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Research Article

Strength of strongest dominating sets in fuzzy graphs

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Abstract: A set S of vertices in a graph $G = (V, E)$ is a dominating set of G if every vertex of $V - S$ is adjacent to some vertex of S. For an integer $k \geq 1$, a set S of vertices is a k-step dominating set if any vertex of G is at distance k from some vertex of S. In this paper, using membership values of vertices and edges in fuzzy graphs, we introduce the concepts of strength of strongest dominating set as well as strength of strongest k-step dominating set in fuzzy graphs. We determine various bounds for these parameters in fuzzy graphs. We also determine the strength of strongest dominating set in some families of fuzzy graphs including complete fuzzy graphs and complete bipartite fuzzy graphs.

Keywords: Dominating set, Exact 1-step dominating set, Strongest dominating set in fuzzy graphs, Nordhaus-Gaddum type bound.

AMS Subject classification: 05C69, 03E72, 03F55

1. Introduction

Zadeh's paper developed a theory which proposed making the grade of membership of an element in a subset of a universal set a value in the closed interval of real numbers. Zadeh's ideas have found applications in many areas of science and technology. Theoretical mathematics have also been touched by fuzzy set theory. The ideas of fuzzy set theory have been introduced into topology, abstract algebra, geometry, graph theory,

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and analysis [14]. Fuzzy graphs were introduced by Rosenfeld, who has described the fuzzy analogue of several graph theoretic concepts like paths, cycles, trees and connectedness [15]. Nagoor Gani and Chandrasekaran defined dominating set and domination number in fuzzy graphs in [5].
In this paper we study domination in fuzzy graphs. We introduce the concept of strength of strongest dominating set by using membership values of vertices and edges in fuzzy graphs. We present some bounds for the strength of strongest dominating set in fuzzy graphs and then determine the strength of strongest dominating set in some families of fuzzy graphs including complete fuzzy graphs and complete bipartite fuzzy graphs. We also introduce the concept of strength of strongest k-step dominating set in fuzzy graphs, and present various bounds for the strength of strength of strongest 1-step dominating set in fuzzy graphs.

2. Definitions and Notations

In this section we introduce some graph theory as well as fuzzy graph theory definitions and notations.
Graph Theory: Let $G = (V, E)$ be a graph of order $n(G) = |V|$ and size $m(G) = |E(G)| = |E|$. The degree of a vertex v in G is number of edges that are adjacent to v and denote it by $\deg_G(v) = d_G(v)$. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G)$ ($\delta(G)$, respectively). A path P_n of G is a sequence $v_1, \ldots, v_n$ of vertices of G in which v_i is adjacent to v_{i+1} for $i = 1, \ldots, n-1$. The distance between two vertices u and v in a graph is the number of the edges in a shortest path connecting them and in denoted by $d(u, v)$. A subset S of V is called a *dominating set* in G if every vertex in $V \setminus S$ is adjacent to some vertex in S. The *domination number* of G is the minimum cardinality taken over all dominating sets in G and is denoted by $\gamma(G)$, or simply γ [2]. A subset S of V is called a total dominating set in G if every vertex in V is adjacent to some vertex in S. The *total domination number* of G is the minimum cardinality taken over all total dominating sets in G and is denoted by $\gamma_t(G)$, or simply γ_t. The literature on the subject of domination and total domination parameters in graphs have been surveyed and detailed in the books [8, 9].
Two vertices u and v in a graph G for which $d(u, v) = k$, are said to k-step dominate each other. The set $N_k(v)$ denotes the set of vertices of G that are k-step dominated by v. Schultz [16] defined a set $S = \{v_1, v_2, \ldots, v_r\}$ of vertices in a graph G, for some integer $r \leq n$, as a *step dominating set* for G if there exist nonnegative integers $k_1, \ldots, k_r$ such that the sets $\{N_{k_i}(v_i)\}$ form a partition of $V(G)$. The minimum cardinality of a step dominating set is called the *step domination number* of G. A set S of vertices of G is called a *k-step dominating set* if $\bigcup_{v \in S} N_k(v) = V(G)$. A k-step dominating set S such that the sets $N_k(v), v \in S$, are pairwise disjoint, is called an *exact k-step dominating set*. If a graph G has an exact k-step dominating set, then G is called an *exact k-step domination graph*. The concepts of step domination and exact k-step domination were further studied in [3, 4, 10].

