarrow \{-1, 1, 2\}$ by $g(v_{4i+1}) = g(v_{4i+2}) = 2$ for $0 \leq i \leq t$, $g(v_{4i+3}) = g(v_{4i+4}) = 1$ for $0 \leq i \leq t - 1$ and $g(v_{4t+3}) = 1$. Then $g$ is an STR3DF on $C_n$ of weight $\omega(g) = 6t + 5 = \lceil \frac{3n}{2} \rceil$ and thus $\gamma_{stR}^3(C_n) = \gamma_{stI}^3(C_n) = \lceil \frac{3n}{2} \rceil$ in the third case.

Let now $n = 4t + 2$ for an integer $t \geq 1$. If $f$ is a $\gamma_{stI}^3(C_n)$-function, then we observe that $f(v_i) \geq 1$ for each $1 \leq i \leq n$ and $f(v_1) + f(v_{i+1}) + f(v_{i+2}) + f(v_{i+3}) \geq 6$, where the indices are taken modulo $n$. Assume, without loss of generality, that $f(v_1) = f(v_2) = 2$ and $f(v_3) = f(v_4) = 1$. Then it follows that $f(v_5) = f(v_6) = 2$, and we obtain

$$\omega(f) = f(v_1) + f(v_2) + f(v_3) + f(v_4) + f(v_5) + f(v_6)$$
$$+ \sum_{i=1}^{t-1} (f(v_{4i+3}) + f(v_{4i+4}) + f(v_{4i+5}) + f(v_{4i+6}))$$
$$\geq 10 + 6(t - 1) = 6t + 4.$$ 

Therefore $\omega(f) \geq 6t + 4 = \lceil \frac{3n}{2} \rceil + 1$ and thus $\gamma_{stR}^3(C_n) \geq \gamma_{stI}^3(C_n) \geq \lceil \frac{3n}{2} \rceil + 1$ in this case. Conversely, if we define $g : V(C_n) \rightarrow \{-1, 1, 2\}$ by $g(v_{4i+1}) = g(v_{4i+2}) = 2$ for $0 \leq i \leq t$ and $g(v_{4i+3}) = g(v_{4i+4}) = 1$ for $0 \leq i \leq t - 1$, then $g$ is an STR3DF on $C_n$ of weight $\omega(g) = 6t + 4 = \lceil \frac{3n}{2} \rceil + 1$ and thus $\gamma_{stR}^3(C_n) = \gamma_{stI}^3(C_n) = \lceil \frac{3n}{2} \rceil + 1$ in the last case.
Example 7. If \( k \geq 1 \) and \( q \geq p \geq 2 \) are integers such that \( p \geq k/2 \), then \( \gamma_{st}^k(K_{p,q}) = 2k \).

Proof. Let \( G = K_{p,q} \), and let \( g \) be a \( \gamma_{st}^k(G) \)-function. In addition, let \( X \) and \( Y \) be a bipartition of the complete bipartite graph \( G \) with \( |X| = p \) and \( |Y| = q \). If \( x \in X \) and \( y \in Y \), then

\[
\gamma_{st}^k(G) = g(V(G)) = g(N(x)) + g(N(y)) \geq k + k = 2k.
\]

For the converse inequality we distinguish different cases.

a) Assume that \( k \geq q \). Let the function \( f : V(G) \rightarrow \{-1,1,2\} \) assign to \( 2p - k \) vertices of \( X \) the value 1, to the remaining \( k - p \) vertices of \( X \) the value 2, to \( 2q - k \) vertices of \( Y \) the value 1 and to the remaining \( k - q \) vertices of \( Y \) the value 2. Then \( f \) is an STIkDF on \( G \) of weight \( \omega(f) = 2k \) and so \( \gamma_{st}^k(G) \leq 2k \). Thus \( \gamma_{st}^k(G) = 2k \) in this case.

b) Assume that \( q > k \geq p \) and \( q - k \) is even. Let the function \( f : V(G) \rightarrow \{-1,1,2\} \) assign to \( 2p - k \) vertices of \( X \) the value 1, to the remaining \( k - p \) vertices of \( X \) the value 2, to \( \frac{k+q}{2} \) vertices of \( Y \) the value 1 and to the remaining \( \frac{2-k}{2} \) vertices of \( Y \) the value -1. Then \( f \) is an STIkDF on \( G \) of weight \( \omega(f) = 2k \) and so \( \gamma_{st}^k(G) \leq 2k \). Thus \( \gamma_{st}^k(G) = 2k \) in the second case.

c) Assume that \( q > k \geq p \) and \( q - k \) is odd. Let the function \( f : V(G) \rightarrow \{-1,1,2\} \) assign to \( 2p - k \) vertices of \( X \) the value 1, to the remaining \( k - p \) vertices of \( X \) the value 2, to one vertex of \( Y \) the value 2, to \( \frac{k+q-3}{2} \) vertices of \( Y \) the value 1 and to the remaining \( \frac{q-k+1}{2} \) vertices of \( Y \) the value -1. Then \( f \) is an STIkDF on \( G \) of weight \( \omega(f) = 2k \) and thus \( \gamma_{st}^k(G) = 2k \) in the third case too.

