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Two upper bounds on the A_α-spectral radius of a connected graph

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Abstract: If $A(G)$ and $D(G)$ are respectively the adjacency matrix and the diagonal matrix of vertex degrees of a connected graph G, the generalized adjacency matrix $A_\alpha(G)$ is defined as $A_\alpha(G) = \alpha\ D(G) + (1-\alpha)\ A(G)$, where $0 \leq \alpha \leq 1$. The A_α (or generalized) spectral radius $\lambda(A_\alpha(G))$ (or simply λ_α) is the largest eigenvalue of $A_\alpha(G)$. In this paper, we show that

$$\lambda_\alpha \leq \alpha\ \Delta + (1-\alpha)\sqrt{2m\left(1-\frac{1}{\omega}\right)},$$

where m, Δ and $\omega = \omega(G)$ are respectively the size, the largest degree and the clique number of G. Further, if G has order n, then we show that

$$\lambda_\alpha \leq \frac{1}{2}\max_{1\leq i\leq n}\left[\alpha d_i + \sqrt{\alpha^2 d_i^2 + 4m_i(1-\alpha)[\alpha + (1-\alpha)m_j]}\right],$$

where d_i and m_i are respectively the degree and the average 2-degree of the vertex v_i.

Keywords: Adjacency matrix, generalized adjacency matrix, spectral radius, clique number

AMS Subject classification: 05C12, 05C50, 15A18

1. Introduction

Let $G(V,E)$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and order $|V(G)| = n$. The degree $d(v_i)$ or d_i of a vertex v_i is the number of edges incident on v_i. A graph is regular if each of its vertices has the same degree. If vertex i is adjacent to vertex j, we write $i \sim j$. A clique is a maximal complete subgraph of G. Also, the clique number $\omega = \omega(G)$ of G is the order of the largest clique in G.
The generalized adjacency matrix $A_\alpha(G)$ [6] is defined as $A_\alpha(G) = \alpha\ D(G) + (1-\alpha)\ A(G)$, where $0 \leq \alpha \leq 1$, $A(G)$ and $D(G)$ are respectively

the adjacency matrix and the diagonal matrix of vertex degrees of G. The largest eigenvalue of $A_\alpha(G)$ is called the A_α (or generalized adjacency) spectral radius of G and we denote it by $\lambda(A_\alpha(G))$ (or simply λ_α). If G is connected, $A_\alpha(G)$ is non-negative and irreducible for $\alpha \neq 1$, and by the Perron-Frobenius theorem, λ_α is unique and there is a unique positive unit eigenvector X corresponding to λ_α, which is called the *generalized adjacency Perron vector* of G. From the last few years an enormous activity has been observed in the study of the spectral properties of the matrix $A_\alpha(G)$. For some recent results on λ_α, we refer to [2–4, 8] and the references therein. For standard definitions, we refer to [1, 7].

2. Upper bounds for $\lambda(A_\alpha(G))$

We consider a column vector $X = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$ to be a function defined on $V(G)$ which maps vertex v_i to x_i, that is, $X(v_i) = x_i$ for i, $1 \leq i \leq n$. Therefore,

$$X^T A_\alpha(G) X = \alpha \sum_{i=1}^{n} d_i x_i^2 + 2(1-\alpha) \sum_{j \sim i} x_i x_j,$$

and λ_α is an eigenvalue of $A_\alpha(G)$ corresponding to the eigenvector X if and only if $X \neq 0$ and

$$\lambda_\alpha x_i = \alpha d_i x_i + (1-\alpha) \sum_{j \sim i} x_i x_j.$$

These equations are called as (λ_α, x)-*eigenequations* of G. For a normalized column vector $X \in \mathbb{R}^n$ with at least one non-negative component, by the Rayleigh's principle, we have

$$\lambda_\alpha \geq X^T A_\alpha(G) X,$$

with equality if and only if X is the generalized adjacency Perron vector of G.

We have the following lemma [5].

Lemma 1. *If* $F = \{x = (x_1, x_2, \ldots, , x_n)^T \ : \ x_i \geq 0, \ \sum_{i=1}^{n} x_i = 1\}$, *then*

$$1 - \frac{1}{\omega} = \max_{x \in F} \ (x, Ax) \tag{1}$$

Now, we obtain an upper bound for λ_α which involves the clique number and the largest degree of G.

Theorem 1. *If λ_α is the A_α-spectral radius (G), then*

$$\lambda_\alpha \le \alpha\ \Delta + (1-\alpha)\sqrt{2m\left(1-\frac{1}{\omega}\right)}, \tag{2}$$

where m, Δ and $\omega = \omega(G)$ are respectively the size, the largest degree and the clique number of G.