We next introduce fuzzy graph theory definitions and notations.

Fuzzy Graph Theory: We use the notations $\vee$ for supremum and $\wedge$ for infimum. A fuzzy subset of S is a mapping $\mu : S \to [0,1]$, where $[0,1]$ denotes the set $\{t \in \mathbb{R} : 0 \leq t \leq 1\}$. We purpose μ as assigning to each element $x \in S$, a degree of membership, $0 \leq \mu(x) \leq 1$, [14]. A fuzzy graph $G = (V, \sigma, \mu)$ is a nonempty set V together with a pair of functions $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$ such that for all $x, y \in V$, we have $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$. For the sake of notational convenience, we omit V in the sequel and use the notation $G = (\sigma, \mu)$. The underlying graph of fuzzy graph $G = (\sigma, \mu)$, is the graph with vertices and edges of $G = (\sigma, \mu)$ such that $\sigma(x) = 1$, for every vertex x of the fuzzy vertices of $G = (\sigma, \mu)$, and $\mu(x,y) = 1$, for every edge (x,y) of the fuzzy edges of $G = (\sigma, \mu)$ and is denoted by $G^* = (\sigma, \mu)$ or G^*. The order p and size q of a fuzzy graph $G = (\sigma, \mu)$ are defined as $p = \sum_{x \in V} \sigma(x)$ and $q = \sum_{xy \in E} \mu(x,y)$. Let $\sigma : V \to [0,1]$ be a fuzzy subset of V. Then the complete fuzzy graph on σ is defined as $G(\sigma, \mu)$, where $\mu(x,y) > 0$, for all $x, y \in E$ and is denoted by K_σ. A fuzzy graph $G = (\sigma, \mu)$ is said to be bipartite if the set of vertices V can be partitioned into two nonempty sets V_1 and V_2 such that $\mu(v_1, v_2) = 0$ if $v_1 \in V_1$ and $v_2 \in V_1$ or $v_1 \in V_2$ and $v_2 \in V_2$. Further if $\mu(u,v) > 0$ for all $u \in V_1$ and $v \in V_2$, then G is called a complete bipartite fuzzy graph and is denoted by K_{σ_1, σ_2}, where σ_1 and σ_2 are the restrictions of σ to V_1 and V_2, respectively.

Let $G = (\sigma, \mu)$ be a fuzzy graph on V and $S \subseteq_f V$. Then the fuzzy cardinality of S is defined to be $\sum_{v \in S} \sigma(v)$ and is denoted by $|S|_f$. For a vertex x, the set $N(x) = \{y \in V | \mu(x,y) > 0\}$ is called the neighborhood of x and $N[x] = \{N(x) \cup \{x\}\}$ is the closed neighborhood of x. A vertex u of a fuzzy graph G is said to be an isolated vertex if $\mu(u,v) = 0$ for all $v \in V \setminus \{u\}$, that is, $N(u) = \emptyset$, [14]. Also $\mu(v, S)$ is equal to the minimum membership of edges between v and vertices of the set S. The degree of a vertex v is $\sum \sigma(u)$, where $u \in N(v)$. We denote by $\Delta_f(G)$ and $\delta_f(G)$ the maximum and minimum degree in fuzzy graph $G = (\sigma, \mu)$, respectively. For a vertex v, we define the depth of v as the minimum membership of edges adjacent to v and denote it by $d(v)$. A vertex with maximum membership in a fuzzy graph G is denoted by v_s and a vertex with minimum membership in fuzzy graph G is denoted by v_w. Also an edge with maximum membership is denoted by e_s and an edge by minimum membership is denoted by e_w. The complement of a fuzzy graph $G = (\sigma, \mu)$, denoted by $\bar{G} = (\sigma, \bar{\mu})$, is $\bar{G} = (\sigma, \bar{\mu})$, where $\bar{\mu}(x,y) = \sigma(x) \wedge \sigma(y) - \mu(x,y)$, for all $x, y \in V$, [11].

