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Signed total Italian k-domination in graphs

Lutz Volkmann

Lehrstuhl II für Mathematik, RWTH Aachen University, 52056 Aachen, Germany
volkm@math2.rwth-aachen.de

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Abstract: Let $k \geq 1$ be an integer, and let G be a finite and simple graph with vertex set $V(G)$. A signed total Italian k-dominating function (STIkDF) on a graph G is a function $f : V(G) \to \{-1, 1, 2\}$ satisfying the conditions that $\sum_{x \in N(v)} f(x) \geq k$ for each vertex $v \in V(G)$, where $N(v)$ is the neighborhood of v, and each vertex u with $f(u) = -1$ is adjacent to a vertex v with $f(v) = 2$ or to two vertices w and z with $f(w) = f(z) = 1$. The weight of an STIkDF f is $\omega(f) = \sum_{v \in V(G)} f(v)$. The signed total Italian k-domination number $\gamma^k_{stI}(G)$ of G is the minimum weight of an STIkDF on G. In this paper we initiate the study of the signed total Italian k-domination number of graphs, and we present different bounds on $\gamma^k_{stI}(G)$. In addition, we determine the signed total Italian k-domination number of some classes of graphs. Some of our results are extensions of well-known properties of the signed total Roman k-domination number $\gamma^k_{stR}(G)$, introduced and investigated by Volkmann [8, 10].

Keywords: Signed total Italian k-dominating function, Signed total Italian k-domination number, Signed total Roman k-dominating function, Signed total Roman k-domination number.

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1. Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [4]. Specifically, let G be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the *order* and the *size* of the graph G, respectively. The *open neighborhood* of a vertex v is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d_G(v) = d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. For a set $X \subseteq V(G)$, its *open neighborhood* is the set $N_G(X) = N(X) =$

$\bigcup_{v\in X} N(v)$, and its *closed neighborhood* is the set $N_G[X] = N[X] = N(X) \cup X$. The *complement* of a graph G is denoted by $\overline{G}$. A *leaf* of a graph G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. Denote by $S(G)$ the set of support vertices of G. An edge incident with a leaf is called a *pendant edge*. Let P_n, C_n and K_n be the path, cycle and complete graph of order n. A complete bipartite graph with partite sets X and Y, where $|X| = p$ and $|Y| = q$ is denoted by $K_{p,q}$. Specifically, $K_{1,q}$ is called a *star*. Let $S(r, s)$ be the *double star* with exactly two adjacent vertices u and v that are not leaves such that u is adjacent to $r \geq 1$ leaves and v is adjacent to $s \geq 1$ leaves. A *spider graph* is a tree with one vertex of degree at least 3 and all others of degree at most two.
All along this paper we will assume that k is a positive integer. In this paper we continue the study of signed total Roman and signed total Italian dominating functions in graphs and digraphs. For a subset $S \subseteq V(G)$ of vertices of a graph G and a function $f\colon V(G) \longrightarrow \mathbb{R}$, we define $f(S) = \sum_{x\in S} f(x)$.

If $k \geq 1$ is an integer, then Volkmann [10] defined the *signed total Roman k-dominating function* (STRkDF) on a graph G as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $f(N(v)) = \sum_{x\in N(v)} f(x) \geq k$ for every $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$. The weight of an STRkDF f on a graph G is $\omega(f) = \sum_{v\in V(G)} f(v)$. The *signed total Roman k-domination number* $\gamma^k_{stR}(G)$ of G is the minimum weight of an STRkDF on G. The special case $k = 1$ was introduced and investigated by Volkmann [8] and [7]. A $\gamma^k_{stR}(G)$-function is a signed total Roman k-dominating function on G of weight $\gamma^k_{stR}(G)$. Amjadi and Soroudi [1], Dehgardi and Volkmann [3] and Volkmann [9] studied the signed total Roman domination number in digraphs.
A *signed total Italian k-dominating function* (STIkDF) on a graph G is defined as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ having the property $f(N(v)) \geq k$ for every $v \in V(G)$, and each vertex u with $f(u) = -1$ is adjacent to a vertex v with $f(v) = 2$ or to two vertices w and z with $f(w) = f(z) = 1$. Note that in the case $k \geq 2$, the second condition is superfluous. The weight of an STIkDF f is $\omega(f) = \sum_{v\in V(G)} f(v)$. The signed total Italian k-domination number $\gamma^k_{stI}(G)$ of G is the minimum weight of an STIkDF on G. The special case $k = 1$ was introduced and investigated by Volkmann [6]. A $\gamma^k_{stI}(G)$-function is a signed total Italian k-dominating function on G of weight $\gamma^k_{stI}(G)$. For an STIkDF f on G, let $V_i = V_i(f) = \{v \in V(G) : f(v) = i\}$ for $i = -1, 1, 2$. A signed total Italian k-dominating function $f : V(G) \longrightarrow \{-1, 1, 2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(G)$.
The signed total Italian k-domination number exists when $\delta \geq \frac{k}{2}$. The definitions lead to $\gamma^k_{stI}(G) \leq \gamma^k_{stR}(G)$. Therefore each lower bound of $\gamma^k_{stI}(G)$ is also a lower bound of $\gamma^k_{stR}(G)$. Note that the signed total Italian k-domination number is the total version of the weak signed Roman k-domination number, see Volkmann [11, 12].
For an integer $q \geq 1$, Kulli [5] called a subset D of vertices of a graph G a *total q-dominating set* if every vertex $x \in V(G)$ has at least q neighbors in D. The *total q-domination number* $\gamma_{tq}(G)$ is the minimum cardinality of a total q-dominating set

of G. The special case $q = 1$ is the usual *total domination number* $\gamma_t(G)$, introduced by Cockayne, Dawes and Hedetniemi [2].

