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Research Article

Weak signed Roman k-domatic number of a graph

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Abstract: Let $k \geq 1$ be an integer. A weak signed Roman k-dominating function on a graph G is a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{u \in N[v]} f(u) \geq k$ for every $v \in V(G)$, where $N[v]$ is the closed neighborhood of v. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct weak signed Roman k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called a weak signed Roman k-dominating family (of functions) on G. The maximum number of functions in a weak signed Roman k-dominating family on G is the weak signed Roman k-domatic number of G, denoted by $d^k_{wsR}(G)$. In this paper we initiate the study of the weak signed Roman k-domatic number in graphs, and we present sharp bounds for $d^k_{wsR}(G)$. In addition, we determine the weak signed Roman k-domatic number of some graphs.

Keywords: weak signed Roman k-dominating function, weak signed Roman k-domination number, weak signed Roman k-domatic number.

AMS Subject classification: 05C69

1. Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [4]. Specifically, let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A graph G is *regular* or *r-regular* if $d(v) = r$ for each vertex v of G. The complement of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n, $K_{p,q}$ for the *complete bipartite graph* with partite sets X and Y, where $|X| = p$ and $|Y| = q$, and C_n for the *cycle* of length n.

In this paper we continue the study of Roman dominating functions in graphs and digraphs. If $k \geq 1$ is an integer, then the *signed Roman k-dominating function* (SRkDF) on a graph G is defined in [5] as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{u \in N[v]} f(u) \geq k$ for each $v \in V(G)$, and such that every vertex $u \in V(G)$ for which $f(u) = -1$ is adjacent to at least one vertex w for which $f(w) = 2$. The *weight* of an SRkDF f is the value $\omega(f) = \sum_{v \in V} f(v)$. The *signed Roman k-domination number* of a graph G, denoted by $\gamma^k_{sR}(G)$, equals the minimum weight of an SRkDF on G. A $\gamma^k_{sR}(G)$-*function* is a signed Roman k-dominating function of G with weight $\gamma^k_{sR}(G)$. If $k = 1$, then we write $\gamma^1_{sR}(G) = \gamma_{sR}(G)$. This case was introduced and studied in [1]. The signed Roman domination number of digraphs was investigated in [8].

A *weak signed Roman k-dominating function* (WSRkDF) on a graph G is defined in [12] as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{u \in N[v]} f(u) \geq k$ for each $v \in V(G)$. The *weight* of a WSRkDF f is the value $\omega(f) = \sum_{v \in V} f(v)$. The *weak signed Roman k-domination number* of a graph G, denoted by $\gamma^k_{wsR}(G)$, equals the minimum weight of a WSRkDF on G. A $\gamma^k_{wsR}(G)$-*function* is a weak signed Roman k-dominating function of G with weight $\gamma^k_{wsR}(G)$. The special case $k = 1$ was introduced and investigated by Volkmann [11].

The weak signed Roman k-domination number exists when $\delta \geq \frac{k}{2} - 1$. Therefore we assume in this paper that $\delta \geq \frac{k}{2} - 1$. The definitions lead to $\gamma^k_{wsR}(G) \leq \gamma^k_{sR}(G)$.

A concept dual in a certain sense to the domination number is the domatic number, introduced by Cockayne and Hedetniemi [3]. They have defined the domatic number $d(G)$ of a graph G by means of sets. A partition of $V(G)$, all of whose classes are dominating sets in G, is called a *domatic partition*. The maximum number of classes of a domatic partition of G is the *domatic number* $d(G)$ of G. But Rall has defined a variant of the domatic number of G, namely the *fractional domatic number* of G, using functions on $V(G)$. (This was mentioned by Slater and Trees in [9].) Analogous to the fractional domatic number we may define the (weak) signed Roman k-domatic number.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct (weak) signed Roman k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called a *(weak) signed Roman k-dominating family* (of functions) on G. The maximum number of functions in a (weak) signed Roman k-dominating family ((W)SRkD family) on G is the *(weak) signed Roman k-domatic number* of G, denoted by ($d^k_{wsR}(G)$) $d^k_{sR}(G)$. The (weak) signed Roman k-domatic number is well-defined and $d^k_{wsR}(G) \geq d^k_{sR}(G) \geq 1$ for all graphs G with $\delta(G) \geq \frac{k}{2} - 1$, since the set consisting of any (W)SRkDF forms a (W)SRkD family on G. For more information on the Roman domatic problem, we refer the reader to the survey article [2].