A path P in a fuzzy graph $G = (\sigma, \mu)$ is a sequence of distinct vertices $x_0, x_1, ..., x_n$ (except possibly x_0 and x_n) such that $\mu(x_{i-1}, x_i) > 0, 1 \leq i \leq n$. Here $n \geq 1$ is called the length of the path P. The consecutive pairs (x_{i-1}, x_i) are called the edges of the path. The strength of the path of length k from x_0 to x_k, is defined as $\wedge_{i=1}^{k} \mu(x_{i-1}, x_i)$ and is denoted by $\mu^k(x_0, x_k)$. In other words, the strength of a path is defined to be the weight of the weakest edge of the path.

A fuzzy graph $G = (\sigma, \mu)$ is connected if there exists a path with positive strength between every two arbitrary vertices of G. The strength of connectedness between two vertices x and y is defined as the maximum strength among all paths between x and y and is denoted by $CONN_G(x,y)$, [11]. A single vertex x may also be considered as a path. In this case, the path is of length 0. If a path has length 0, it is convenient to define its strength to be $\mu(x_0)$. It may be noted that any path of length $n > 0$ can be defined as a sequence of edges (x_{i-1}, x_i), $1 \leq i \leq n$, satisfying the condition $\mu(x_{i-1}, x_i) > 0$ for $1 \leq i \leq n$, [14].

A vertex x in a fuzzy graph $G = (\sigma, \mu)$, dominates a vertex y if $\mu(x, y) > 0$. A subset S of vertices is called a *dominating set* in $G = (\sigma, \mu)$ if for every vertex $v \notin S$, there exists a vertex $u \in S$ such that u dominates v. The minimum fuzzy cardinality of a dominating set in $G = (\sigma, \mu)$ is called the *domination number* of fuzzy graph $G = (\sigma, \mu)$ and is denoted by $\gamma_f(G)$ or γ_f. A subset S of vertices is called a *total dominating set* in $G = (\sigma, \mu)$ if for every vertex v of $G = (\sigma, \mu)$, there exists $u \in S$ such that u dominates v. The minimum fuzzy cardinality of a total dominating set in $G = (\sigma, \mu)$ is called the *total domination number* of fuzzy graph $G = (\sigma, \mu)$ and is denoted by $\gamma_{tf}(G)$ or γ_{tf}. Domination in fuzzy graphs are studied in, for example [5, 6, 12, 13].

Let $G = (\sigma, \mu)$ be a fuzzy graph. Analogue to k-step domination in graphs we can define k-step domination in fuzzy graphs as follows. A set $S \subseteq V(G)$ is a *k-step dominating set* of G if for every vertex v there exists at least one path of length k between v and the vertices of S, that is, there exists at least one vertex u of S such that $\mu^k(v, u) > 0$. A k-step dominating set S of G such that the sets $N_k(v) = \{u \in V(G) | \mu^k(v, u) > 0\}$, $v \in S$, are pair-wise disjoint, is called an *exact k-step dominating set*. If a fuzzy graph G has an exact k-step dominating set, then G is called an *exact k-step domination fuzzy graph*.

3. Strongest dominating set in fuzzy graphs

Following the definition of dominating sets in fuzzy graphs, we note that every vertex has a degree of membership in a fuzzy graph. So two different minimum dominating sets of a fuzzy graph may have non-equal fuzzy cardinality. Furthermore, given a dominating set S in a fuzzy graph $G = (\sigma, \mu)$, a vertex may be dominated by several vertices of S with different memberships. This motivate us to define the best dominating set for a fuzzy graph $G = (\sigma, \mu)$ by contemplate degree of membership of vertices and edges as follows. For a dominating set S and a vertex $v \in G \setminus S$, we define $max\{\mu(u, v) | u \in S, v \in G \setminus S\}$ as the *strength of dominance* on v and denote it by $sdom(v, S)$. We also define $\min\{sdom(v, S) | v \in G \setminus S\}$ as the *dominate strength* of S and denote it by $sdom(G \setminus S, S)$. We denote by $S_s(G)$ the set of minimum dominating sets with maximum $sdom(G \setminus S, S)$. A set with maximum fuzzy cardinality between all minimum dominating sets of $S_s(G)$ is called the *strongest dominating set* and its fuzzy cardinality is called *strength of strongest dominating set* in $G = (\sigma, \mu)$ and is denoted it by $ssd(G)$.

Let us to give an example to represent these concepts.

