Weak signed Roman $k$-domatic number of a graph

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Abstract: Let $k \geq 1$ be an integer. A weak signed Roman $k$-dominating function on a graph $G$ is a function $f : V(G) \rightarrow \{-1,1,2\}$ such that $\sum_{u \in N[v]} f(u) \geq k$ for every $v \in V(G)$, where $N[v]$ is the closed neighborhood of $v$. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct weak signed Roman $k$-dominating functions on $G$ with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called a weak signed Roman $k$-dominating family (of functions) on $G$. The maximum number of functions in a weak signed Roman $k$-dominating family on $G$ is the weak signed Roman $k$-domatic number of $G$, denoted by $d_{w,R}^{k}(G)$. In this paper we initiate the study of the weak signed Roman $k$-domatic number in graphs, and we present sharp bounds for $d_{w,R}^{k}(G)$. In addition, we determine the weak signed Roman $k$-domatic number of some graphs.

Keywords: weak signed Roman $k$-dominating function, weak signed Roman $k$-domination number, weak signed Roman $k$-domatic number.

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1. Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [4]. Specifically, let $G$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of $G$ is denoted by $n = n(G)$. For every vertex $v \in V$, the open neighborhood $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $d(v) = |N(v)|$. The minimum and maximum degree of a graph $G$ are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A graph $G$ is regular or $r$-regular if $d(v) = r$ for each vertex $v$ of $G$. The complement of a graph $G$ is denoted by $\overline{G}$. We write $K_n$ for the complete graph of order $n$, $K_{p,q}$ for the complete bipartite graph with partite sets $X$ and $Y$, where $|X| = p$ and $|Y| = q$, and $C_n$ for the cycle of length $n$. 

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In this paper we continue the study of Roman dominating functions in graphs and digraphs. If \( k \geq 1 \) is an integer, then the signed Roman \( k \)-dominating function (SRkDF) on a graph \( G \) is defined in [5] as a function \( f : V(G) \to \{-1,1,2\} \) such that \( \sum_{u \in N[v]} f(u) \geq k \) for each \( v \in V(G) \), and such that every vertex \( u \in V(G) \) for which \( f(u) = -1 \) is adjacent to at least one vertex \( w \) for which \( f(w) = 2 \). The weight of an SRkDF \( f \) is the value \( \omega(f) = \sum_{v \in V} f(v) \). The signed Roman \( k \)-domination number of a graph \( G \), denoted by \( \gamma_{sR}^k(G) \), equals the minimum weight of an SRkDF on \( G \). A \( \gamma_{sR}^k(G) \)-function is a signed Roman \( k \)-dominating function of \( G \) with weight \( \gamma_{sR}^k(G) \). If \( k = 1 \), then we write \( \gamma_{sR}^1(G) = \gamma_{sR}(G) \). This case was introduced and studied in [1]. The signed Roman domination number of digraphs was investigated in [8].

A weak signed Roman \( k \)-dominating function (WSRkDF) on a graph \( G \) is defined in [12] as a function \( f : V(G) \to \{-1,1,2\} \) such that \( \sum_{u \in N[v]} f(u) \geq k \) for each \( v \in V(G) \). The weight of a WSRkDF \( f \) is the value \( \omega(f) = \sum_{v \in V} f(v) \). The weak signed Roman \( k \)-domination number of a graph \( G \), denoted by \( \gamma_{wsR}^k(G) \), equals the minimum weight of a WSRkDF on \( G \). A \( \gamma_{wsR}^k(G) \)-function is a weak signed Roman \( k \)-dominating function of \( G \) with weight \( \gamma_{wsR}^k(G) \). The special case \( k = 1 \) was introduced and investigated by Volkmann [11].

The weak signed Roman \( k \)-domination number exists when \( \delta \geq \frac{k}{2} - 1 \). Therefore we assume in this paper that \( \delta \geq \frac{k}{2} - 1 \). The definitions lead to \( \gamma_{wsR}^k(G) \leq \gamma_{sR}^k(G) \).