Our purpose in this paper is to initiate the study of the weak signed Roman k-domatic number in graphs. We first derive basic properties and bounds for the weak signed Roman k-domatic number of a graph. In addition, we determine the weak signed Roman k-domatic number of some classes of graphs.

We make use of the following known results in this paper.

Proposition A. ([7, 10],) *If $n \geq k \geq 1$ are integers, then $d^k_{sR}(K_n) = n$, unless $n = 3$ and $k = 1$ or $n = k = 2$, in which cases $d^1_{sR}(K_3) = 1$ or $d^2_{sR}(K_2) = 1$.*

Proposition B. ([12]) *If $n \geq k \geq 2$ are integers, then $\gamma^k_{wsR}(K_n) = k$.*

Proposition C. ([12]) *If G is an r-regular graph of order n with $r \geq \frac{k}{2} - 1$, then*

$$\gamma^k_{wsR}(G) \geq \frac{kn}{r+1}.$$

Proposition D. ([5]) *If G is a graph of order n with $\delta(G) \geq k - 1$, then*

$$\gamma^k_{wsR}(G) \leq \gamma^k_{sR}(G) \leq n.$$

Proposition E. ([12]) *Let G be a graph of order n with $\delta(G) \geq \lceil \frac{k}{2} \rceil - 1$. Then $\gamma^k_{wsR}(G) \leq 2n$, with equality if and only if k is even, $\delta(G) = \frac{k}{2} - 1$, and each vertex of G is of minimum degree or adjacent to a vertex of minimum degree.*

2. Bounds on the weak signed Roman k-domatic number

In this section we present basic properties of $d^k_{wsR}(G)$ and sharp bounds on the weak signed Roman k-domatic number of a graph.

Theorem 1. If G is a graph, then $d^k_{wsR}(G) \leq \delta(G)+1$. Moreover, if $d^k_{wsR}(G) = \delta(G)+1$, then for each WSRkD family $\{f_1, f_2, \ldots, f_d\}$ on G with $d = d^k_{wsR}(G)$ and each vertex v of minimum degree, $\sum_{x\in N[v]} f_i(x) = k$ for each function f_i and $\sum_{i=1}^{d} f_i(x) = k$ for all $x \in N[v]$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a WSRkD family on G such that $d = d^k_{wsR}(G)$. If v is a vertex of minimum degree $\delta(G)$, then we deduce that

$$\begin{aligned} kd &\leq \sum_{i=1}^{d} \sum_{x\in N[v]} f_i(x) = \sum_{x\in N[v]} \sum_{i=1}^{d} f_i(x) \\ &\leq \sum_{x\in N[v]} k = k(\delta(G)+1) \end{aligned}$$

and thus $d^k_{wsR}(G) \leq \delta(G) + 1$.
If $d^k_{wsR}(G) = \delta(G) + 1$, then the two inequalities occurring in the proof become equalities. Hence for the WSRkD family $\{f_1, f_2, \ldots, f_d\}$ on G and for each vertex v of minimum degree, $\sum_{x\in N[v]} f_i(x) = k$ for each function f_i and $\sum_{i=1}^{d} f_i(x) = k$ for all $x \in N[v]$. □

Example 1. *If $n \geq k \geq 1$ are integers, then $d^k_{wsR}(K_n) = n$, unless $n = k = 2$, in which case $d^2_{wsR}(K_2) = 1$.*

Proof. Theorem 1 implies $d^k_{wsR}(K_n) \le n$. It is easy to see that $d^2_{wsR}(K_2) = 1$ and $d^1_{wsR}(K_3) = 3$. In all other cases, Proposition A leads to $d^k_{wsR}(K_n) \ge d^k_{sR}(K_n) = n$, and the proof is complete. □

Theorem 2. If G is a graph of order n, then

$$\gamma^k_{wsR}(G) \cdot d^k_{wsR}(G) \le kn.$$

Moreover, if $\gamma^k_{wsR}(G) \cdot d^k_{wsR}(G) = kn$, then for each WSRkD family $\{f_1, f_2, \dots, f_d\}$ on G with $d = d^k_{wsR}(G)$, each function f_i is a $\gamma^k_{wsR}(G)$-function and $\sum_{i=1}^d f_i(v) = k$ for all $v \in V(G)$.