Example 1. Consider the fuzzy graph $G = (\sigma, \mu)$ given in Figure 1. For this fuzzy graph, the subsets $S_1 = \{a_1, a_6\}$, $S_2 = \{a_1, a_5\}$, $S_3 = \{a_2, a_6\}$ and $S_4 = \{a_2, a_5\}$ are minimum

dominating sets. Now we calculate $sdom(G \setminus S_i, S_i)$ for $i = 1, 2, 3, 4$.
For S_1 we have $sdom(a_2, S_1) = 0.2$, $sdom(a_3, S_1) = 0.2$, $sdom(a_4, S_1) = 0.5$, $sdom(a_5, S_1) = 0.4$, $sdom(a_7, S_1) = 0.2$, $sdom(a_8, S_1) = 0.3$. So $sdom(G \setminus S_1, S_1) = 0.2$. For S_2 we have $sdom(a_2, S_2) = 0.2$, $sdom(a_3, S_2) = 0.2$, $sdom(a_4, S_2) = 0.2$, $sdom(a_6, S_2) = 0.4$, $sdom(a_7, S_2) = 0.2$, $sdom(a_8, S_2) = 0.1$. So $sdom(G \setminus S_2, S_2) = 0.1$. For S_3 we have $sdom(a_1, S_3) = 0.2$, $sdom(a_3, S_3) = 0.2$, $sdom(a_4, S_3) = 0.5$, $sdom(a_5, S_3) = 0.4$, $sdom(a_7, S_3) = 0.3$, $sdom(a_8, S_3) = 0.3$. So $sdom(G \setminus S_3, S_3) = 0.2$. Finally, for S_4 we have $sdom(a_1, S_4) = 0.2$, $sdom(a_3, S_4) = 0.2$, $sdom(a_4, S_4) = 0.2$, $sdom(a_6, S_4) = 0.4$, $sdom(a_7, S_4) = 0.3$, $sdom(a_8, S_4) = 0.1$. So $sdom(G \setminus S_1, S_1) = 0.1$. Then $S_s = \{S_1, S_3\}$, $|S_1|_f = 0.2+0.6 = 0.8$ and $|S_3|_f = 0.3+0.6 = 0.9$. Therefore, S_3 is the strongest dominating set of fuzzy graph G and $ssd(G) = 0.9$.

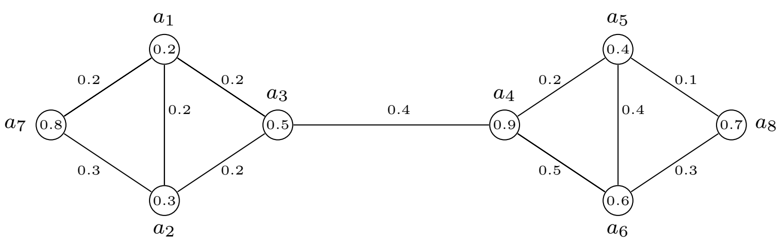

Figure 1. **A graph $G = (\sigma, \mu)$ with $ssd(G) = 0.9$**

In the following, we give an upper bound for the strength of strongest dominating set in fuzzy graphs.

Theorem 1. For any fuzzy graph $G = (\sigma, \mu)$ of order p, we have $ssd(G) \leq p$, and equality holds if and only if each vertex of G is an isolated vertex.

Proof. Let $G = (\sigma, \mu)$ be a fuzzy graph of order p and S a strongest dominating set of $G = (\sigma, \mu)$. Because $ssd(G)$ is the fuzzy cardinality of S and $S \subseteq V(G)$, we have $|S|_f \leq |V(G)|_f = \sum_{x \in V} \sigma(x) = p$. We next prove the equality part. Assume that all vertices of the fuzzy graph $G = (\sigma, \mu)$ are isolated vertices. Then $V(G)$ is only dominating set of G, and so $S = V(G)$ is the strongest dominating set of G. Thus, $ssd(G) = |S|_f = \sum_{x \in S} \sigma(x) = \sum_{x \in V} \sigma(x) = p$. Conversely, let $ssd(G) = p$. If x is a vertex of G that is not an isolated vertex and y is a vertex adjacent to x, then x is dominated by y and the set $V(G) \setminus \{x\}$ is a dominating set of G, implying that $ssd(G) < p$, a contradiction. Thus each vertex of G is an isolated vertex. $\square$

We next prove that $ssd(G)$ is bounded bellow by $\gamma_f(G)$ in any fuzzy graph $G = (\sigma, \mu)$.