Our purpose in this work is to initiate the study of the signed total Italian k-domination number. We present basic properties and sharp bounds on $\gamma^k_{stI}(G)$. In particular, we show that many lower bounds on $\gamma^k_{stR}(G)$ are also valid for $\gamma^k_{stI}(G)$. Some of our results are extensions of well-known properties of the signed total Roman k-domination number and the signed total Italian domination number $\gamma_{stI}(G) = \gamma^1_{stI}(G)$, given by Volkmann [6, 8, 10].

2. Preliminary results

In this section we present basic properties of the signed total Italian k-dominating functions and the signed total Italian k-domination numbers.

Proposition 1. *Let $k \geq 1$ be an integer, and let G be a graph of order n with $\delta(G) \geq \lceil k/2 \rceil$. If $f = (V_{-1}, V_1, V_2)$ is an STIkDF on G, then*

(a) $|V_{-1}| + |V_1| + |V_2| = n$.

(b) $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$.

(c) *If $\delta(G) \geq \lceil (k+s)/2 \rceil$ with $s \in \{0, 1\}$, then $V_1 \cup V_2$ is a total $\lceil (k+s)/2 \rceil$-dominating set of G.*

Proof. Since (a) and (b) are immediate, we only prove (c). Suppose on the contrary, that there exists a vertex v with at most $\lceil (k+s)/2 \rceil - 1$ neighbors in $V_1 \cup V_2$. Then v has at least

$$\delta(G) - (\lceil (k+s)/2 \rceil - 1) \geq \lceil (k+s)/2 \rceil - (\lceil (k+s)/2 \rceil - 1) = 1$$

neighbors in V_{-1}. Hence the definition implies the contradiction

$$k \leq f(N(v)) \leq 2(\lceil (k+s)/2 \rceil - 1) - 1 \leq \frac{2(k+s+1)}{2} - 3 = k + s - 2 \leq k - 1.$$

Consequently, $V_1 \cup V_2$ is a total $\lceil (k+s)/2 \rceil$-dominating set of G. □

Corollary 1. *If G is a graph of order n and minimum degree $\delta \geq \lceil \frac{k+s}{2} \rceil$ with $s \in \{0, 1\}$, then $\gamma^k_{stI}(G) \geq 2\gamma_{t\lceil \frac{k+s}{2} \rceil} - n$.*

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma^k_{stI}(G)$-function. Then it follows from Proposition 1 that

$$\gamma^k_{stI}(G) = |V_1| + 2|V_2| - |V_{-1}| = 2|V_1| + 3|V_2| - n \geq 2|V_1 \cup V_2| - n \geq 2\gamma_{t\lceil \frac{k+s}{2} \rceil} - n.$$

□

The graphs qK_2 and qK_3 show that Corollary 1 is sharp for $k = 1$ and $k = 2$. The proof of the next proposition is identically with the proof of Proposition 2 in [10] and is therefore omitted.

Proposition 2. Let $k \geq 1$ be an integer, and let G be a graph of order n with minimum degree $\delta \geq k/2$ and maximum degree Δ. If $f = (V_{-1}, V_1, V_2)$ is an STIkDF on G, then

(i) $(2\Delta - k)|V_2| + (\Delta - k)|V_1| \geq (\delta + k)|V_{-1}|$.

(ii) $(2\Delta + \delta)|V_2| + (\Delta + \delta)|V_1| \geq (\delta + k)n$.

(iii) $(\Delta + \delta)\omega(f) \geq (\delta - \Delta + 2k)n + (\delta - \Delta)|V_2|$.

(iv) $\omega(f) \geq (\delta - 2\Delta + 2k)n/(2\Delta + \delta) + |V_2|$.

3. Bounds on the signed total Italian k-domination number

We start with a general upper bound, and we characterize all extremal graphs.

Theorem 1. Let G be a graph of order n with $\delta(G) \geq \lceil \frac{k}{2} \rceil$. Then $\gamma^k_{stI}(G) \leq 2n$, with equality if and only if k is even, $\delta(G) = \frac{k}{2}$, and each vertex of G is adjacent to a vertex of minimum degeree.