Proof. Let $\{f_1, f_2, \dots, f_d\}$ be a WSRkD family on G such that $d = d^k_{wsR}(G)$ and let $v \in V(G)$. Then

$$\begin{aligned} d \cdot \gamma^k_{wsR}(G) &= \sum_{i=1}^{d} \gamma^k_{wsR}(G) \le \sum_{i=1}^{d} \sum_{v \in V(G)} f_i(v) \\ &= \sum_{v \in V(G)} \sum_{i=1}^{d} f_i(v) \le \sum_{v \in V(G)} k = kn. \end{aligned}$$

If $\gamma^k_{wsR}(G) \cdot d^k_{wsR}(G) = kn$, then the two inequalities occurring in the proof become equalities. Hence for the WSRkD family $\{f_1, f_2, \dots, f_d\}$ on G and for each i, $\sum_{v \in V(G)} f_i(v) = \gamma^k_{wsR}(G)$. Thus each function f_i is a $\gamma^k_{wsR}(G)$-function, and $\sum_{i=1}^d f_i(v) = k$ for all $v \in V(G)$. □

Example 2. *If $k, n \ge 1$ are integers such that $n + 1 \le k \le 2n - 1$, then $d^k_{wsR}(K_n) = n$.*

Proof. Theorem 1 implies $d^k_{wsR}(K_n) \le n$. Now let $x_1, x_2, \dots, x_n$ be the vertices of K_n, and let $k = n + t$ for an integer $1 \le t \le n - 1$. For $1 \le i \le n$, define the function $f_i : V(K_n) \longrightarrow \{-1, 1, 2\}$ by

$$f_i(x_i) = f_i(x_{i+1}) = \ldots = f_i(x_{i+t-1}) = 2$$

and

$$f_i(x_{i+t}) = f_i(x_{i+t+1}) = \ldots = f_i(x_{i+n-1}) = 1,$$

where the indices are taken modulo n. It is easy to verify that f_i is a weak signed Roman k-dominating function on K_n for $1 \le i \le n$ and $\{f_1, f_2, \dots, f_n\}$ is a weak signed Roman k-dominating family on K_n. Hence $d^k_{wsR}(K_n) \ge n$ and thus $d^k_{wsR}(K_n) = n$. □

Examples 1, 2 and Proposition B demonstrate that Theorems 1 and 2 are both sharp.

Theorem 3. Let G be a graph of order $n \geq 2$ with $\delta(G) \geq \lceil \frac{k}{2} \rceil - 1$. Then $d^k_{wsR}(G) = n$ if and only if $G = K_n$, with exception of the cases $k = 2n$ or $k = n = 2$, in which cases $d^{2n}_{wsR}(K_n) = 1$ or $d^2_{wsR}(K_2) = 1$.

Proof. Let $G = K_n$. If $k = 2n$, then the function f with $f(x) = 2$ for each vertex $x \in V(G)$ is the unique weak signed Roman dominating function on G and so $d^{2n}_{wsR}(K_n) = 1$. In addition, it follows from Examples 1 and 2 that $d^2_{wsR}(K_2) = 1$, $d^1_{wsR}(K_3) = 3$ and $d^k_{wsR}(K_n) = n$ in the remaining cases.
Conversely, assume that $d^k_{wsR}(G) = n$. Then we deduce from Theorem 1 that $n = d^k_{wsR}(G) \leq \delta(G) + 1$, and so $\delta(G) \geq n - 1$. Thus $G = K_n$, and the proof is complete. □

Theorem 3 shows that Theorem 1 is sharp.

Theorem 4. Let $k \geq 3$ be an integer, and let G be a graph of order n with $\delta(G) \geq \lceil \frac{k}{2} \rceil - 1$. If $\gamma^k_{wsR}(G) \leq 2n - 1$, then $d^k_{wsR}(G) \geq 2$.

