On the powers of signed graphs

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Abstract: A signed graph is an ordered pair \( \Sigma = (G, \sigma) \), where \( G = (V,E) \) is the underlying graph of \( \Sigma \) with a signature function \( \sigma : E \to \{1, -1\} \). In this article, we define the \( n^{th} \) power of a signed graph and discuss some properties of these powers of signed graphs. As we can define two types of signed graphs as the power of a signed graph, necessary and sufficient conditions are given for an \( n^{th} \) power of a signed graph to be unique. Also, we characterize balanced power signed graphs.

Keywords: Signed graph, signed distance, distance compatibility, power of signed graphs

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1. Introduction

In this paper, we will treat only simple, finite and connected signed graphs. A signature on a graph \( G = (V,E) \) is a function \( \sigma : E \to \{1, -1\} \). A signed graph is a graph \( G = (V,E) \) with a signature \( \sigma \) and is denoted as \( \Sigma = (G, \sigma) \), where \( G \) is called the underlying graph of \( \Sigma \). The sign of a cycle in a signed graph is the product of the signs of its edges. A signed graph is said to be balanced if no negative cycle exists and \( \Sigma \) is unbalanced, otherwise [2].

To begin with, we recall and adopt some definitions and notations from [3], in which the concept of signed distance in signed graphs and distance compatible signed graphs are introduced.

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Let \( u \) and \( v \) be any two vertices in a connected graph \( G \). As usual, \( d(u, v) \) denotes the distance (the length of the shortest path) between \( u \) and \( v \). If \( u \) and \( v \) are adjacent vertices, then \( d(u, v) = 1 \). Let \( P_{(u,v)} \) denote the collection of all shortest paths \( P_{(u,v)} \) between them. Then, the distance between \( u \) and \( v \) in a signed graph is defined as:

\[
d_{\text{max}}(u,v) = \sigma_{\text{max}}(u,v) \cdot d(u,v) = \max\{\sigma(P_{(u,v)}) : P_{(u,v)} \in P_{(u,v)}\} \cdot d(u,v)
\]

and

\[
d_{\text{min}}(u,v) = \sigma_{\text{min}}(u,v) \cdot d(u,v) = \min\{\sigma(P_{(u,v)}) : P_{(u,v)} \in P_{(u,v)}\} \cdot d(u,v),
\]

where the sign of a path \( P \) in \( \Sigma \) is defined as

\[
\sigma(P) = \prod_{e \in E(P)} \sigma(e).
\]

Two vertices \( u \) and \( v \) in \( \Sigma \) are said to be distance-compatible (briefly, compatible) if

\[
d_{\text{min}}(u,v) = d_{\text{max}}(u,v).
\]

A signed graph \( \Sigma \) is said to be (distance)-compatible or simply compatible, if every pair of vertices is compatible and \( \Sigma \) is incompatible, otherwise. Corresponding to the functions \( d_{\text{max}} \) and \( d_{\text{min}} \), there are two types of distance matrices in a signed graph called signed distance matrices [3] as given below.

(D1) \( D_{\text{max}}(\Sigma) = (d_{\text{max}}(u,v))_{n \times n} \).

(D2) \( D_{\text{min}}(\Sigma) = (d_{\text{min}}(u,v))_{n \times n} \).

Also, the concept of associated signed complete graphs associated with \( D_{\text{max}}(\Sigma) \) and \( D_{\text{min}}(\Sigma) \) are introduced in [3], as follows.

**Definition 1.** ([3]) The associated signed complete graph \( K^{D_{\text{max}}}(\Sigma) \) with respect to \( D_{\text{max}}(\Sigma) \) is obtained by joining the non-adjacent vertices of \( \Sigma \) with edges having signs

\[
\sigma(uv) = \sigma_{\text{max}}(uv)
\]

The associated signed complete graph \( K^{D_{\text{min}}}(\Sigma) \) with respect to \( D_{\text{min}}(\Sigma) \) is obtained by joining the non-adjacent vertices of \( \Sigma \) with edges having signs

\[
\sigma(uv) = \sigma_{\text{min}}(uv)
\]

whenever \( D_{\text{max}} = D_{\text{min}} = D^{\pm} \), say, the associated signed complete graph of \( \Sigma \) is denoted by \( K^{D^{\pm}}(\Sigma) \).