Proof. Define the function $g : V(G) \longrightarrow \{-1, 1, 2\}$ by $g(x) = 2$ for each vertex $x \in V(G)$. Since $\delta(G) \geq \lceil \frac{k}{2} \rceil$, the function g is an STIkDF on G of weight $2n$ and thus $\gamma^k_{stI}(G) \leq 2n$.
Now let k be even, $\delta(G) = \frac{k}{2}$, and assume that each vertex of G is adjacent to a vertex of minimum degeree. Let f be an STIkDF on G, and let $x \in V(G)$ be an arbitray vertex. Then x has a neighbor v with $d(v) = \frac{k}{2}$. Therefore the condition $f(N(v)) \geq k$ implies $f(x) = 2$. Thus f is of weight $2n$, and we obtain $\gamma^k_{stI}(G) = 2n$.
Conversely, assume that $\gamma^k_{stI}(G) = 2n$. If $k = 2p + 1$ is odd, then $\delta(G) \geq p + 1$. Define the function $h : V(G) \longrightarrow \{-1, 1, 2\}$ by $h(w) = 1$ for an arbitrary vertex w and $h(x) = 2$ for each vertex $x \in V(G) \setminus \{w\}$. Then

$$h(N(v)) = \sum_{x \in N(v)} f(x) \geq 2(p + 1) - 1 = 2p + 1 = k$$

for each vertex $v \in V(G)$. Thus the function h is an STIkDF on G of weight $2n - 1$, and we obtain the contradiction $\gamma^k_{stI}(G) \leq 2n - 1$.
Let now k even, and assume that there exists a vertex w such that $d(x) \geq \frac{k}{2} + 1$ for each $x \in N(w)$. Define the function $h_1 : V(G) \longrightarrow \{-1, 1, 2\}$ by $h_1(w) = 1$ and $h_1(x) = 2$ for each vertex $x \in V(G) \setminus \{w\}$. Then $h_1(N(w)) \geq k$, $h_1(N(x)) \geq 2(\frac{k}{2} + 1) - 1 = k + 1$ for each $x \in N(w)$ and $h_1(N(y)) \geq k$ for each $y \notin N[w]$. Hence the function h_1 is an STIkDF on G of weight $2n - 1$, a contradiction to the assumption $\gamma^k_{stI}(G) = 2n$. This completes the proof. □

The proof of Theorem 1 also leads to the next result.

Theorem 2. Let G be a graph of order n with $\delta(G) \geq \lceil \frac{k}{2} \rceil$. Then $\gamma^k_{stR}(G) \leq 2n$, with equality if and only if k is even, $\delta(G) = \frac{k}{2}$, and each vertex of G is adjacent to a vertex of minimum degeree.

Let $k \geq 2$ be an even integer. If H is a $\frac{k}{2}$-regular graph of order n, then Theorems 1 and 2 imply that $\gamma^k_{stI}(H) = \gamma^k_{stR}(H) = 2n$. The next corollaries follow from Theorems 1 and 2 immediately.

Corollary 2. Let G be a connected graph of order $n \geq 2$ with $\delta(G) = 1$. Then $\gamma^2_{stI}(G) = 2n$ if and only if $n = 2$.

Corollary 3. Let G be a connected graph of order $n \geq 2$ with $\delta(G) = 1$. Then $\gamma^2_{stR}(G) = 2n$ if and only if $n = 2$.

Observation 3. *If G is a graph of order n with $\delta(G) \geq k$, then $\gamma^k_{stI}(G) \leq \gamma^k_{stR}(G) \leq n$.*

Proof. Define the function $f : V(G) \longrightarrow \{-1, 1, 2\}$ by $f(x) = 1$ for each vertex $x \in V(G)$. Since $\delta(G) \geq k$, the function f is an STRkDF on G of weight n and thus $\gamma^k_{stI}(G) \leq \gamma^k_{stR}(G) \leq n$. □

As an application of Proposition 2 (iii), we obtain a lower bound on the signed total Italian k-domination number for r-regular graphs.

Corollary 4. If G is an r-regular graph of order n with $r \geq \frac{k}{2}$, then

$$\gamma^k_{stI}(G) \geq \frac{kn}{r}.$$

Example 1. *If H is a k-regular graph of order n, then it follows from Corollary 4 that $\gamma^k_{stI}(H) \geq n$ and thus $\gamma^k_{stI}(H) = n$, according to Observation 3.*

Example 1 shows that Observation 3 and Corollary 4 are both sharp.

Theorem 4. If G is a graph of order n with $\delta(G) \geq \frac{k}{2}$, then

$$\gamma^k_{stI}(G) \geq k + 2 + \delta(G) - n.$$

If in addition $\delta(G) - k$ is odd, then $\gamma^k_{stI}(G) \geq k + 3 + \delta(G) - n$.

Proof. Let f be a $\gamma^k_{stI}(G)$-function. Then there exists a vertex w with $f(w) \geq 1$. It follows from the the definitions that

$$\begin{aligned}\gamma^k_{stI}(G) &= \sum_{x\in V(G)} f(x) = f(w) + \sum_{x\in N(w)} f(x) + \sum_{x\in V(G)-N[w]} f(x)\\ &\geq 1 + k - (n - d(w) - 1) = k + 2 + d(w) - n \geq k + 2 + \delta(G) - n.\end{aligned}$$