Proof. Assume first that $k \geq 4$. Since $\gamma^k_{wsR}(G) \leq 2n - 1$, there exists a WSRkDF f_1 with $f_1(v) \leq 1$ for at least one vertex $v \in V(G)$. Note that $f_2 : V(G) \longrightarrow \{-1, 1, 2\}$ with $f_2(x) = 2$ for each vertex $x \in V(G)$ is another WSRkDF on G. As $f_1(x) + f_2(x) \leq 4 \leq k$ for each vertex $x \in V(G)$, $\{f_1, f_2\}$ is a weak signed Roman k-dominating family on G and thus $d^k_{wsR}(G) \geq 2$.
Assume next that $k = 3$. If H is a component of G of order two with the vertices u and v, then define $f_1(u) = 1$ and $f_1(v) = 2$ and $f_2(u) = 2$ and $f_2(v) = 1$. If x is a vertex of a component of order at least three, then define $f_1(x) = 1$ if x is a leaf and $f_1(x) = 2$ if x is not a leaf and $f_2(x) = 2$ if x is a leaf and $f_2(x) = 1$ if x is not a leaf. Then f_1, f_2 are WSR3DF on G such that $f_1(x) + f_2(x) = 3$ for each $x \in V(G)$. Therefore $\{f_1, f_2\}$ is a weak signed Roman 3-dominating family on G and so $d^3_{wsR}(G) \geq 2$. □

The next examples demonstrate that Theorem 4 is not valid for $k = 1$ or $k = 2$ in general.

Example 3. *Let $G = H \circ K_1$ be the graph constructed from a graph H, where for each vertex $v \in V(H)$, a new vertex v' and a pendant edge vv' are added. If f is WSR2DF on G, then it is easy to see that $f(x) \geq 1$ for each vertex $x \in V(G)$. Theorem 1 implies $d^2_{wsR}(G) \leq \delta(G)+1 = 2$. Suppose that $d^2_{wsR}(G) = 2$, and let $\{f_1, f_2\}$ be a weak signed Roman 2-dominating family on G. The condition $f_1(x) + f_2(x) \leq 2$ leads to $f_1(x) = f_2(x) = 1$ for each vertex $x \in V(G)$, a contradiction. Consequently, $d^2_{wsR}(G) = 1$.*

Example 4. *Let $G = K_{1,n-1}$ for an integer $n \geq 3$ with the center vertex w and the leaves $v_1, v_2, \ldots, v_{n-1}$. According to Theorem 1, we note that $d^1_{wsR}(G) \leq \delta(G) + 1 = 2$. Suppose that $d^1_{wsR}(G) = 2$, and let $\{f_1, f_2\}$ be a weak signed Roman 1-dominating family on G. The condition $f_1(x) + f_2(x) \leq 1$ leads to $f_1(x) = -1$ or $f_2(x) = -1$ for each vertex $x \in V(G)$. Assume, without loss of generality, that $f_1(w) = -1$. The condition $\sum_{x \in N[v_i]} f_1(x) \geq 1$*

implies $f_1(v_i) = 2$ *for each* $1 \leq i \leq n-1$. *Using this fact, we observe that* $f_2(v_i) = -1$ *for each* $1 \leq i \leq n-1$. *Since* $n \geq 3$, *we conclude that* $\sum_{x \in N[w]} f_2(x) \leq 2 - (n-1) \leq 0$, *a contradiction. Thus* $d^1_{wsR}(G) = 1$.

Corollary 1. Let G be a graph of order n with $\delta(G) \geq 1$. If $2n-1 \geq \gamma^3_{wsR}(G) \geq n+1$, then $d^3_{wsR}(G) = 2$.

Proof. Theorem 4 implies $d^3_{wsR}(G) \geq 2$.
Conversely, it follows from Theorem 2 that

$$d^3_{wsR}(G) \leq \frac{3n}{\gamma^3_{wsR}(G)} \leq \frac{3n}{n+1} < 3.$$

Thus $d^3_{wsR}(G) \leq 2$, and the proof is complete. □

If C_{3t} is a cycle of length $3t$ with an integer $t \geq 1$, then Volkmann [10] showed that $d^3_{sR}(C_{3t}) = 3$. We deduce that $d^3_{wsR}(C_{3t}) \geq d^3_{sR}(C_{3t}) = 3$, and therefore $d^3_{wsR}(C_{3t}) = 3$ according to Theorem 1. Since $\gamma^3_{wsR}(C_{3t}) = n = 3t$ (see [12]), we note that the condition $\gamma^3_{wsR}(G) \geq n+1$ in Corollary 1 is best possible in some sense.

Corollary 2. Let G be a graph of order n with $\delta(G) \geq 1$. If $2n-1 \geq \gamma^4_{wsR}(G) > \frac{4n}{3}$, then $d^4_{wsR}(G) = 2$.