A concept dual in a certain sense to the domination number is the domatic number, introduced by Cockayne and Hedetniemi [3]. They have defined the domatic number \( d(G) \) of a graph \( G \) by means of sets. A partition of \( V(G) \), all of whose classes are dominating sets in \( G \), is called a domatic partition. The maximum number of classes of a domatic partition of \( G \) is the domatic number \( d(G) \) of \( G \). But Rall has defined a variant of the domatic number of \( G \), namely the fractional domatic number of \( G \), using functions on \( V(G) \). (This was mentioned by Slater and Trees in [9].) Analogous to the fractional domatic number we may define the (weak) signed Roman \( k \)-domatic number.

A set \( \{f_1,f_2,\ldots,f_d\} \) of distinct (weak) signed Roman \( k \)-dominating functions on \( G \) with the property that \( \sum_{i=1}^d f_i(v) \leq k \) for each \( v \in V(G) \), is called a (weak) signed Roman \( k \)-dominating family (of functions) on \( G \). The maximum number of functions in a (weak) signed Roman \( k \)-dominating family ((W)SRkD family) on \( G \) is the (weak) signed Roman \( k \)-domatic number of \( G \), denoted by \( (d^k_{wsR}(G)) \) \( d^k_{sR}(G) \). The (weak) signed Roman \( k \)-domatic number is well-defined and \( d^k_{wsR}(G) \geq d^k_{sR}(G) \geq 1 \) for all graphs \( G \) with \( \delta(G) \geq \frac{k}{2} - 1 \), since the set consisting of any (W)SRkDF forms a (W)SRkD family on \( G \). For more information on the Roman domatic problem, we refer the reader to the survey article [2].

Our purpose in this paper is to initiate the study of the weak signed Roman \( k \)-domatic number in graphs. We first derive basic properties and bounds for the weak signed Roman \( k \)-domatic number of a graph. In addition, we determine the weak signed Roman \( k \)-domatic number of some classes of graphs.

We make use of the following known results in this paper.
Proposition A. ([7, 10]) If $n \geq k \geq 1$ are integers, then $d^k_G(K_n) = n$, unless $n = 3$ and $k = 1$ or $n = k = 2$, in which cases $d^1_G(K_3) = 1$ or $d^2_G(K_2) = 1$.

Proposition B. ([12]) If $n \geq k \geq 2$ are integers, then $\gamma^k_G(K_n) = k$.

Proposition C. ([12]) If $G$ is an $r$-regular graph of order $n$ with $r \geq \frac{k}{2} - 1$, then

$$\gamma^k_{wsR}(G) \geq \frac{kn}{r+1}.$$ 

Proposition D. ([5]) If $G$ is a graph of order $n$ with $\delta(G) \geq k - 1$, then

$$\gamma^k_{wsR}(G) \leq \gamma^k_{sR}(G) \leq n.$$ 

Proposition E. ([12]) Let $G$ be a graph of order $n$ with $\delta(G) \geq \lceil \frac{k}{2} \rceil$. Then $\gamma^k_{wsR}(G) \leq 2n$, with equality if and only if $k$ is even, $\delta(G) = \frac{k}{2} - 1$, and each vertex of $G$ is of minimum degree or adjacent to a vertex of minimum degree.

2. Bounds on the weak signed Roman $k$-domatic number

In this section we present basic properties of $d^k_{wsR}(G)$ and sharp bounds on the weak signed Roman $k$-domatic number of a graph.

Theorem 1. If $G$ is a graph, then $d^k_{wsR}(G) \leq \delta(G) + 1$. Moreover, if $d^k_{wsR}(G) = \delta(G) + 1$, then for each WSRkD family $\{f_1, f_2, \ldots, f_d\}$ on $G$ with $d = d^k_{wsR}(G)$ and each vertex $v$ of minimum degree, $\sum_{x \in N[v]} f_i(x) = k$ for each function $f_i$ and $\sum_{i=1}^d f_i(x) = k$ for all $x \in N[v]$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a WSRkD family on $G$ such that $d = d^k_{wsR}(G)$. If $v$ is a vertex of minimum degree $\delta(G)$, then we deduce that

$$kd \leq \sum_{i=1}^d \sum_{x \in N[v]} f_i(x) = \sum_{x \in N[v]} \sum_{i=1}^d f_i(x) \leq \sum_{x \in N[v]} k = k(\delta(G) + 1)$$

and thus $d^k_{wsR}(G) \leq \delta(G) + 1$.