Now assume that $\delta(G) - k$ is odd. If there exists a vertex w with $f(w) = 2$ or a vertex v with $f(v) = 1$ and $d(v) \geq \delta(G) + 1$, then the inequality chain above leads to $\gamma^k_{stI}(G) \geq k + 3 + \delta(G) - n$. So assume that $f(x) \in \{-1, 1\}$ for each vertex $x \in V(G)$, and each vertex y with $f(y) = 1$ has degree $d(y) = \delta(G)$. Let now u be a vertex of minimum degree with $f(u) = 1$. Since $\delta(G) - k$ is odd, we observe that

$$\sum_{x\in N(u)} f(x) \geq k + 1$$

and thus

$$\begin{aligned}\gamma^k_{stI}(G) &= \sum_{x\in V(G)} f(x) = f(u) + \sum_{x\in N(u)} f(x) + \sum_{x\in V(G)-N[u]} f(x)\\ &\geq 1 + k + 1 - (n - d(u) - 1) = k + 3 + d(u) - n = k + 3 + \delta(G) - n.\end{aligned}$$

This completes the proof. □

Example 2. *If $k \geq 1$ and $n \geq 2$ are integers such that $2n - 2 \geq k$, then it holds:*

(i) If $k \geq n$, then $\gamma^k_{stI}(K_n) = k + 2$.

(ii) If $k \leq n - 1$ and $n - k$ is odd, then $\gamma^k_{stI}(K_n) = k + 1$.

(iii) If $k \leq n - 1$ and $n - k$ is even, then $\gamma^k_{stI}(K_n) = k + 2$.

Proof. Let f be a $\gamma^k_{stI}(K_n)$-function.
(i) Since $k \geq n$, there exists a vertex w with $f(w) \geq 2$. This implies $\gamma^k_{stI}(K_n) = f(w) + f(N(w)) \geq 2 + k$.
For $\gamma^k_{stI}(K_n) \leq k + 2$, let the function $g : V(K_n) \longrightarrow \{-1, 1, 2\}$ assign to $2n - k - 2$ vertices the value 1 and to the remaining $k + 2 - n$ vertices the value 2. Then g is an STIkDF on K_n of weight $\omega(g) = k+2$ and so $\gamma^k_{stI}(K_n) \leq k+2$. Thus $\gamma^k_{stI}(K_n) = k+2$ in this case.
(ii) Since there exists a vertex w with $f(w) \geq 1$, we note that $\gamma^k_{stI}(K_n) = f(w) + f(N(w)) \geq 1 + k$.
For $\gamma^k_{stI}(K_n) \leq k + 1$, let the function $g : V(K_n) \longrightarrow \{-1, 1, 2\}$ assign to $\frac{n+k+1}{2}$ vertices the value 1 and to the remaining $\frac{n-k-1}{2}$ vertices the value -1. Then g is an STIkDF on K_n of weight $\omega(g) = k+1$ and so $\gamma^k_{stI}(K_n) \leq k+1$. Thus $\gamma^k_{stI}(K_n) = k+1$ in the second case.

(iii) If there exists a vertex w with $f(w) = 2$, then it follws that $\gamma^k_{stI}(K_n) = f(w) + f(N(w)) \geq 2 + k$. If $f(x) \in \{-1, 1\}$ for all $x \in V(K_n)$, then $f(N(x))$ is even when n is odd and $f(N(x))$ is odd when n is even. Since $f(N(x)) \geq k$, whe observe that $f(N(x)) \geq k + 1$ when $n - k$ is even. If w is a vertex with $f(w) = 1$, we therefore deduce that $\gamma^k_{stI}(K_n) = f(w) + f(N(w)) \geq 1 + k + 1 = k + 2$.
For $\gamma^k_{stI}(K_n) \leq k + 2$, let the function $g : V(K_n) \longrightarrow \{-1, 1, 2\}$ assign to $\frac{n+k+2}{2}$ vertices the value 1 and to the remaining $\frac{n-k-2}{2}$ vertices the value -1. Then g is an STIkDF on K_n of weight $\omega(g) = k+2$ and so $\gamma^k_{stI}(K_n) \leq k+2$. Thus $\gamma^k_{stI}(K_n) = k+2$ in the last case. □

Example 2 (ii) and (iii) show that Theorem 4 is sharp for $k \leq n-1$. If $\Delta(G) \geq \delta(G)+3$, then the next lower bound is an improvement of Theorem 4.

Proposition 3. If G is a graph of order n with $\delta(G) \geq \frac{k}{2}$, then

$$\gamma^k_{stI}(G) \geq k + \Delta(G) - n.$$

Proof. Let $w \in V(G)$ be a vertex of maximum degree, and let f be a $\gamma^k_{stI}(G)$-function. Then the definitions imply

$$\begin{aligned}
\gamma^k_{stI}(G) &= \sum_{x\in V(G)} f(x) = \sum_{x\in N(w)} f(x) + \sum_{x\in V(G)-N(w)} f(x) \\
&\geq k + \sum_{x\in V(G)-N(w)} f(x) \geq k - (n - \Delta(G)) \\
&= k + \Delta(G) - n,
\end{aligned}$$

and the proof of the desired lower bound is complete. □

Corollary 5. Let G be a graph of order n, minimum degree $\delta \geq \frac{k}{2}$ and maximum degree Δ. If $\delta < \Delta$, then

$$\gamma^k_{stI}(G) \geq \left\lceil \frac{2\delta + 3k - 2\Delta}{2\Delta + \delta} n \right\rceil.$$

Proof. Multiplying both sides of the inequality in Proposition 2 (iv) by $\Delta - \delta$ and adding the resulting inequality to the inequality in Proposition 2 (iii), we obtain the desired lower bound. □

Since $\gamma^k_{stR}(G) \geq \gamma^k_{stI}(G)$, Corollary 5 leads to the next known lower bound immediately.