Proof. Theorem 4 implies $d^4_{wsR}(G) \geq 2$.
Conversely, it follows from Theorem 2 that

$$d^4_{wsR}(G) \leq \frac{4n}{\gamma^4_{wsR}(G)} < \frac{4n}{\frac{4n}{3}} = 3.$$

Thus $d^4_{wsR}(G) \leq 2$, and the proof is complete. □

Example 5. *If* C_{3t} *is a cycle of length* $3t$ *with an integer* $t \geq 1$, *then* $d^4_{wsR}(C_{3t}) = 3$.

Proof. According to Theorem 1, $d^4_{wsR}(C_{3t}) \leq 3$. Let $C_{3t} = v_0 v_1 \dots v_{3t-1} v_0$. Define the functions f_1, f_2 and f_3 by

$$f_1(v_{3i+1}) = 1,\ f_1(v_{3i+2}) = 1,\ f_1(v_{3i}) = 2,$$

$$f_2(v_{3i+1}) = 2,\ f_2(v_{3i+2}) = 1,\ f_2(v_{3i}) = 1,$$

$$f_3(v_{3i+1}) = 1,\ f_3(v_{3i+2}) = 2,\ f_3(v_{3i}) = 1$$

for $0 \leq i \leq t-1$. It is easy to see that f_i is a weak signed Roman 4-dominating function on C_{3t} of weight $4t$ for $1 \leq i \leq 3$, and $\{f_1, f_2, f_3\}$ is a weak signed Roman 4-dominating family on C_{3t}. Therefore $d^4_{wsR}(C_{3t}) \geq 3$ and so $d^4_{wsR}(C_{3t}) = 3$. □

Since $\gamma^4_{wsR}(C_{3t}) = \lceil \frac{4n}{3} \rceil = \lceil \frac{12t}{3} \rceil = 4t$ (see [12]), Example 5 shows that the condition $\gamma^4_{wsR}(G) > \frac{4n}{3}$ in Corollary 2 is best possible in some sense.

For some regular graphs we will improve the upper bound given in Theorem 1.

Theorem 5. Let G be a δ-regular graph of order n with $\delta \geq \frac{k}{2} - 1$ such that $n = p(\delta+1)+r$ with integers $p \geq 1$ and $1 \leq r \leq \delta$ and $kr = t(\delta+1)+s$ with integers $t \geq 0$ and $1 \leq s \leq \delta$. Then $d^k_{wsR}(G) \leq \delta$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a WSRkD family on G such that $d = d^k_{wsR}(G)$. It follows that

$$\sum_{i=1}^{d} \omega(f_i) = \sum_{i=1}^{d} \sum_{v \in V(G)} f_i(v) = \sum_{v \in V(G)} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V(G)} k = kn.$$

Proposition C implies

$$\begin{aligned} \omega(f_i) &\geq \gamma^k_{wsR}(G) \geq \left\lceil \frac{kn}{\delta+1} \right\rceil = \left\lceil \frac{kp(\delta+1)+kr}{\delta+1} \right\rceil \\ &= kp + \left\lceil \frac{kr}{\delta+1} \right\rceil = kp + \left\lceil \frac{t(\delta+1)+s}{\delta+1} \right\rceil = kp+t+1, \end{aligned}$$

for each $i \in \{1, 2, \ldots, d\}$. If we suppose to the contrary that $d = \delta+1$, then the above inequality chains lead to the contradiction

$$\begin{aligned} kn &\geq \sum_{i=1}^{d} \omega(f_i) \geq d(kp+t+1) = (\delta+1)(kp+t+1) \\ &= kp(\delta+1) + (\delta+1)(t+1) = kp(\delta+1) + t(\delta+1) + \delta + 1 \\ &= kp(\delta+1) + kr - s + \delta + 1 > kp(\delta+1) + kr = k(p(\delta+1)+r) = kn. \end{aligned}$$

Thus $d \leq \delta$, and the proof is complete. □

Examples 1, 2 and 5 demonstrate that Theorem 5 is not valid in general.