The concept of \( n^{th} \) power of graph is discussed in [1]. The \( n^{th} \) power of a graph \( G = (V,E) \) is denoted as \( G^n \) and is defined as the graph having the same vertex set as that of \( G \) and any two vertices \( u \) and \( v \) are adjacent in \( G^n \) if their distance \( d(u,v) \) is less than or equal to \( n \).

In this paper, we define the \( n^{th} \) power of a signed graph by using the concept of signed distance in signed graphs and discuss some properties of \( n^{th} \) power of signed graphs. Also, we characterize the balanced power signed graphs.

**2. Main Results**

Corresponding to \( \sigma_{\text{max}} \) and \( \sigma_{\text{min}} \), two types of \( n^{th} \) powers of signed graph for a given signed graph \( \Sigma \), can be defined as follows.
The following result immediately comes from the definitions.

**Proposition 1.** Let $\Sigma = (G, \sigma)$ be a signed graph. Then, the $n^{th}$ power of $\Sigma$ is unique if and only if there exists no incompatible pair of vertices at a distance less than or equal to $n$.

**Proposition 2.** Let $\Sigma = (G, \sigma)$ be a signed graph with diameter less than or equal to $n$. Then, the $n^{th}$ power signed graph $\Sigma_{\text{max}}^{n} = (G^{n}, \sigma')$ (or $\Sigma_{\text{min}}^{n} = (G^{n}, \sigma'')$) is the associated signed complete graph $K^{D_{\text{max}}^{n}}(\Sigma)$ (or $K^{D_{\text{min}}^{n}}(\Sigma)$). Moreover, if $\Sigma$ is compatible then, $\Sigma^{n} = (G^{n}, \sigma')$ is the associated signed complete graph $K^{D^{n}}(\Sigma)$.

**Proof.** Let $\Sigma = (G, \sigma)$ be a signed graph with diameter less than or equal to $n$. That is, the maximum distance between any two vertices in $\Sigma$ is $n$. Therefore, while taking $\Sigma_{\text{max}}^{n}$ all the non-adjacent vertices $u$ and $v$ will form an edge $uv$ with sign $\sigma'(uv) = \sigma_{\text{max}}(u, v)$. Hence, $\Sigma_{\text{max}}^{n}$ is the associated signed complete graph $K^{D_{\text{max}}^{n}}(\Sigma)$. Similarly, if we consider $\Sigma_{\text{min}}^{n}$ all the non-adjacent vertices $u$ and $v$ will form an edge $uv$ with sign $\sigma''(uv) = \sigma_{\text{min}}(u, v)$. Hence, $\Sigma_{\text{min}}^{n}$ is the associated signed complete graph $K^{D_{\text{min}}^{n}}(\Sigma)$.

If $\Sigma$ is compatible, then $\Sigma_{\text{max}}^{n} = \Sigma_{\text{min}}^{n} = \Sigma^{n}$. Since, the diameter is less than or equal to $n$, all the non-adjacent vertices in $\Sigma$ will form an edge in $\Sigma^{n}$ and sign of these edges are $\sigma'(uv) = \sigma_{\text{max}}(u, v) = \sigma_{\text{min}}(u, v)$. Hence, $\Sigma^{n}$ is the associated signed complete graph $K^{D^{n}}(\Sigma)$.

**Lemma 1.** Let $\Sigma = (G, \sigma)$ be a signed graph and $u, v$ be two vertices in $\Sigma$. If $\Sigma^{n} = (G^{n}, \sigma')$ exists, then for any $u-v$ path $P$ of length $k$ in $\Sigma$, there exists a $u-v$ path $P'$ of length $\lceil \frac{k}{n} \rceil$ in $\Sigma^{n}$ such that $\sigma(P) = \sigma'(P')$.

**Proof.** Let $\Sigma = (G, \sigma)$ be a signed graph and let $P$ be a $u-v$ path of length $k$ in $\Sigma$. In $\Sigma^{n}$ all the vertices with distance at most $n$ in $\Sigma$ will form an edge. By considering division algorithm on $k$ and $n$, we get $k = nq + r$, where $0 \leq r < n$.

**Case 1:** $r = 0$.

Then, $k = nq$. Hence, there will be $q$ edges between $u$ and $v$ in $\Sigma^{n}$.