Corollary 6. ([10]) Let G be a graph of order n, minimum degree $\delta \geq \frac{k}{2}$ and maximum degree Δ. If $\delta < \Delta$, then

$$\gamma^k_{stR}(G) \geq \left\lceil \frac{2\delta + 3k - 2\Delta}{2\Delta + \delta} n \right\rceil.$$

Examples 12 and 13 in [10] demonstrate that Corollary 6 is sharp and therefore Corollary 5 too. The special case $k = 1$ of Corollaries 4 and 5 can be found in [6].
A set $S \subseteq V(G)$ is a *2-packing* of the graph G if $N[u] \cap N[v] = \emptyset$ for any two distinct vertices $u, v \in S$. The *2-packing number* $\rho(G)$ of G is defined by

$$\rho(G) = \max\{|S| : S \text{ is a } 2-\text{packing of } G\}.$$

Analogously to Theorem 14 in [10], one can prove the next lower bound on the signed total Italian k-domination number.

Theorem 5. If G is a graph of order n with $\delta(G) \geq \frac{k}{2}$, then

$$\gamma^k_{stI}(G) \geq \rho(G)(k + \delta(G)) - n.$$

Corollary 7. ([10]) If G is a graph of order n with $\delta(G) \geq \frac{k}{2}$, then

$$\gamma^k_{stR}(G) \geq \rho(G)(k + \delta(G)) - n.$$

In [10], the author presented an infinite family of graphs achieving equality in Corollary 7. Thus Corollary 7 and Theorem 5 are sharp. Using Corollary 4, one can prove the following Nordhaus-Gaddum type inequality analogously to Theorem 15 in [10].

Theorem 6. If G is an r-regular graph of order n such that $r \geq \frac{k}{2}$ and $n - r - 1 \geq \frac{k}{2} - 1$, then

$$\gamma^k_{stI}(G) + \gamma^k_{stI}(\overline{G}) \geq \frac{4kn}{n-1}.$$

If n is even, then $\gamma^k_{stI}(G) + \gamma^k_{stI}(\overline{G}) \geq 4k(n-1)/(n-2)$.

Let $k \geq 2$ be an even integer, and let H and $\overline{H}$ be k-regular graphs of order $n = 2k+1$. By Example 1, we have $\gamma^k_{stI}(H) = \gamma^k_{stI}(\overline{H}) = n$. Consequently,

$$\gamma^k_{stI}(H) + \gamma^k_{stI}(\overline{H}) = 2n = \frac{4kn}{n-1}.$$

Thus the Nordhaus-Gaddum bound of Theorem 6 is sharp for even k.

Theorem 7. Let G be connected graph of order $n \geq 3$ and let $S(G)$ be the set of its support vertices. Then $\gamma^2_{stI}(G) \leq \gamma^2_{stR}(G) \leq n + |S(G)|$.

Proof. Define the function $f : V(G) \longrightarrow \{-1, 1, 2\}$ by $f(x) = 2$ when x is a support vertex and $f(x) = 1$ otherwise. If v is a leaf, then v is ajacent to a support vertex and so $f(N(v)) = 2$. If v is not a leaf, then $d(v) \geq 2$ and thus $f(N(v)) \geq 2$. Therefore f is an STR2DF on G of weight $n+|S(G)|$ and hence $\gamma^2_{stI}(G) \leq \gamma^2_{stR}(G) \leq n+|S(G)|$. □

Example 3. *For $n \geq 4$, we have $\gamma^2_{stR}(P_n) = \gamma^2_{stI}(P_n) = n + 2$.*

Proof. Let $P_n = v_1v_2 \ldots v_n$, and let f be a $\gamma^2_{stI}(P_n)$-function. According to Theorem 7, we have $\gamma^2_{stI}(P_n) \leq \gamma^2_{stR}(P_n) \leq n + 2$. Conversely, it is easy to see that $f(v_i) \geq 1$ for $1 \leq i \leq n$ and $f(v_2) = f(v_{n-1}) = 2$. Therefore $\gamma^2_{stR}(P_n) \geq \gamma^2_{stI}(P_n) \geq n + 2$ and thus $\gamma^2_{stR}(P_n) = \gamma^2_{stI}(P_n) = n + 2$. □

Since $\gamma^2_{stR}(P_3) = \gamma^2_{stI}(P_3) = 4$, the condition $n \geq 4$ in Example 3 is necessary.

Example 4. *Let H be a spider graph, and let $w \in V(H)$ with $d(w) \geq 3$. If every leaf of H has distance at least two from w, then $\gamma^2_{stR}(H) = \gamma^2_{stI}(H) = n + |S(H)|$.*

Proof. According to Theorem 7, we have $\gamma^2_{stI}(H) \leq \gamma^2_{stR}(H) \leq n + |S(H)|$. Converesely, let f be a $\gamma^2_{stI}(H)$-function. Since every leaf of H has distance at least two from w, we note that $f(x) \geq 1$ for $x \in V(H)$. In addition, we have $f(x) = 2$ for each $x \in S(H)$. Therefore $\gamma^2_{stR}(H) \geq \gamma^2_{stI}(H) \geq n + |S(H)|$ and thus $\gamma^2_{stR}(H) = \gamma^2_{stI}(H) = n + |S(H)|$. □

Examples 3 and 4 demonstrate that Theorem 7 is sharp.