If $d^k_{wsR}(G) = \delta(G) + 1$, then the two inequalities occurring in the proof become equalities. Hence for the WSRkD family $\{f_1, f_2, \ldots, f_d\}$ on $G$ and for each vertex $v$ of minimum degree, $\sum_{x \in N[v]} f_i(x) = k$ for each function $f_i$ and $\sum_{i=1}^d f_i(x) = k$ for all $x \in N[v].$ \hfill $\square$

Example 1. If $n \geq k \geq 1$ are integers, then $d^k_{wsR}(K_n) = n$, unless $n = k = 2$, in which case $d^2_{wsR}(K_2) = 1$. 

Proof. Theorem 1 implies $d_{wsR}^k(K_n) \leq n$. It is easy to see that $d_{wsR}^2(K_2) = 1$ and $d_{wsR}^1(K_3) = 3$. In all other cases, Proposition A leads to $d_{wsR}^k(K_n) \leq d_{sR}^k(K_n) = n$, and the proof is complete.

**Theorem 2.** If $G$ is a graph of order $n$, then

$$\gamma_{wsR}^k(G) \cdot d_{wsR}^k(G) \leq kn.$$  

Moreover, if $\gamma_{wsR}^k(G) \cdot d_{wsR}^k(G) = kn$, then for each WSRkD family $\{f_1, f_2, \ldots, f_d\}$ on $G$ with $d = d_{wsR}^k(G)$, each function $f_i$ is a $\gamma_{wsR}^k(G)$-function and $\sum_{i=1}^d f_i(v) = k$ for all $v \in V(G)$.

**Example 2.** If $k, n \geq 1$ are integers such that $n + 1 \leq k \leq 2n - 1$, then $d_{wsR}^k(K_n) = n$.

Proof. Theorem 1 implies $d_{wsR}^k(K_n) \leq n$. Now let $x_1, x_2, \ldots, x_n$ be the vertices of $K_n$, and let $k = n + t$ for an integer $1 \leq t \leq n - 1$. For $1 \leq i \leq n$, define the function $f_i : V(K_n) \rightarrow \{-1, 1, 2\}$ by

$$f_i(x_1) = f_i(x_{i+1}) = \ldots = f_i(x_{i+t-1}) = 2$$

and

$$f_i(x_{i+t}) = f_i(x_{i+t+1}) = \ldots = f_i(x_{i+n-1}) = 1,$$

where the indices are taken modulo $n$. It is easy to verify that $f_i$ is a weak signed Roman $k$-dominating function on $K_n$ for $1 \leq i \leq n$ and $\{f_1, f_2, \ldots, f_n\}$ is a weak signed Roman $k$-dominating family on $K_n$. Hence $d_{wsR}^k(K_n) \geq n$ and thus $d_{wsR}^k(K_n) = n$.

Examples 1, 2 and Proposition B demonstrate that Theorems 1 and 2 are both sharp.
Theorem 3. Let $G$ be a graph of order $n \geq 2$ with $\delta(G) \geq \lceil \frac{n}{2} \rceil - 1$. Then $d_{wsR}^k(G) = n$ if and only if $G = K_n$, with exception of the cases $k = 2n$ or $k = n = 2$, in which cases $d_{wsR}^{2n}(K_n) = 1$ or $d_{wsR}^n(K_2) = 1$.

Proof. Let $G = K_n$. If $k = 2n$, then the function $f$ with $f(x) = 2$ for each vertex $x \in V(G)$ is the unique weak signed Roman dominating function on $G$ and so $d_{wsR}^{2n}(K_n) = 1$. In addition, it follows from Examples 1 and 2 that $d_{wsR}^2(K_2) = 1$, $d_{wsR}^1(K_3) = 3$ and $d_{wsR}^k(K_n) = n$ in the remaining cases.