**Case 2:** $r \neq 0$.

Then, $k = nq + r$. Since, $r < n$ the path of length $r$ will form an edge in $\Sigma^{n}$.
there will be \( q + 1 \) edges between \( u \) and \( v \) in \( \Sigma^n \).

From the above cases, it can be concluded that, corresponding to the path \( P \) of length \( k \) in \( \Sigma \) there is a path \( P' \) from \( u \) to \( v \) in \( \Sigma^n \) of length \( \lceil \frac{k}{n} \rceil \).

To prove \( \sigma(P) = \sigma'(P') \). By the definition of \( \Sigma^n \) any path \( P \) of length \( k \leq n \) will form an edge \( e \in \Sigma^n \) and here \( \sigma(P) = \sigma'(e) \).

Let \( k > n \) and \( k = nq + r \) and let \( \{e_1, e_2, \ldots, e_{nq+r}\} \) be the edge set of \( P \). Then, \( \sigma(P) = \prod_{i=1}^{nq+r} \sigma(e_i) \). Suppose that \( r \neq 0 \). Then, \( P \) can be written as the union of edge disjoint paths, \( P'_i = e_{(i-1)n+1}, e_{(i-1)n+2}, \ldots, e_{in}, 1 \leq i \leq q \) and \( P'_{q+1} = e_{nq+1}, e_{nq+2}, \ldots, e_{nq+r} \) whose length is at most \( n \). Since, every path of length at most \( n \) will form an edge in \( \Sigma^n \), \( \{e'_1, e'_2, \ldots, e'_{q+1}\} \) be the edges in \( \Sigma^n \) corresponding to paths \( P'_1, \ldots, P'_{q+1} \).

Let \( P' \) be the path from \( u \) to \( v \) with edges \( e'_1, e'_2, \ldots, e'_{q+1} \) in \( \Sigma^n \). Then, \( \sigma(P') = \prod_{i=1}^{q+1} \sigma'(e'_i) = \prod_{i=1}^{q+1} \sigma(P'_i) = \sigma(P) \). When \( r = 0 \), in a similar way we can see that \( \sigma(P) = \sigma'(P') \).

\[ \text{Lemma 2.} \quad \text{Let } \Sigma = (G, \sigma) \text{ be a signed graph and } u, v \text{ be two vertices in } \Sigma. \text{ If } \Sigma^n = (G^n, \sigma') \text{ exists, then for any } u-v \text{ path } P \text{ of length } k \text{ in } \Sigma^n, \text{ there exists a } u-v \text{ path } P' \text{ of length } k', \text{ where } (k-1)n+1 \leq k' \leq kn \text{ in } \Sigma \text{ such that } \sigma(P) = \sigma(P'). \]

\[ \text{Proof.} \quad \text{Let } \Sigma^n = (G^n, \sigma') \text{ be the } n^{th} \text{ power of } \Sigma = (G, \sigma) \text{ and } P \text{ be a } uv \text{ path of length } k \text{ in } \Sigma^n. \]

\[ \text{Case 1: } k = 1. \]

Then, \( uv \) is an edge in \( \Sigma^n \). By the definition, each edge in \( \Sigma^n \) corresponds to a path \( P' \) of length \( k' \leq n \) in \( \Sigma \), where \( \sigma'(uv) = \sigma(P') \).

\[ \text{Case 2: } k > 1. \]

Let \( \{e_1, e_2, \ldots, e_k\} \) be the edge set of \( P \) in \( \Sigma^n \), where each edge \( e_i \), \( 1 \leq i \leq k \) corresponds to a path \( P_i \) of length \( l_i \leq n \) in \( \Sigma \). Then, the concatenation \( P' = \cup_i P_i \) will form a \( uv \) path of length \( k' = \sum_{i=1}^{k} l_i \leq kn \) in \( \Sigma \) and the sign of \( P' \) is given by \( \sigma(P') = \prod_{i=1}^{k} \sigma(P_i) = \prod_{i=1}^{k} \sigma'(e_i) = \sigma'(P) \).