Example 5. *If $n \geq 2$, then $\gamma^2_{stI}(K_{1,n-1}) = \gamma^2_{stR}(K_{1,n-1}) = 4$.*

Proof. Let $G = K_{1,n-1}$, and let f be a $\gamma^2_{stI}(G)$-function. If $n = 2$, then the result is obviously. Let now $n \geq 3$, and let w be the central vertex of the star G. Clearly, $f(w) = 2$. Hence it follows that

$$\gamma^2_{stR}(G) \geq \gamma^2_{stI}(G) = f(w) + f(N(w)) \geq 2 + 2 = 4.$$

For the converse inequality $\gamma^2_{stI}(G) \leq \gamma^2_{stR}(G) \leq 4$, let $v_1, v_2, \ldots, v_{n-1}$ be the leaves of G. If $n - 1 = 2p$ is even, define $g : V(G) \longrightarrow \{-1, 1, 2\}$ by $g(w) = 2$, $g(v_i) = 1$ for $1 \leq i \leq p + 1$ and $g(v_i) = -1$ for $p + 2 \leq i \leq 2p$. Then g is an STR2DF on G of weight 4 and thus $\gamma^2_{stI}(G) \leq \gamma^2_{stR}(G) \leq 4$ in this case. If $n = 2p + 1$ is odd, then define $h : V(G) \longrightarrow \{-1, 1, 2\}$ by $h(w) = h(v_{2p+1}) = 2$, $h(v_i) = 1$ for $1 \leq i \leq p$ and $h(v_i) = -1$ for $p + 1 \leq i \leq 2p$. Then h is an STR2DF on G of weight 4 and thus $\gamma^2_{stI}(G) \leq \gamma^2_{stR}(G) \leq 4$ also in this case. □

Example 5 shows that $\gamma^2_{stR}(H) = \gamma^2_{stI}(H) = n + |S(H)|$ is not valid for each spider graph H. Therefore the condition that every leaf of H has distance at least two from w is important in Example 4.

4. Special classes of graphs

Let C_n be a cycle of length $n \geq 3$. In [6], the author has shown that $\gamma_{stI}(C_n) = n/2$ when $n \equiv 0 \pmod 4$, $\gamma_{stI}(C_n) = (n+3)/2$ when $n \equiv 1, 3 \pmod 4$ and $\gamma_{stI}(C_n) = (n+6)/2$ when $n \equiv 2 \pmod 4$. Now we determine $\gamma^k_{stI}(C_n)$ as well as $\gamma^k_{stR}(C_n)$ for $2 \leq k \leq 4$.

Corollary 4 and Observation 3 immediately lead to $\gamma^2_{stI}(C_n) = \gamma^2_{stR}(C_n) = n$. In addition, Theorems 1 and 2 imply $\gamma^4_{stI}(C_n) = \gamma^4_{stR}(C_n) = 2n$.

Example 6. *For $n \geq 3$, we have $\gamma^3_{stR}(C_n) = \gamma^3_{stI}(C_n) = \lceil \frac{3n}{2} \rceil + 1$ when $n \equiv 2 \pmod 4$ and $\gamma^3_{stR}(C_n) = \gamma^3_{stI}(C_n) = \lceil \frac{3n}{2} \rceil$ otherwise.*

Proof. Let $C_n = v_1v_2 \ldots v_nv_1$. Assume first that $n \equiv 0, 1, 3 \pmod 4$. Applying Corollary 4, we obtain $\gamma^3_{stR}(C_n) \geq \gamma^3_{stI}(C_n) \geq \lceil \frac{3n}{2} \rceil$.

If $n = 4t$ for an integer $t \geq 1$, then define $g : V(C_n) \longrightarrow \{-1, 1, 2\}$ by $g(v_{4i+1}) = g(v_{4i+2}) = 2$ and $g(v_{4i+3}) = g(v_{4i+4}) = 1$ for $0 \leq i \leq t-1$. Then g is an STR3DF on C_n of weight $\omega(g) = 6t = \frac{3n}{2}$ and so $\gamma^3_{stI}(C_n) \leq \gamma^3_{stR}(C_n) \leq \lceil \frac{3n}{2} \rceil$ and thus $\gamma^3_{stI}(C_n) = \gamma^3_{stR}(C_n) = \lceil \frac{3n}{2} \rceil$ in this case.

If $n = 4t+1$ for an integer $t \geq 1$, then define $g : V(C_n) \longrightarrow \{-1, 1, 2\}$ by $g(v_{4i+1}) = g(v_{4i+2}) = 2$ and $g(v_{4i+3}) = g(v_{4i+4}) = 1$ for $0 \leq i \leq t-1$ and $g(v_{4t+1}) = 2$. Then g is an STR3DF on C_n of weight $\omega(g) = 6t+2 = \lceil \frac{3n}{2} \rceil$ and thus $\gamma^3_{stR}(C_n) = \gamma^3_{stI}(C_n) = \lceil \frac{3n}{2} \rceil$ in this case too.