3. Upper bounds on the sum $\gamma^k_{wsR}(G) + d^k_{wsR}(G)$

Theorem 6. If G is a graph of order $n \geq 1$ and $\delta(G) \geq k-1$, then

$$\gamma^k_{wsR}(G) + d^k_{wsR}(G) \leq n + k.$$

Proof. If $d^k_{wsR}(G) \le k$, then Proposition D implies $\gamma^k_{wsR}(G) + d^k_{wsR}(G) \le n + k$ immediately. Let now $d^k_{wsR}(G) \ge k$. It follows from Theorem 2 that

$$\gamma^k_{wsR}(G) + d^k_{wsR}(G) \le \frac{kn}{d^k_{wsR}(G)} + d^k_{wsR}(G).$$

According to Theorem 1, we have $k \le d^k_{wsR}(G) \le n$. Using these bounds, and the fact that the function $g(x) = x + (kn)/x$ is decreasing for $k \le x \le \sqrt{kn}$ and increasing for $\sqrt{kn} \le x \le n$, we obtain

$$\gamma^k_{wsR}(G) + d^k_{wsR}(G) \le \frac{kn}{d^k_{wsR}(G)} + d^k_{wsR}(G) \le \max\{n + k, k + n\} = n + k,$$

and the desired bound is proved. □

Theorem 7. Let G be a graph of order $n \ge 2$ and $\delta(G) \ge \lceil \frac{k}{2} \rceil - 1$. Then

$$\gamma^k_{wsR}(G) + d^k_{wsR}(G) \le 2n + k - 1,$$

with equality if and only if $k = 2$ and $G = \overline{K_n}$.

Proof. If $\delta = \delta(G) \ge k - 1$, then Theorem 6 implies

$$\gamma^k_{wsR}(G) + d^k_{wsR}(G) \le n + k < 2n + k - 1.$$

Assume next that $\lceil \frac{k}{2} \rceil - 1 \le \delta \le k - 2$. Then $k \ge 2$ and according to Proposition E and Theorem 1, we obtain

$$\gamma^k_{wsR}(G) + d^k_{wsR}(G) \le 2n + \delta + 1 \le 2n + k - 1. \tag{1}$$

If we have equality in (1), then $\gamma^k_{wsR}(G) = 2n$ and $d^k_{wsR}(G) = k - 1$. Therefore Theorem 2 leads to $2n(k-1) = \gamma^k_{wsR}(G) \cdot d^k_{wsR}(G) \le kn$ and so $k = 2$. Thus Proposition E yields to $\delta = 0$ and hence $G = \overline{K_n}$.
Clearly, if $G = \overline{K_n}$, then $\gamma^2_{wsR}(G) = 2n$ and $d^2_{wsR}(G) = 1$ and thus $\gamma^2_{wsR}(G) + d^2_{wsR}(G) = 2n + 1 = 2n + 2 - 1$. □

Theorem 8. Let $k \ge 3$ be an integer, and let G be a graph of order n with $\delta(G) \ge \lceil \frac{k}{2} \rceil - 1$. If $k = 2n$, then $G = K_n$ and $\gamma^k_{wsR}(G) + d^k_{wsR}(G) = 2n + 1$. If $k \le 2n - 1$, then

$$\gamma^k_{wsR}(G) + d^k_{wsR}(G) \le 2n + \left\lceil \frac{k}{2} \right\rceil - 1.$$

Proof. Since $n \geq \delta(G)+1 \geq \lceil \frac{k}{2} \rceil \geq \frac{k}{2}$, we observe that $k \leq 2n$.
If $k = 2n$, then $\delta(G)+1 = n$ and thus $G = K_n$. Proposition E implies $\gamma_{wsR}^k(G) = 2n$. Clearly, $d_{wsR}^k(G) = 1$ and therefore $\gamma_{wsR}^k(G) + d_{wsR}^k(G) = 2n+1$.
Let now $k \leq 2n-1$. In this case, it is straightforward to verify that $n+k \leq 2n+\lceil \frac{k}{2} \rceil - 1$. If $\delta = \delta(G) \geq k-1$, then the last inequality and Theorem 6 lead to the desired bound. Assume next that $\lceil \frac{k}{2} \rceil - 1 \leq \delta \leq k-1$. If $\gamma_{wsR}^k(G) = 2n$, then the definitions lead to $d_{wsR}^k(G) = 1$ and thus

$$\gamma_{wsR}^k(G) + d_{wsR}^k(G) = 2n+1 \leq 2n + \left\lceil \frac{k}{2} \right\rceil - 1.$$