Theorem 2. For every fuzzy graph $G = (\sigma, \mu)$, we have $\gamma_f(G) \leq ssd(G)$.

Proof. Let S be a dominating set of $G = (\sigma, \mu)$ such that $|S|_f = \gamma_f(G)$. So S has minimum fuzzy cardinality among all dominating sets of $G = (\sigma, \mu)$. Let S' be a minimum dominating set of $S_s(G)$ with maximum fuzzy cardinality, and $|S'|_f = ssd(G)$. Then $\gamma_f(G) = |S|_f \leq \min\{|w|_f | w \in S_s\} \leq \max\{|w|_f | w \in S_s\} = |S'|_f = ssd(G)$. $\square$

The following theorem provides lower and upper bounds for the $ssd(G)$ in terms of the domination number of the underlying graph of G.

Theorem 3. For every fuzzy graph $G = (\sigma, \mu)$, we have $\gamma(G^*).\sigma(v_w) \leq ssd(G) \leq \gamma(G^*).\sigma(v_s)$.

Proof. Let S be a minimum dominating set of underlying graph of $G = (\sigma, \mu)$ such that $|S| = \gamma(G^*)$. Since v_s has maximum membership in G, we have $ssd(G) = |S|_f \leq |S|.\sigma(v_s) = \gamma(G^*).\sigma(v_s)$. Similarly, since v_w has minimum membership in G, we have $\gamma(G^*).\sigma(v_w) = |S|.\sigma(v_w) \leq |S|_f = ssd(G)$. Thus, $\gamma(G^*).\sigma(v_w) \leq ssd(G) \leq \gamma(G^*).\sigma(v_s)$. $\square$

Now, we obtain strength of strongest dominating set for complete fuzzy graphs as well as complete bipartite fuzzy graphs.

Theorem 4. Let $G = (\sigma, \mu)$ be a complete fuzzy graph K_σ. Then $ssd(K_\sigma) = max\{\sigma(v) | v \in S\}$, where S is the set of vertices of K_σ with maximum depth.

Proof. Let K_σ a the complete fuzzy graph. For every minimum dominating set $D = \{v\}$ of K_σ, and any vertex $u \in V \setminus D$ we have $sdom(u, D) = \mu(u, v)$. The strength of $D = \{v\}$ is $sdom(K_\sigma \setminus D, D) = sdom(K_\sigma \setminus \{v\}, v) = min\{\mu(u, v) | u \in K_\sigma \setminus \{v\}\} = d(v)$. So the set S_s contains vertices of K_σ with maximum depth among all vertices of K_σ, and the strength of strongest dominating set in K_σ is equal to $ssd(K_\sigma) = max\{|v|_f | v \in S_s\} = max\{\sigma(v) | v \in S_s\}$. $\square$

Theorem 5. Let $G = (\sigma, \mu)$ be a complete bipartite fuzzy graph K_{σ_1,σ_2}. Then $ssd(K_{\sigma_1,\sigma_2}) = max\{\sigma_1(v) + \sigma_2(w) | \{v, w\} \in S\}$, where S is the set contains every pair of vertices $\{v, w\}$ such that $v \in \sigma_1$, $w \in \sigma_2$ and $d(v) \wedge d(w)$ is maximum among all pair of vertices $\{v, w\}$ with $v \in \sigma_1$ and $w \in \sigma_2$.