Since, \( P \) is of length \( k \) by Lemma 1 \( k' \) should be greater than \( (k-1)n \).

\[ \text{Theorem 1.} \quad \text{Let } \Sigma = (G, \sigma) \text{ be a signed graph with diameter greater than } n \text{ and the } n^{th} \text{ power of } \Sigma \text{ exists and unique. Then, } \Sigma^n = (G^n, \sigma') \text{ is compatible implies } \Sigma \text{ is compatible.} \]

\[ \text{Proof.} \quad \text{Suppose that } \Sigma \text{ is incompatible. Since, the } n^{th} \text{ power of } \Sigma \text{ is unique, by Proposition 1, there exist no incompatible pair of vertices at a distance less than or equal to } n. \text{ Let } u \text{ and } v \text{ be an incompatible pair of vertices at a distance } k > n. \text{ Then, there exist two shortest path } P \text{ and } Q \text{ from } u \text{ to } v \text{ of length } k \text{ with } \sigma(P) \text{ is positive and } \sigma(Q) \text{ is negative. Then, by Lemma 1, there exists two paths } P' \text{ and } Q' \text{ from } u \text{ to } v \text{ in } \Sigma^n \text{ of length } \lceil \frac{k}{n} \rceil \text{ with } \sigma(P) = \sigma'(P') \text{ and } \sigma(Q) = \sigma'(Q'). \text{ Thus, } u \text{ and } v \text{ will form an incompatible pair of vertices in } \Sigma^n, \text{ a contradiction. Hence, the signed graph } \Sigma \text{ is compatible.} \]

\[ \text{Remark 2.} \quad \text{The converse of the above theorem is not generally true. For example,} \]
consider the cycle $C_7^-$ and its square signed graph given in Figure 1. The cycle $C_7^-$ is compatible, where its square signed graph contains incompatible vertices $u_1$ and $u_4$.

![Figure 1. The signed graph $C_7^-$ and its square signed graph.](image)

2.1. Balance criterion in the power of a signed graph

In this section we give a characterization for balance in the $n^{th}$ power of a signed graph. First we recall some results from [3, 4].

**Theorem 2** ([3]). For a signed graph $\Sigma$ the following statements are equivalent:

1. $\Sigma$ is balanced
2. The associated signed complete graph $K^{D^\max}(\Sigma)$ is balanced.
3. The associated signed complete graph $K^{D^\min}(\Sigma)$ is balanced.
4. $D^\max(\Sigma) = D^\min(\Sigma)$ and the associated signed complete graph $K^{D^\pm}(\Sigma)$ is balanced.

**Theorem 3** ([3]). A signed graph $\Sigma$ is balanced if and only if the associated signed complete graph $K^{D^\pm}(\Sigma)$ has the spectrum $\left(\begin{array}{cc} n-1 & -1 \\ 1 & n-1 \end{array}\right)$.

**Lemma 3** ([4]). Let $u$ and $v$ be incompatible pair of vertices with least distance in a 2-connected non-geodetic signed graph. Then, there will be two internally disjoint shortest paths from $u$ to $v$ of opposite signs.

A connected graph $G$ is called 2-connected, if for every vertex $x \in V(G)$, $G - x$ is connected. The following lemma discuss the compatibility of the $n^{th}$ power of a 2-connected signed graph.
**Lemma 4.** Let $\Sigma = (G, \sigma)$ be a 2-connected signed graph. If $\Sigma^n = (G^n, \sigma')$ exists, then $\Sigma$ is balanced implies $\Sigma^n$ is compatible.

**Proof.** Let $\Sigma = (G, \sigma)$ be a 2-connected balanced signed graph. If possible, let $u$ and $v$ be an incompatible pair of vertices in $\Sigma^n$. Since, $\Sigma^n$ is 2-connected by using Lemma 3, we get two internally disjoint shortest $uv$ paths $P$ and $Q$ of opposite signs. Then, by Lemma 2, we can find two $uv$ paths $P'$ and $Q'$ in $\Sigma$ with $\sigma(P') = \sigma'(P)$ and $\sigma(Q') = \sigma'(Q)$. Therefore, $P'$ and $Q'$ can not be the same. If $P'$ and $Q'$ are internally disjoint, then the concatenation $(P' \cup (Q')^{-1}$ will be a negative cycle in $\Sigma$, a contradiction. Suppose that $P'$ and $Q'$ are not internally disjoint. Consider the cycles $C_1, C_2, \ldots, C_m$ formed by the common points of $P'$ and $Q'$. Since, $P'$ and $Q'$ are not the same, there exists at least one such cycle. Also, since $\sigma(P') \neq \sigma(Q')$, we can find at least one cycle among $C_1, C_2, \ldots, C_m$, say $C_i$ with common points $u_i$ and $v_i$ of $P'$ and $Q'$, where the sign of $u_iv_i$ path along $P'$ and along $Q'$ are distinct. Then, the cycle $C_i$ will be a negative cycle in $\Sigma$, a contradiction. Hence, $\Sigma^n$ should be compatible. \qed