Proof. Corresponding to the eigenvalue λ_α of $A_\alpha(G)$, let $X = [x_1, x_2, \ldots, x_n]^T$ be the normalized eigenvector. Therefore,

$$\begin{aligned}\lambda_\alpha &= \alpha \sum_{i=1}^{n} d_i x_i^2\ + 2(1-\alpha)\sum_{j\sim i} x_i x_j \\ &\le \alpha\ \Delta \sum_{i=1}^{n} x_i^2\ + 2(1-\alpha)\sum_{j\sim i} x_i x_j \\ &= \alpha\Delta\ + 2(1-\alpha)\sum_{j\sim i} x_i x_j.\end{aligned}$$

From [6], for $0 \le \alpha \le \frac{1}{2}$, clearly we have $\lambda_\alpha \ge \alpha\ (\Delta+1)$.
Therefore, applying Cauchy's inequality, we obtain

$$(\lambda_\alpha - \alpha\ \Delta)^2 \le \left(2(1-\alpha)\sum_{j\sim i} x_i x_j\right)^2 \le 2(1-\alpha)^2\ m\ \left(2\sum_{j\sim i} x_i^2 x_j^2\right). \tag{3}$$

Now, evidently $(x_1, x_2, \ldots, x_n)^T \ge 0$ and $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$.
Thus, using (1), we get

$$2\sum_{j\sim i} x_i^2 x_j^2 \le 1 - \frac{1}{\omega}. \tag{4}$$

Using (4) in (3), we obtain

$$(\lambda_\alpha - \alpha\ \Delta)^2 \le 2(1-\alpha)^2\ m\ \left(1-\frac{1}{\omega}\right)$$

which implies that

$$\lambda_\alpha \le \alpha\ \Delta + (1-\alpha)\sqrt{2m\left(1-\frac{1}{\omega}\right)}$$

and the proof is complete. □

In G, the average 2-degree of a vertex v_i, denoted my m_i, is defined as $m_i = \sum_{v_j v_i \in E(G)} \frac{d_j}{d_i}$, with d_i being the degree of the vertex v_i. Now, we obtain an upper bound of λ_α in terms of vertex degrees and average 2-degrees.

Theorem 2. *Let G be a connected graph of order n and let d_i and m_i be respectively the degree and the average 2-degree of the vertex v_i. Then*

$$\lambda_\alpha \leq \frac{1}{2} \max_{1 \leq i \leq n} \left[\alpha d_i + \sqrt{\alpha^2 d_i^2 + 4m_i(1-\alpha)[\alpha + (1-\alpha)m_j]} \right]. \tag{5}$$

Proof. Consider the matrix $D = diag(d_1, d_2, \ldots, d_n)$, where $d_1 \geq d_2 \geq \cdots \geq d_n$ are the degrees of the vertices of G. Obviously $A_\alpha(G)$ and $D^{-1}A_\alpha(G)D$ are similar matrices and therefore have the same spectrum. Thus, λ_α is also the largest eigenvalue of $D^{-1}A_\alpha(G)D$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector corresponding to the eigenvalue λ_α of $D^{-1}A_\alpha(G)D$. Assume that one of the eigencomponent, say x_i equals 1 and the remaining eigencomponents are less than or equal to 1.
Now, the (i, j)th entry of $D^{-1}A_\alpha(G)D$ is given by

$$[D^{-1}A_\alpha(G)D]_{ij} = \begin{cases} d_j, & \text{if } \ i = j \\ \frac{d_j}{d_i}, & \text{if } \ j \sim i. \end{cases}$$

We have

$$D^{-1}A_\alpha(G)DX \ = \ A_\alpha X. \tag{6}$$

Taking the ith and jth equations of (6), we obtain

$$\lambda_\alpha x_i = \alpha d_i x_i + (1-\alpha) \left(\sum_{j \sim i} \frac{d_j}{d_i} x_j \right),$$

or

$$\lambda_\alpha = \alpha d_i + (1-\alpha) \left(\sum_{j \sim i} \frac{d_j}{d_i} x_j \right) \tag{7}$$

and

$$\lambda_\alpha x_j = \alpha d_i x_j + (1-\alpha) \left(\sum_{t \sim j} \frac{d_t}{d_j} x_t \right). \tag{8}$$

From (7), we get

$$\lambda_\alpha^2 = \alpha d_i \lambda_\alpha + (1-\alpha) \left(\sum_{j \sim i} \frac{d_j}{d_i} x_j \right) \lambda_\alpha. \tag{9}$$

Now, using (8) in (9), we have

$$\begin{aligned}
\lambda_\alpha^2 &= \alpha d_i \lambda_\alpha + (1-\alpha)\sum_{j\sim i}\left(\frac{d_j}{d_i}\left(\alpha d_j x_j + (1-\alpha)\sum_{t\sim j}\frac{d_t}{d_j}x_t\right)\right)\\
&= \alpha d_i \lambda_\alpha + \alpha(1-\alpha)\sum_{j\sim i}\frac{d_j^2}{d_i}x_j + (1-\alpha)^2\sum_{j\sim i}\sum_{t\sim j}\frac{d_t}{d_j}x_t\\
&\le \alpha d_i \lambda_\alpha + \alpha(1-\alpha)m_i + (1-\alpha)^2\sum_{j\sim i}\frac{d_j}{d_i}\sum_{t\sim j}\frac{d_t}{d_j}\\
&= \alpha d_i \lambda_\alpha + \alpha(1-\alpha)m_i + (1-\alpha)^2 m_i m_j.
\end{aligned}$$

Therefore,

$$\lambda_\alpha^2 - \alpha d_i \lambda_\alpha - \alpha(1-\alpha)m_i - (1-\alpha)^2 m_i m_j \le 0. \tag{10}$$

From (10), it follows that

$$\lambda_\alpha \le \frac{1}{2}\max_{1\le i\le n}\left[\alpha d_i + \sqrt{\alpha^2 d_i^2 + 4m_i(1-\alpha)[\alpha + (1-\alpha)m_j]}\right],$$

completing the proof. □

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