Proof. Let K_{σ_1,σ_2} be a complete bipartite fuzzy graph and $D = \{v, w\}$ be a subset of $V(K_{\sigma_1,\sigma_2})$ such that $v \in \sigma_1$ and $w \in \sigma_2$. It is clear that D is a minimum dominating set for K_{σ_1,σ_2}. For every vertex $u \in V \setminus D$ if $u \in \sigma_1$, then $sdom(u, D) = \mu(u, w)$, while if $u \in \sigma_2$, then $sdom(u, D) = \mu(u, v)$. The strength of D is $sdom(K_{\sigma_1,\sigma_2} \setminus D, D) = sdom(K_{\sigma_1,\sigma_2} \setminus \{v, w\}, \{v, w\}) = d(v) \wedge d(w)$. So the set S_s contains every pair of vertices $\{v, w\}$ such that $v \in \sigma_1$, $w \in \sigma_2$ and $d(v) \wedge d(w)$ is maximum among all pair vertices $\{v, w\}$ of K_{σ_1,σ_2} with $v \in \sigma_1$ and $w \in \sigma_2$. Thus the strength of strongest dominating set in K_{σ_1,σ_2} is equal to $ssd(K_{\sigma_1,\sigma_2}) = max\{|\{v, w\}|_f | \{v, w\} \in S_s\} = max\{\sigma_1(v) + \sigma_2(w) | \{v, w\} \in S_s\}$. $\square$

4. Strongest k-step dominating set in fuzzy graphs

In these section, we introduce and study strongest k-step dominating set in fuzzy graphs. Let $G = (\sigma, \mu)$ be a fuzzy graph, S a k-step dominating set of G, and v an arbitrary vertex of G. There exists at least one vertex of S at distance k of v. Let there exists r vertices u_i, i=1,2, …,r, of S at distance k of v. Assume r paths, $Q_1, \ldots, Q_r$, of length k between v and S, where $Q_i = (v, u_i)$, for $i = 1, 2, \ldots, r$. Now for every $v \in V(G)$, we define the *strength of dominance* on v as $\vee_{i=1}^{r} \mu^k(v, u_i)$, $u_i \in S$, and denote it by $sdom_k(v, S)$. Also we define $\min\{sdom_k(v, S)|v \in G\}$ as the *dominate strength* of the k-step dominating set S and denote it by $sdom_k(G, S)$. We denote the set of minimum k-step dominating sets with maximum $sdom_k(G, S)$ by $S_{sk}(G)$. A set with maximum fuzzy cardinality between all minimum k-step dominating sets of S_{sk} is called the *strongest k-step dominating set* of the fuzzy graph $G = (\sigma, \mu)$ and its fuzzy cardinality is called strength of strongest k-step dominating set in fuzzy graph $G = (\sigma, \mu)$ and is denoted by $ssd_k(G)$.

Now let $G = (\sigma, \mu)$ be an exact k-step domination fuzzy graph and S be an exact k-step dominating set of G, and v an arbitrary vertex of G. There exists exactly one vertex as x_i of S at distance k of v. So for every $v \in V(G)$, we define the *strength of dominance* on v with exact k-step dominating set S as $\vee_{i=1}^{r} \mu^k(v, u_i) = \mu^k(v, x_i)$, $u_i \in S$, and denote it by $sdom_{ek}(v, S)$. Also we define $\min\{sdom_{ek}(v, S)|v \in G\} = \min\{\mu^k(v, x_i)|v \in G\}$ as the *dominate strength* of the exact k-step dominating set S and denote it by $sdom_{ek}(G, S)$. We denote the set of minimum exact k-step dominating sets with maximum $sdom_{ek}(G, S)$ by $S_{esk}(G)$. A set with maximum fuzzy cardinality between all minimum exact k-step dominating sets of S_{esk} is called the *strongest exact k-step dominating set* of the fuzzy graph $G = (\sigma, \mu)$ and its fuzzy cardinality is called strength of strongest exact k-step dominating set in fuzzy graph $G = (\sigma, \mu)$ and is denoted by $ssd_{ek}(G)$.
We next obtain new bounds for the strength of dominance on a vertex v and the dominate strength of a k-step dominating set S of a fuzzy graph $G = (\sigma, \mu)$.

Theorem 6. Let $G = (\sigma, \mu)$ be a fuzzy graph and v an arbitrary vertex of G. If S is a k-step dominating set of $G = (\sigma, \mu)$, then $\mu(e_w) \leq sdom_k(v, S) \leq \mu(e_s)$.