Conversely, assume that $d_{wsR}^k(G) = n$. Then we deduce from Theorem 1 that $n = d_{wsR}^k(G) \leq \delta(G) + 1$, and so $\delta(G) \geq n - 1$. Thus $G = K_n$, and the proof is complete.

Theorem 3 shows that Theorem 1 is sharp.

Theorem 4. Let $k \geq 3$ be an integer, and let $G$ be a graph of order $n$ with $\delta(G) \geq \lceil \frac{n}{2} \rceil - 1$. If $\gamma_{wsR}^k(G) \leq 2n - 1$, then $d_{wsR}^k(G) \geq 2$.

Proof. Assume first that $k \geq 4$. Since $\gamma_{wsR}^k(G) \leq 2n - 1$, there exists a WSR$k$DF $f_1$ with $f_1(v) \leq 1$ for at least one vertex $v \in V(G)$. Note that $f_2 : V(G) \rightarrow \{-1, 1, 2\}$ with $f_2(x) = 2$ for each vertex $x \in V(G)$ is another WSR$k$DF on $G$. As $f_1(x) + f_2(x) \leq 4 \leq k$ for each vertex $x \in V(G)$, the family $\{f_1, f_2\}$ is a weak signed Roman $k$-dominating family on $G$ and thus $d_{wsR}^k(G) \geq 2$.

Assume next that $k = 3$. If $H$ is a component of $G$ of order two with the vertices $u$ and $v$, then define $f_1(u) = 1$ and $f_1(v) = 2$ and $f_2(u) = 2$ and $f_2(v) = 1$. If $x$ is a vertex of a component of order at least three, then define $f_1(x) = 1$ if $x$ is a leaf and $f_1(x) = 2$ if $x$ is not a leaf and $f_2(x) = 2$ if $x$ is a leaf and $f_2(x) = 1$ if $x$ is not a leaf. Then $f_1, f_2$ are WSR$3$DF on $G$ such that $f_1(x) + f_2(x) = 3$ for each $x \in V(G)$. Therefore $\{f_1, f_2\}$ is a weak signed Roman 3-dominating family on $G$ and so $d_{wsR}^3(G) \geq 2$.

The next examples demonstrate that Theorem 4 is not valid for $k = 1$ or $k = 2$ in general.

Example 3. Let $G = H \circ K_1$ be the graph constructed from a graph $H$, where for each vertex $v \in V(H)$, a new vertex $v'$ and a pendant edge $vv'$ are added. If $f$ is WSR$2$DF on $G$, then it is easy to see that $f(x) \geq 1$ for each vertex $x \in V(G)$. Theorem 1 implies $d_{wsR}^1(G) \leq \delta(G) + 1 = 2$. Suppose that $d_{wsR}^1(G) = 2$, and let $\{f_1, f_2\}$ be a weak signed Roman 2-dominating family on $G$. The condition $f_1(x) + f_2(x) \leq 2$ leads to $f_1(x) = f_2(x) = 1$ for each vertex $x \in V(G)$, a contradiction. Consequently, $d_{wsR}^1(G) = 1$.

Example 4. Let $G = K_{1,n-1}$ for an integer $n \geq 3$ with the center vertex $w$ and the leaves $v_1, v_2, \ldots, v_{n-1}$. According to Theorem 1, we note that $d_{wsR}^1(G) \leq \delta(G) + 1 = 2$. Suppose that $d_{wsR}^1(G) = 2$, and let $\{f_1, f_2\}$ be a weak signed Roman 1-dominating family on $G$. The condition $f_1(x) + f_2(x) \leq 1$ leads to $f_1(x) = -1$ or $f_2(x) = -1$ for each vertex $x \in V(G)$. Assume, without loss of generality, that $f_1(w) = -1$. The condition $\sum_{x \in N[v_1]} f_1(x) \geq 1$
implies \( f_1(v_i) = 2 \) for each \( 1 \leq i \leq n - 1 \). Using this fact, we observe that \( f_2(v_i) = -1 \) for each \( 1 \leq i \leq n - 1 \). Since \( n \geq 3 \), we conclude that \( \sum_{x \in N[w]} f_2(x) \leq 2 - (n - 1) \leq 0 \), a contradiction. Thus \( d_{wsR}^1(G) = 1 \).