Let now $\gamma_{wsR}^k(G) \leq 2n-1$. If $d_{wsR}^k(G) \leq \lceil \frac{k}{2} \rceil$, then the desired bound is immediate. Finally, let $d_{wsR}^k(G) \geq \lceil \frac{k}{2} \rceil + 1$. Using Theorem 1, we observe that

$$\left\lceil \frac{k}{2} \right\rceil + 1 \leq d_{wsR}^k(G) \leq \delta + 1 \leq k.$$

We deduce from Theorem 2 that

$$\gamma_{wsR}^k(G) + d_{wsR}^k(G) \leq \frac{kn}{d_{wsR}^k(G)} + d_{wsR}^k(G).$$

Using these bounds, we obtain analogously to the proof of Theorem 6 that

$$\gamma_{wsR}^k(G) + d_{wsR}^k(G) \leq \max\left\{ \frac{kn}{\lceil k/2 \rceil + 1} + \left\lceil \frac{k}{2} \right\rceil + 1, n+k \right\}.$$

Since $n \geq \delta + 1 \geq \lceil \frac{k}{2} \rceil + 1$, it is straightforward to verify that

$$\frac{kn}{\lceil k/2 \rceil + 1} + \left\lceil \frac{k}{2} \right\rceil + 1 \leq 2n + \left\lceil \frac{k}{2} \right\rceil - 1,$$

and this leads to the desired bound. □

Let k and n be integers such that $n \geq 3$ and $2n-2 \leq k \leq 2n-1$. Example 2 implies $d_{wsR}^k(K_n) = n$, and it follows from Proposition C that $\gamma_{wsR}^k(K_n) \geq k$. Thus

$$\gamma_{wsR}^k(K_n) + d_{wsR}^k(K_n) \geq n+k. \qquad (2)$$

If $k = 2n-1$, then we deduce from inequality (2) and Theorem 8 that

$$3n-1 = n+k \leq \gamma_{wsR}^k(K_n) + d_{wsR}^k(K_n) \leq 2n + \left\lceil \frac{k}{2} \right\rceil - 1 = 3n-1$$

and therefore $\gamma_{wsR}^k(K_n) + d_{wsR}^k(K_n) = 2n + \lceil \frac{k}{2} \rceil - 1$ and $\gamma_{wsR}^k(K_n) = k$.
If $k = 2n-2$, then we deduce from inequality (2) and Theorem 8 that

$$3n-2 = n+k \leq \gamma_{wsR}^k(K_n) + d_{wsR}^k(K_n) \leq 2n + \left\lceil \frac{k}{2} \right\rceil - 1 = 3n-2$$

and therefore $\gamma_{wsR}^k(K_n) + d_{wsR}^k(K_n) = 2n + \lceil \frac{k}{2} \rceil - 1$ and $\gamma_{wsR}^k(K_n) = k$.
These examples demonstrate that the upper bound in Theorem 8 is sharp.

4. Nordhaus-Gaddum type results

Results of Nordhaus-Gaddum type study the extreme values of the sum or the product of a parameter on a graph and its complement. In their current classical paper [6], Nordhaus and Gaddum discussed this problem for the chromatic number. We note such inequalities for the weak signed Roman k-domatic number. Using Theorems 1, 2 and 5, one can prove the nexts results analogue to the corresponding one in [10].

Theorem 9. If G is a graph of order n with $\delta(G), \delta(\overline{G}) \geq \lceil \frac{k}{2} \rceil - 1$, then $d_{wsR}^k(G) + d_{wsR}^k(\overline{G}) \leq n+1$. Furthermore, if $d_{wsR}^k(G) + d_{wsR}^k(\overline{G}) = n+1$, then G is regular.

Theorem 10. If G is a graph of order n such that $\delta(G), \delta(\overline{G}) \geq 1$, then $d_{wsR}^2(G) + d_{wsR}^2(\overline{G}) \leq n$.

If $n \geq 3$, then $d_{wsR}^2(K_n) = n$ by Example 1 and $d_{wsR}^2(\overline{K_n}) = 1$ and therefore $d_{wsR}^2(K_n) + d_{wsR}^2(\overline{K_n}) = n+1$. This example shows that the condition $\delta(G), \delta(\overline{G}) \geq 1$ in Theorem 10 is necessary.

Theorem 11. Let $k \geq 3$ be an integer. Then there is only a finite number of graphs G with $\delta(G), \delta(\overline{G}) \geq k-1$ such that $d_{wsR}^k(G) + d_{wsR}^k(\overline{G}) = n(G) + 1$.

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