Proof. Let $G = (\sigma, \mu)$ be a fuzzy graph, S be a k-step dominating set of G and v a vertex of G. There exists at least one path of length k between v and a vertex of S. Let there exists r paths, $Q_1, \ldots, Q_r$, of length k between v and the vertices $u_i \in S$, $i = 1, 2, \ldots, r$, where the vertices $u_i \in S$ are not necessarily distinct. So we have $sdom_k(v, S) = \vee_{i=1}^{r} \mu^k(v, u_i) = \vee_{i=1}^{r}\{\wedge_{j=1}^{k} \mu(x_{j-1}, x_j)|x_0 = v \in G, x_k = u_i \in S\}$. Since the edge e_s has the maximum membership between all edges, so for every $j = 1, ..., k$ we have $\mu(x_{j-1}, x_j) \leq \mu(e_s)$ and so $\wedge_{j=1}^{k} \mu(x_{j-1}, x_j) \leq \mu(e_s)$. Therefore, $sdom_k(v, S) \leq \mu(e_s)$. Moreover, since the edge e_w has the minimum membership between all edges, we have $\mu(e_w) \leq \wedge_{j=1}^{k} \mu(x_{j-1}, x_j)$. Therefore, $\mu(e_w) \leq sdom_k(v, S)$.

□

Corollary 1. If $G = (\sigma, \mu)$ is a fuzzy graph and S is a k-step dominating set of G, then $\mu(e_w) \leq sdom_k(G, S) \leq \mu(e_s)$.

Theorem 7. If $G = (\sigma, \mu)$ is a fuzzy graph and S is a k-step dominating set of G, then for every vertex $v \in G$ we have

$$sdom_k(v, S) \leq \vee_{i=1}^{r} \{CONN_G(v, u_i) | u_i \in S\}.$$

Proof. Let $G = (\sigma, \mu)$ be a fuzzy graph and S be a k-step dominating set of G. Let v be an arbitrary vertex of G. Since S is a k-step dominating set, there exists at least one path of length k between v and a vertex like u of S. Let there exists r paths, $Q_1, \ldots, Q_r$, of length k between v and the vertices u_i of the k-step dominating set S, $i = 1, 2, \ldots, r$, where the vertices u_i are not necessarily distinct. Since $CONN_G(v, u_i)$ is maximum strength of all paths between v and u_i we have $\mu^k(v, u_i) \leq CONN_G(v, u_i)$ for every $i = 1, ...r$. So

$$sdom_k(v, S) = \vee_{i=1}^{r} \mu^k(v, u_i) \leq \vee_{i=1}^{r} \{CONN_G(v, u_i) | u_i \in S\}.$$

□

We next provide a relation between the total domination number and strength of strongest exact 1-step dominating set of exact 1-step domination fuzzy graphs $G = (\sigma, \mu)$.

Theorem 8. For every exact 1-step domination fuzzy graph $G = (\sigma, \mu)$, we have $\gamma_{tf}(G) \leq ssd_{e1}(G)$.

Proof. Let S be a total dominating set of $G = (\sigma, \mu)$ such that $|S|_f = \gamma_f(G)$. So S has minimum fuzzy cardinality between all total dominating sets of $G = (\sigma, \mu)$. Let S' be a minimum total dominating set of $S_{es1}(G)$ with maximum fuzzy cardinality. So $|S'|_f = ssd_{e1}(G)$. Therefore, we have

$$\gamma_{tf}(G) = |S|_f \leq min\{|w|_f | w \in S_{es1}\} \leq \max\{|w|_f | w \in S_{es1}\} = |S'|_f = ssd_{e1}(G).$$

□

Gavlas and Schultz Proved in [7] that all exact 1-step dominating sets of a graph G have the same cardinality.

Theorem 9. ***(Gavlas and Schultz [7])*** Let G be an exact 1-step domination graph. Every exact 1-step dominating set of G has cardinality $\gamma_t(G)$.

Theorem 10. For every exact 1-step domination fuzzy graph $G = (\sigma, \mu)$, we have

$$\gamma_t(G^*).\sigma(v_w) \leq ssd_{e1}(G) \leq \gamma_t(G^*).\sigma(v_s).$$

Proof. Let S be an exact 1-step dominating set of the fuzzy graph $G = (\sigma, \mu)$ and $D \in S_{es1}(G)$ be an exact 1-step dominating set of G such that $|D|_f = ssd_{e1}(G)$. By Theorem 9, $|D| = \gamma_t(G^*)$. Since v_s is the vertex with maximum membership in G, we have $ssd_{e1}(G) = |D|_f \leq |D|.\sigma(v_s) = \gamma_t(G^*).\sigma(v_s)$. Thus the upper bound follows. For the lower bound, since v_w is the vertex by minimum membership in G, by Theorem 9, we have $\gamma_t(G^*).\sigma(v_w) = |D|.\sigma(v_w) \leq |D|_f = ssd_{e1}(G)$, as desired. □

Cockayne, Dams and Hedetniemi proved the following in [1] that relates total domination number and the maximum degree $\Delta(G)$.