If $n = 4t+3$ for an integer $t \geq 0$, then define $g : V(C_n) \longrightarrow \{-1, 1, 2\}$ by $g(v_{4i+1}) = g(v_{4i+2}) = 2$ for $0 \leq i \leq t$, $g(v_{4i+3}) = g(v_{4i+4}) = 1$ for $0 \leq i \leq t-1$ and $g(v_{4t+3}) = 1$. Then g is an STR3DF on C_n of weight $\omega(g) = 6t+5 = \lceil \frac{3n}{2} \rceil$ and thus $\gamma^3_{stR}(C_n) = \gamma^3_{stI}(C_n) = \lceil \frac{3n}{2} \rceil$ in the third case.

Let now $n = 4t+2$ for an integer $t \geq 1$. If f is a $\gamma^3_{stI}(C_n)$-function, then we observe that $f(v_i) \geq 1$ for each $1 \leq i \leq n$ and $f(v_i) + f(v_{i+1}) + f(v_{i+2}) + f(v_{i+3}) \geq 6$, where the indices are taken modulo n. Assume, without loss of generality, that $f(v_1) = f(v_2) = 2$ and $f(v_3) = f(v_4) = 1$. Then it follows that $f(v_5) = f(v_6) = 2$, and we obtain

$$\begin{aligned}\omega(f) &= f(v_1) + f(v_2) + f(v_3) + f(v_4) + f(v_5) + f(v_6) \\ &+ \sum_{i=1}^{t-1} (f(v_{4i+3}) + f(v_{4i+4}) + f(v_{4i+5}) + f(v_{4i+6})) \\ &\geq 10 + 6(t-1) = 6t + 4.\end{aligned}$$

Therefore $\omega(f) \geq 6t+4 = \lceil \frac{3n}{2} \rceil + 1$ and thus $\gamma^3_{stR}(C_n) \geq \gamma^3_{stI}(C_n) \geq \lceil \frac{3n}{2} \rceil + 1$ in this case. Conversely, if we define $g : V(C_n) \longrightarrow \{-1, 1, 2\}$ by $g(v_{4i+1}) = g(v_{4i+2}) = 2$ for $0 \leq i \leq t$ and $g(v_{4i+3}) = g(v_{4i+4}) = 1$ for $0 \leq i \leq t-1$, then g is an STR3DF on C_n of weight $\omega(g) = 6t+4 = \lceil \frac{3n}{2} \rceil + 1$ and thus $\gamma^3_{stR}(C_n) = \gamma^3_{stI}(C_n) = \lceil \frac{3n}{2} \rceil + 1$ in the last case. □

Example 7. *If $k \geq 1$ and $q \geq p \geq 2$ are integers such that $p \geq k/2$, then $\gamma_{stI}^k(K_{p,q}) = 2k$.*

Proof. Let $G = K_{p,q}$, and let g be a $\gamma_{stI}^k(G)$-function. In addition, let X and Y be a bipartition of the complete bipartite graph G with $|X| = p$ and $|Y| = q$. If $x \in X$ and $y \in Y$, then

$$\gamma_{stI}^k(G) = g(V(G)) = g(N(x)) + g(N(y)) \geq k + k = 2k.$$

For the converse inequality we distinguish different cases.
a) Assume that $k \geq q$. Let the function $f : V(G) \longrightarrow \{-1, 1, 2\}$ assign to $2p - k$ vertices of X the value 1, to the remaining $k - p$ vertices of X the value 2, to $2q - k$ vertices of Y the value 1 and to the remaining $k - q$ vertices of Y the value 2. Then f is an STIkDF on G of weight $\omega(f) = 2k$ and so $\gamma_{stI}^k(G) \leq 2k$. Thus $\gamma_{stI}^k(G) = 2k$ in this case.
b) Assume that $q > k \geq p$ and $q - k$ is even. Let the function $f : V(G) \longrightarrow \{-1, 1, 2\}$ assign to $2p - k$ vertices of X the value 1, to the remaining $k - p$ vertices of X the value 2, to $\frac{k+q}{2}$ vertices of Y the value 1 and to the remaining $\frac{q-k}{2}$ vertices of Y the value -1. Then f is an STIkDF on G of weight $\omega(f) = 2k$ and so $\gamma_{stI}^k(G) \leq 2k$. Thus $\gamma_{stI}^k(G) = 2k$ in the second case.
c) Assume that $q > k \geq p$ and $q - k$ is odd. Let the function $f : V(G) \longrightarrow \{-1, 1, 2\}$ assign to $2p - k$ vertices of X the value 1, to the remaining $k - p$ vertices of X the value 2, to one vertex of Y the value 2, to $\frac{k+q-3}{2}$ vertices of Y the value 1 and to the remaining $\frac{q-k+1}{2}$ vertices of Y the value -1. Then f is an STIkDF on G of weight $\omega(f) = 2k$ and so $\gamma_{stI}^k(G) \leq 2k$. Thus $\gamma_{stI}^k(G) = 2k$ in the third case.
d) Assume that $k < p$, $p - k$ is even and $q - k$ is even. Let the function $f : V(G) \longrightarrow \{-1, 1, 2\}$ assign to $\frac{k+p}{2}$ vertices of X the value 1, to the remaining $\frac{p-k}{2}$ vertices of X the value -1, to $\frac{k+q}{2}$ vertices of Y the value 1 and to the remaining $\frac{q-k}{2}$ vertices of Y the value -1. Then f is an STIkDF on G of weight $\omega(f) = 2k$ and thus $\gamma_{stI}^k(G) = 2k$ in this case too.
e) Assume that $k < p$, $p - k$ is odd and $q - k$ is odd. Let the function $f : V(G) \longrightarrow \{-1, 1, 2\}$ assign to one vertex of X the value 2, to $\frac{k+p-3}{2}$ vertices of X the value 1, to the remaining $\frac{p-k+1}{2}$ vertices of X the value -1, to one vertex of Y the value 2, to $\frac{k+q-3}{2}$ vertices of Y the value 1 and to the remaining $\frac{q-k+1}{2}$ vertices of Y the value -1. Then f is an STIkDF on G of weight $\omega(f) = 2k$ and thus $\gamma_{stI}^k(G) = 2k$ also in this case.
The cases $k < p$, $p - k$ even and $q - k$ odd or $k < p$, $p - k$ odd and $q - k$ even are analogously, and are therefore omitted. □