**Lemma 5.** Let $\Sigma = (G, \sigma)$ be a signed graph. Then, the associated signed complete graphs $K^{D^\max}(\Sigma) = K^{D^\max}(\Sigma^n_{\max})$ and $K^{D^\min}(\Sigma) = K^{D^\min}(\Sigma^n_{\min})$. Moreover, if $\Sigma$ is balanced, then $K^{D^\pm}(\Sigma) = K^{D^\pm}(\Sigma^n)$.

**Proof.** Let $\Sigma = (G, \sigma)$ be a signed graph and $\Sigma^n_{\max} = (G^n, \sigma')$ be the $n^{th}$ power of $\Sigma$. Then, by Lemma 1, corresponding to the shortest $uv$ path $P$ in $\Sigma$, there is a shortest $uv$ path $P'$ in $\Sigma^n_{\max}$ where, $\sigma_{\max}(P'_{u,v}) = \sigma'_{\max}(P'_{u,v})$. The associated signed complete graph $K^{D^\max}(\Sigma)$ is obtained by joining all the non-adjacent vertices in $\Sigma$, with edges having signs $\sigma(u,v) = \sigma_{\max}(u,v)$. Similarly, the associated signed complete graph $K^{D^\max}(\Sigma^n_{\max})$ is obtained by joining all the non-adjacent vertices in $\Sigma^n_{\max}$, with edges having signs $\sigma'(u,v) = \sigma'_{\max}(u,v)$. Also, if $u$ and $v$ are adjacent in $\Sigma$, we have $\sigma'(u,v) = \sigma_{\max}(u,v)$. Hence, $K^{D^\max}(\Sigma) = K^{D^\max}(\Sigma^n_{\max})$. Similarly, we get $K^{D^\min}(\Sigma) = K^{D^\min}(\Sigma^n_{\min})$.

If $\Sigma$ is balanced, then $\Sigma^n_{\max} = \Sigma^n_{\min} = \Sigma^n$ and by Lemma 4 $\Sigma^n$ is compatible. Therefore, $K^{D^\max}(\Sigma^n_{\max}) = K^{D^\min}(\Sigma^n_{\min}) = K^{D^\pm}(\Sigma^n)$. Also, $\Sigma$ is balanced implies $\sigma(u,v) = \sigma_{\max}(u,v) = \sigma_{\min}(u,v)$ and $K^{D^\max}(\Sigma) = K^{D^\min}(\Sigma) = K^{D^\pm}(\Sigma)$. Hence, $K^{D^\pm}(\Sigma) = K^{D^\pm}(\Sigma^n)$. \qed

**Theorem 4.** A 2-connected signed graph $\Sigma = (G, \sigma)$ is balanced if and only if $\Sigma^n = (G^n, \sigma')$ is balanced.

**Proof.** Suppose that $\Sigma$ is balanced. Then, by Theorem 2 we get $D^\max(\Sigma) = D^\min(\Sigma)$ and the associated signed complete graph $K^{D^\pm}(\Sigma)$ is balanced. Since, $\Sigma$ is balanced by using Lemma 4 and Lemma 5 we get $\Sigma^n$ is compatible and $K^{D^\pm}(\Sigma^n)$ is balanced. Again, by using Theorem 2, we get $\Sigma^n$ is balanced.
Conversely, suppose that $\Sigma^n$ is balanced. Being a subgraph of $\Sigma^n$, $\Sigma$ should be balanced.

The following Corollary is an immediate consequence of Theorem 3 and Theorem 4.

**Corollary 1.** Let $\Sigma = (G, \sigma)$ be a 2-connected compatible signed graph of order $m$. Then, the $n^{th}$ power signed graph $\Sigma^n$ is balanced if and only if the associated signed complete graph $K^{b\pm}(\Sigma)$ has the spectrum $\left(\begin{array}{cc} m-1 & -1 \\ 1 & m-1 \end{array}\right)$.

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**References**