**Corollary 1.** Let \( G \) be a graph of order \( n \) with \( \delta(G) \geq 1 \). If \( 2n - 1 \geq \gamma_{wsR}^3(G) \geq n + 1 \), then \( d_{wsR}^3(G) = 2 \).

**Proof.** Theorem 4 implies \( d_{wsR}^3(G) \geq 2 \). Conversely, it follows from Theorem 2 that

\[
d_{wsR}^3(G) \leq \frac{3n}{\gamma_{wsR}^3(G)} \leq \frac{3n}{n + 1} < 3.
\]

Thus \( d_{wsR}^3(G) \leq 2 \), and the proof is complete.

If \( C_{3t} \) is a cycle of length \( 3t \) with an integer \( t \geq 1 \), then Volkmann [10] showed that \( d_{wsR}^3(C_{3t}) = 3 \). We deduce that \( d_{wsR}^3(C_{3t}) \geq d_{sR}^3(C_{3t}) = 3 \), and therefore \( d_{wsR}^3(C_{3t}) = 3 \) according to Theorem 1. Since \( \gamma_{wsR}^3(C_{3t}) = n = 3t \) (see [12]), we note that the condition \( \gamma_{wsR}^3(G) \geq n + 1 \) in Corollary 1 is best possible in some sense.

**Corollary 2.** Let \( G \) be a graph of order \( n \) with \( \delta(G) \geq 1 \). If \( 2n - 1 \geq \gamma_{wsR}^4(G) > \frac{4n}{3} \), then \( d_{wsR}^4(G) = 2 \).

**Proof.** Theorem 4 implies \( d_{wsR}^4(G) \geq 2 \). Conversely, it follows from Theorem 2 that

\[
d_{wsR}^4(G) \leq \frac{4n}{\gamma_{wsR}^4(G)} < \frac{4n}{\frac{4n}{3}} = 3.
\]

Thus \( d_{wsR}^4(G) \leq 2 \), and the proof is complete.

**Example 5.** If \( C_{3t} \) is a cycle of length \( 3t \) with an integer \( t \geq 1 \), then \( d_{wsR}^4(C_{3t}) = 3 \).

**Proof.** According to Theorem 1, \( d_{wsR}^4(C_{3t}) \leq 3 \). Let \( C_{3t} = v_0v_1 \ldots v_{3t-1}v_0 \). Define the functions \( f_1, f_2 \) and \( f_3 \) by

\[
\begin{align*}
f_1(v_{3i+1}) &= 1, 
&f_1(v_{3i+2}) &= 1, 
&f_1(v_{3i}) &= 2, 
\end{align*}
\[
\begin{align*}
f_2(v_{3i+1}) &= 2, 
&f_2(v_{3i+2}) &= 1, 
&f_2(v_{3i}) &= 1, 
\end{align*}
\[
\begin{align*}
f_3(v_{3i+1}) &= 1, 
&f_3(v_{3i+2}) &= 2, 
&f_3(v_{3i}) &= 1
\end{align*}
\]

for \( 0 \leq i \leq t - 1 \). It is easy to see that \( f_i \) is a weak signed Roman 4-dominating function on \( C_{3t} \) of weight \( 4t \) for \( 1 \leq i \leq 3 \), and \( \{f_1, f_2, f_3\} \) is a weak signed Roman 4-dominating family on \( C_{3t} \). Therefore \( d_{wsR}^4(C_{3t}) \geq 3 \) and so \( d_{wsR}^4(C_{3t}) = 3 \).
Since $\gamma_{wsR}(C_{3t}) = \lceil \frac{4n}{3} \rceil = \lceil \frac{12t}{3} \rceil = 4t$ (see [12]), Example 5 shows that the condition $\gamma_{wsR}(G) > \frac{4n}{3}$ in Corollary 2 is best possible in some sense.