d) Assume that \( k < p \), \( p - k \) is even and \( q - k \) is even. Let the function \( f : V(G) \rightarrow \{-1,1,2\} \) assign to \( \frac{k+p}{2} \) vertices of \( X \) the value 1, to the remaining \( \frac{p-k}{2} \) vertices of \( X \) the value -1, to \( \frac{k+q}{2} \) vertices of \( Y \) the value 1 and to the remaining \( \frac{q-k}{2} \) vertices of \( Y \) the value -1. Then \( f \) is an STIkDF on \( G \) of weight \( \omega(f) = 2k \) and thus \( \gamma_{st}^k(G) = 2k \) also in this case.

e) Assume that \( k < p \), \( p - k \) is odd and \( q - k \) is odd. Let the function \( f : V(G) \rightarrow \{-1,1,2\} \) assign to one vertex of \( X \) the value 2, to \( \frac{k+p-3}{2} \) vertices of \( X \) the value 1, to the remaining \( \frac{k+q-3}{2} \) vertices of \( X \) the value -1, to one vertex of \( Y \) the value 2, to \( \frac{k+q-3}{2} \) vertices of \( Y \) the value 1 and to the remaining \( \frac{q-k+1}{2} \) vertices of \( Y \) the value -1. Then \( f \) is an STIkDF on \( G \) of weight \( \omega(f) = 2k \) and thus \( \gamma_{st}^k(G) = 2k \) also in this case.

The cases \( k < p \), \( p - k \) even and \( q - k \) odd or \( k < p \), \( p - k \) odd and \( q - k \) even are analogously, and are therefore omitted.

Example 6 and Example 7 with \( p = q \) show again that Corollary 4 is sharp.

Example 8. If \( S(r,s) \) is a double star with \( r, s \geq 2 \), then \( \gamma_{st}^2(S(r,s)) = \gamma_{stR}^2(S(r,s)) = 4 \).
Proof. Let $u$ and $v$ be two adjacent vertices of $S(r, s)$ such that $u$ is adjacent to $r$ leaves and $v$ is adjacent to $s$ leaves. If $g$ is a $\gamma_{stI}(S(r, s))$-function, then the definition implies

$$\gamma_{stR}^2(S(r, s)) \geq \gamma_{stI}^2(S(r, s)) = \omega(g) = g(N(u)) + g(N(v)) \geq 4.$$ 

Conversely, let $r = 2p + 1$ and $s = 2q + 1$ be odd. Define $f$ by $f(u) = f(v) = 2$. In addition, we assign the weight 2 to one leaf of $u$, the weight -1 to $p + 1$ leaves of $u$, the weight 1 to $p - 1$ leaves of $u$, the weight 2 to one leaf of $v$, the weight -1 to $q + 1$ leaves of $v$, and the weight 1 to $q - 1$ leaves of $v$. Then $f$ is an STR2DF on $S(r, s)$ of weight 4 and thus $\gamma_{stI}^2(S(r, s)) \leq \gamma_{stR}^2(S(r, s)) \leq 4$ in this case.

Now let $r = 2p$ and $s = 2q$ be even. Define $f$ by $f(u) = f(v) = 2$. In addition, we assign the weight -1 to $p$ leaves of $u$, the weight 1 to $p$ leaves of $u$, the weight -1 to $q$ leaves of $v$, and the weight 1 to $q$ leaves of $v$. Then $f$ is an STR2DF on $S(r, s)$ of weight 4 and thus $\gamma_{stI}^2(S(r, s)) \leq \gamma_{stR}^2(S(r, s)) \leq 4$ also in this case.

The case $r$ even and $s$ odd or $r$ odd and $s$ even are similar to the cases above and are therefore omitted.

Example 3 shows that $\gamma_{stI}^2(S(1, 1)) = \gamma_{stI}^2(S(1, 1)) = 6$. For the sake of completeness, we now determine $\gamma_{stI}^2(S(1, s))$ for $s \geq 2$.

Example 9. If $S(1, s)$ is a double star with $s \geq 2$, then $\gamma_{stI}^2(S(1, s)) = \gamma_{stR}^2(S(1, s)) = 5$.

Proof. Let $u$ and $v$ be two adjacent vertices of $S(1, s)$ such that $u$ is adjacent to one leaf $w$ and $v$ is adjacent to $s \geq 2$ leaves. If $g$ is a $\gamma_{stI}^2(S(1, s))$-function, then we observe that $g(v) = 2$ and $g(w) \geq 1$. Hence the definition leads to

$$\gamma_{stR}^2(S(1, s)) \geq \gamma_{stI}^2(S(1, s)) = \omega(g) = g(N(u)) + g(N(v)) \geq 3 + 2 = 5.$$ 

Conversely, let $s = 2q + 1$ be odd. Define $f$ by $f(u) = f(v) = 2$ and $f(w) = 1$. In addition, we assign the weight 2 to one leaf of $v$, the weight -1 to $q + 1$ leaves of $v$, and the weight 1 to $q - 1$ leaves of $v$. Then $f$ is an STR2DF on $S(1, s)$ of weight 5 and thus $\gamma_{stI}^2(S(1, s)) \leq \gamma_{stR}^2(S(1, s)) \leq 5$ in this case.

Now let $s = 2q$ be even. Define $f$ by $f(u) = f(v) = 2$ and $f(w) = 1$. In addition, we assign the weight -1 to $q$ leaves of $v$, and the weight 1 to $q$ leaves of $v$. Then $f$ is an STR2DF on $S(1, s)$ of weight 5 and thus $\gamma_{stI}^2(S(1, s)) \leq \gamma_{stR}^2(S(1, s)) \leq 5$ also in the second case.

References