Theorem 11 (Cockayne et al. [1]). (i) If G has n vertices and no isolates, then $\gamma_t(G) \leq n - \Delta(G) + 1$.
(ii) If G is connected and $\Delta(G) < n - 1$, then $\gamma_t(G) \leq n - \Delta(G)$.

We now prove analogous results for fuzzy graphs.

Theorem 12. (i) If $G = (\sigma, \mu)$ is a fuzzy graph whose underlying graph has order n and with no isolates, then $ssd_{e1}(G) \leq (n - \Delta_f(G) + 1).\sigma(v_s)$.
(ii) If $G = (\sigma, \mu)$ is a fuzzy graph whose underlying graph is connected and $\Delta(G^*) < n - 1$, then $ssd_{e1}(G) \leq (n - \Delta_f(G)).\sigma(v_s)$.

Proof. (i) Let $G = (\sigma, \mu)$ is a fuzzy graph. By Theorem 10, we have $ssd_{e1}(G) \leq \gamma_t(G^*).\sigma(v_s)$. On the other hand, $\Delta_f(G) = max_{v \in V(G)}\{\sum \sigma(u) | u \in N(v)\}$. It is clear that $\Delta_f(G) \leq \Delta(G^*)$. So by Theorem 11 we have

$$ssd_{e1}(G) \leq \gamma_t(G^*).\sigma(v_s) \leq (n - \Delta(G^*) + 1).\sigma(v_s) \leq (n - \Delta_f(G) + 1).\sigma(v_s).$$

(ii) Similar to part (i) we have

$$ssd_{e1}(G) \leq \gamma_t(G^*).\sigma(v_s) \leq (n - \Delta(G^*)).\sigma(v_s) \leq (n - \Delta_f(G)).\sigma(v_s).$$

□

Now by Theorem 12 we have following Nordhaus-Gaddum type bound for the strength of strongest 1-step dominating set in fuzzy graphs.

Theorem 13. If $G = (\sigma, \mu)$ is a fuzzy graph that the underlying graph of it has n vertices and no isolates and $\Delta(G^*) < n - 1$, then

$$ssd_{e1}(G) + ssd_{e1}(\bar{G}) \leq (n(2 - \sigma(v_w)) + 2).\sigma(v_s).$$

Proof. Let $G = (\sigma, \mu)$ is a fuzzy graph with no isolated vertices. Since G or $\bar{G}$ is connected, we will assume without loss of generality, that $\bar{G} = (\sigma, \bar{\mu})$ is connected. By Theorem 12, $ssd_{e1}(\bar{G}) \leq (n - \Delta_f(\bar{G})).\sigma(v_s)$, and also

$$ssd_{e1}(G) \leq (n - \Delta_f(G) + 1).\sigma(v_s) \leq (n - \delta_f(G) + 1).\sigma(v_s).$$

Therefore, $ssd_{e1}(G) + ssd_{e1}(\bar{G}) \leq (2n + 1 - (\delta_f(G) + \Delta_f(\bar{G}))).\sigma(v_s)$. It is clear that $(n-1).\sigma(v_w) \leq \delta_f G) + \Delta_f(\bar{G}) \leq (n-1).\sigma(v_s)$. So we have

$$\begin{aligned} ssd_{e1}(G) + ssd_{e1}(\bar{G}) &\leq (2n + 1 - (\delta_f(G) + \Delta_f(\bar{G}))).\sigma(v_s) \\ &\leq (2n + 1 - ((n-1).\sigma(v_w))).\sigma(v_s) \\ &= (n(2 - \sigma(v_w)) + \sigma(v_w) + 1).\sigma(v_s) \\ &\leq (n(2 - \sigma(v_w)) + 1 + 1).\sigma(v_s) \\ &= (n(2 - \sigma(v_w)) + 2).\sigma(v_s). \end{aligned}$$

□

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