Example 6 and Example 7 with $p = q$ show again that Corollary 4 is sharp.

Example 8. *If $S(r, s)$ is a double star with $r, s \geq 2$, then $\gamma_{stI}^2(S(r, s)) = \gamma_{stR}^2(S(r, s)) = 4$.*

Proof. Let u and v be two adjacent vertices of $S(r,s)$ such that u is adjacent to r leaves and v is adjacent to s leaves. If g is a $\gamma^2_{stI}(S(r,s))$-function, then the definition implies

$$\gamma^2_{stR}(S(r,s)) \geq \gamma^2_{stI}(S(r,s)) = \omega(g) = g(N(u)) + g(N(v)) \geq 4.$$

Conversely, let $r = 2p+1$ and $s = 2q+1$ be odd. Define f by $f(u) = f(v) = 2$. In addition, we assign the weight 2 to one leaf of u, the weight -1 to $p+1$ leaves of u, the weight 1 to $p-1$ leaves of u, the weight 2 to one leaf of v, the weight -1 to $q+1$ leaves of v, and the weight 1 to $q-1$ leaves of v. Then f is an STR2DF on $S(r,s)$ of weight 4 and thus $\gamma^2_{stI}(S(r,s)) \leq \gamma^2_{stR}(S(r,s)) \leq 4$ in this case.
Now let $r = 2p$ and $s = 2q$ be even. Define f by $f(u) = f(v) = 2$. In addition, we assign the weight -1 to p leaves of u, the weight 1 to p leaves of u, the weight -1 to q leaves of v, and the weight 1 to q leaves of v. Then f is an STR2DF on $S(r,s)$ of weight 4 and thus $\gamma^2_{stI}(S(r,s)) \leq \gamma^2_{stR}(S(r,s)) \leq 4$ also in this case.
The case r even and s odd or r odd and s even are similar to the cases above and are therefore omitted. □

Example 3 shows that $\gamma^2_{stR}(S(1,1)) = \gamma^2_{stI}(S(1,1)) = 6$. For the sake of completeness, we now determine $\gamma^2_{stI}(S(1,s))$ for $s \geq 2$.

Example 9. *If $S(1,s)$ is a double star with $s \geq 2$, then $\gamma^2_{stI}(S(1,s)) = \gamma^2_{stR}(S(1,s)) = 5$.*

Proof. Let u and v be two adjacent vertices of $S(1,s)$ such that u is adjacent to one leaf w and v is adjacent to $s \geq 2$ leaves. If g is a $\gamma^2_{stI}(S(1,s))$-function, then we observe that $g(v) = 2$ and $g(w) \geq 1$. Hence the definition leads to

$$\gamma^2_{stR}(S(1,s)) \geq \gamma^2_{stI}(S(1,s)) = \omega(g) = g(N(u)) + g(N(v)) \geq 3 + 2 = 5.$$

Conversely, let $s = 2q+1$ be odd. Define f by $f(u) = f(v) = 2$ and $f(w) = 1$. In addition, we assign the weight 2 to one leaf of v, the weight -1 to $q+1$ leaves of v, and the weight 1 to $q-1$ leaves of v. Then f is an STR2DF on $S(1,s)$ of weight 5 and thus $\gamma^2_{stI}(S(1,s)) \leq \gamma^2_{stR}(S(1,s)) \leq 5$ in this case.
Now let $s = 2q$ be even. Define f by $f(u) = f(v) = 2$ and $f(w) = 1$. In addition, we assign the weight -1 to q leaves of v, and the weight 1 to q leaves of v. Then f is an STR2DF on $S(1,s)$ of weight 5 and thus $\gamma^2_{stI}(S(1,s)) \leq \gamma^2_{stR}(S(1,s)) \leq 5$ also in the second case. □

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