For some regular graphs we will improve the upper bound given in Theorem 1.

**Theorem 5.** Let $G$ be a $\delta$-regular graph of order $n$ with $\delta \geq \frac{k}{2} - 1$ such that $n = p(\delta + 1) + r$ with integers $p \geq 1$ and $1 \leq r \leq \delta$ and $kr = t(\delta + 1) + s$ with integers $t \geq 0$ and $1 \leq s \leq \delta$. Then $d_{wsR}^k(G) \leq \delta$.

**Proof.** Let $\{f_1, f_2, \ldots, f_d\}$ be a WSR$k\text{D}$ family on $G$ such that $d = d_{wsR}^k(G)$. It follows that

$$\sum_{i=1}^{d} \omega(f_i) = \sum_{i=1}^{d} \sum_{v \in V(G)} f_i(v) = \sum_{v \in V(G)} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V(G)} k = kn.$$ 

Proposition C implies

$$\omega(f_i) \geq \gamma_{wsR}^k(G) \geq \lceil \frac{kn}{\delta + 1} \rceil = \lceil \frac{kp(\delta + 1) + kr}{\delta + 1} \rceil = kp + \left\lfloor \frac{kr}{\delta + 1} \right\rfloor = kp + \left\lfloor \frac{t(\delta + 1) + s}{\delta + 1} \right\rfloor = kp + t + 1,$$

for each $i \in \{1, 2, \ldots, d\}$. If we suppose to the contrary that $d = \delta + 1$, then the above inequality chains lead to the contradiction

$$kn \geq \sum_{i=1}^{d} \omega(f_i) \geq d(kp + t + 1) = (\delta + 1)(kp + t + 1) = kp(\delta + 1) + (\delta + 1)(t + 1) = kp(\delta + 1) + t(\delta + 1) + \delta + 1 = kp(\delta + 1) + kr - s + \delta + 1 > kp(\delta + 1) + kr = k(p(\delta + 1) + r) = kn.$$ 

Thus $d \leq \delta$, and the proof is complete.

Examples 1, 2 and 5 demonstrate that Theorem 5 is not valid in general.

### 3. Upper bounds on the sum $\gamma_{wsR}^k(G) + d_{wsR}^k(G)$

**Theorem 6.** If $G$ is a graph of order $n \geq 1$ and $\delta(G) \geq k - 1$, then

$$\gamma_{wsR}^k(G) + d_{wsR}^k(G) \leq n + k.$$
Proof. If \( d_{w_{\mathcal{R}}}^k(G) \leq k \), then Proposition D implies \( \gamma_{w_{\mathcal{R}}}^k(G) + d_{w_{\mathcal{R}}}^k(G) \leq n + k \) immediately. Let now \( d_{w_{\mathcal{R}}}^k(G) \geq k \). It follows from Theorem 2 that

\[
\gamma_{w_{\mathcal{R}}}^k(G) + d_{w_{\mathcal{R}}}^k(G) \leq \frac{kn}{d_{w_{\mathcal{R}}}^k(G)} + d_{w_{\mathcal{R}}}^k(G).
\]

According to Theorem 1, we have \( k \leq d_{w_{\mathcal{R}}}^k(G) \leq n \). Using these bounds, and the fact that the function \( g(x) = x + (kn)/x \) is decreasing for \( k \leq x \leq \sqrt{kn} \) and increasing for \( \sqrt{kn} \leq x \leq n \), we obtain

\[
\gamma_{w_{\mathcal{R}}}^k(G) + d_{w_{\mathcal{R}}}^k(G) \leq \frac{kn}{d_{w_{\mathcal{R}}}^k(G)} + d_{w_{\mathcal{R}}}^k(G) \leq \max\{n + k, k + n\} = n + k,
\]

and the desired bound is proved. □

**Theorem 7.** Let \( G \) be a graph of order \( n \geq 2 \) and \( \delta(G) \geq \lceil \frac{k}{2} \rceil - 1 \). Then

\[
\gamma_{w_{\mathcal{R}}}^k(G) + d_{w_{\mathcal{R}}}^k(G) \leq 2n + k - 1,
\]

with equality if and only if \( k = 2 \) and \( G = K_n \).

Proof. If \( \delta = \delta(G) \geq k - 1 \), then Theorem 6 implies

\[
\gamma_{w_{\mathcal{R}}}^k(G) + d_{w_{\mathcal{R}}}^k(G) \leq n + k < 2n + k - 1.
\]

Assume next that \( \lceil \frac{k}{2} \rceil - 1 \leq \delta \leq k - 2 \). Then \( k \geq 2 \) and according to Proposition E and Theorem 1, we obtain

\[
\gamma_{w_{\mathcal{R}}}^k(G) + d_{w_{\mathcal{R}}}^k(G) \leq 2n + \delta + 1 \leq 2n + k - 1.
\]

(1)

If we have equality in (1), then \( \gamma_{w_{\mathcal{R}}}^k(G) = 2n \) and \( d_{w_{\mathcal{R}}}^k(G) = k - 1 \). Therefore Theorem 2 leads to \( 2n(k - 1) = \gamma_{w_{\mathcal{R}}}^k(G) \cdot d_{w_{\mathcal{R}}}^k(G) \leq kn \) and so \( k = 2 \). Thus Proposition E yields to \( \delta = 0 \) and hence \( G = K_n \). Clearly, if \( G = K_n \), then \( \gamma_{w_{\mathcal{R}}}^2(G) = 2n \) and \( d_{w_{\mathcal{R}}}^2(G) = 1 \) and thus \( \gamma_{w_{\mathcal{R}}}^2(G) + d_{w_{\mathcal{R}}}^2(G) = 2n + 1 = 2n + 2 - 1 \). □

**Theorem 8.** Let \( k \geq 3 \) be an integer, and let \( G \) be a graph of order \( n \) with \( \delta(G) \geq \lceil \frac{k}{2} \rceil - 1 \). If \( k = 2n \), then \( G = K_n \) and \( \gamma_{w_{\mathcal{R}}}^k(G) + d_{w_{\mathcal{R}}}^k(G) = 2n + 1 \). If \( k \leq 2n - 1 \), then

\[
\gamma_{w_{\mathcal{R}}}^k(G) + d_{w_{\mathcal{R}}}^k(G) \leq 2n + \left\lceil \frac{k}{2} \right\rceil - 1.
\]
Proof. Since $n \geq \delta(G) + 1 \geq \lceil \frac{k}{2} \rceil \geq \frac{k}{2}$, we observe that $k \leq 2n$.

If $k = 2n$, then $\delta(G) + 1 = n$ and thus $G = K_n$. Proposition E implies $\gamma_{wsR}^k(G) = 2n$. Clearly, $d_{wsR}^k(G) = 1$ and therefore $\gamma_{wsR}^k(G) + d_{wsR}^k(G) = 2n + 1$.

Let now $k \leq 2n - 1$. In this case, it is straightforward to verify that $n + k \leq 2n + \lceil \frac{k}{2} \rceil - 1$. If $\delta = \delta(G) \geq k - 1$, then the last inequality and Theorem 6 lead to the desired bound. Assume next that $\lceil \frac{k}{2} \rceil - 1 \leq \delta \leq k - 1$. If $\gamma_{wsR}^k(G) = 2n$, then the definitions lead to $d_{wsR}^k(G) = 1$ and thus

$$\gamma_{wsR}^k(G) + d_{wsR}^k(G) = 2n + 1 \leq 2n + \lceil \frac{k}{2} \rceil - 1.$$  

Let now $\gamma_{wsR}^k(G) \leq 2n - 1$. If $d_{wsR}^k(G) \leq \lceil \frac{k}{2} \rceil$, then the desired bound is immediate. Finally, let $d_{wsR}^k(G) \geq \lceil \frac{k}{2} \rceil + 1$. Using Theorem 1, we observe that

$$\lceil \frac{k}{2} \rceil + 1 \leq d_{wsR}^k(G) \leq \delta + 1 \leq k.$$  

We deduce from Theorem 2 that

$$\gamma_{wsR}^k(G) + d_{wsR}^k(G) \leq \frac{kn}{d_{wsR}^k(G)} + d_{wsR}^k(G).$$  

Using these bounds, we obtain analogously to the proof of Theorem 6 that

$$\gamma_{wsR}^k(G) + d_{wsR}^k(G) \leq \max \left\{ \frac{kn}{\lceil k/2 \rceil + 1} + \left\lceil \frac{k}{2} \rceil + 1, n + k \right\}.$$  

Since $n \geq \delta + 1 \geq \lceil \frac{k}{2} \rceil + 1$, it is straightforward to verify that

$$\frac{kn}{\lceil k/2 \rceil + 1} + \left\lceil \frac{k}{2} \rceil + 1 \leq 2n + \left\lceil \frac{k}{2} \rceil - 1,$$

and this leads to the desired bound. $\square$

Let $k$ and $n$ be integers such that $n \geq 3$ and $2n - 2 \leq k \leq 2n - 1$. Example 2 implies $d_{wsR}^k(K_n) = n$, and it follows from Proposition C that $\gamma_{wsR}^k(K_n) \geq k$. Thus

$$\gamma_{wsR}^k(K_n) + d_{wsR}^k(K_n) \geq n + k.$$  

(2)

If $k = 2n - 1$, then we deduce from inequality (2) and Theorem 8 that

$$3n - 1 = n + k \leq \gamma_{wsR}^k(K_n) + d_{wsR}^k(K_n) \leq 2n + \left\lceil \frac{k}{2} \rceil - 1 = 3n - 1$$

and therefore $\gamma_{wsR}^k(K_n) + d_{wsR}^k(K_n) = 2n + \lceil \frac{k}{2} \rceil - 1$ and $\gamma_{wsR}^k(K_n) = k$.

If $k = 2n - 2$, then we deduce from inequality (2) and Theorem 8 that

$$3n - 2 = n + k \leq \gamma_{wsR}^k(K_n) + d_{wsR}^k(K_n) \leq 2n + \left\lceil \frac{k}{2} \rceil - 1 = 3n - 2$$

and therefore $\gamma_{wsR}^k(K_n) + d_{wsR}^k(K_n) = 2n + \lceil \frac{k}{2} \rceil - 1$ and $\gamma_{wsR}^k(K_n) = k$.

These examples demonstrate that the upper bound in Theorem 8 is sharp.
4. Nordhaus-Gaddum type results

Results of Nordhaus-Gaddum type study the extreme values of the sum or the product of a parameter on a graph and its complement. In their current classical paper [6], Nordhaus and Gaddum discussed this problem for the chromatic number. We note such inequalities for the weak signed Roman $k$-domatic number. Using Theorems 1, 2 and 5, one can prove the nexts results analogue to the corresponding one in [10].

**Theorem 9.** If $G$ is a graph of order $n$ with $\delta(G), \delta(\overline{G}) \geq \lceil \frac{k}{2} \rceil - 1$, then $d^k_{wsR}(G) + d^k_{wsR}(\overline{G}) \leq n + 1$. Furthermore, if $d^k_{wsR}(G) + d^k_{wsR}(\overline{G}) = n + 1$, then $G$ is regular.

**Theorem 10.** If $G$ is a graph of order $n$ such that $\delta(G), \delta(\overline{G}) \geq 1$, then $d^2_{wsR}(G) + d^2_{wsR}(\overline{G}) \leq n$.

If $n \geq 3$, then $d^2_{wsR}(K_n) = n$ by Example 1 and $d^2_{wsR}(\overline{K_n}) = 1$ and therefore $d^2_{wsR}(K_n) + d^2_{wsR}(\overline{K_n}) = n + 1$. This example shows that the condition $\delta(G), \delta(\overline{G}) \geq 1$ in Theorem 10 is necessary.

**Theorem 11.** Let $k \geq 3$ be an integer. Then there is only a finite number of graphs $G$ with $\delta(G), \delta(\overline{G}) \geq k - 1$ such that $d^k_{wsR}(G) + d^k_{wsR}(\overline{G}) = n(G) + 1$.

References


