Two upper bounds on the $A_\alpha$-spectral radius of a connected graph

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Abstract: If $A(G)$ and $D(G)$ are respectively the adjacency matrix and the diagonal matrix of vertex degrees of a connected graph $G$, the generalized adjacency matrix $A_\alpha(G)$ is defined as $A_\alpha(G) = \alpha D(G) + (1 - \alpha) A(G)$, where $0 \leq \alpha \leq 1$. The $A_\alpha$ (or generalized) spectral radius $\lambda(A_\alpha(G))$ (or simply $\lambda_\alpha$) is the largest eigenvalue of $A_\alpha(G)$. In this paper, we show that

$$\lambda_\alpha \leq \alpha \Delta + (1 - \alpha) \sqrt{2m(1 - \frac{1}{\omega})},$$

where $m$, $\Delta$ and $\omega = \omega(G)$ are respectively the size, the largest degree and the clique number of $G$. Further, if $G$ has order $n$, then we show that

$$\lambda_\alpha \leq \frac{1}{2} \max_{1 \leq i \leq n} \left[ \alpha d_i + \sqrt{\alpha^2 d_i^2 + 4m_i(1 - \alpha)[\alpha + (1 - \alpha)m_j]} \right],$$

where $d_i$ and $m_i$ are respectively the degree and the average 2-degree of the vertex $v_i$.

Keywords: Adjacency matrix, generalized adjacency matrix, spectral radius, clique number

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1. Introduction

Let $G(V, E)$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and order $|V(G)| = n$. The degree $d(v_i)$ or $d_i$ of a vertex $v_i$ is the number of edges incident on $v_i$. A graph is regular if each of its vertices has the same degree. If vertex $i$ is adjacent to vertex $j$, we write $i \sim j$. A clique is a maximal complete subgraph of $G$. Also, the clique number $\omega = \omega(G)$ of $G$ is the order of the largest clique in $G$.

The generalized adjacency matrix $A_\alpha(G)$ \cite{6} is defined as $A_\alpha(G) = \alpha D(G) + (1 - \alpha) A(G)$, where $0 \leq \alpha \leq 1$, $A(G)$ and $D(G)$ are respectively
the adjacency matrix and the diagonal matrix of vertex degrees of $G$. The largest eigenvalue of $A_\alpha(G)$ is called the $A_\alpha$ (or generalized adjacency) spectral radius of $G$ and we denote it by $\lambda(A_\alpha(G))$ (or simply $\lambda_\alpha$). If $G$ is connected, $A_\alpha(G)$ is non-negative and irreducible for $\alpha \neq 1$, and by the Perron-Frobenius theorem, $\lambda_\alpha$ is unique and there is a unique positive unit eigenvector $X$ corresponding to $\lambda_\alpha$, which is called the generalized adjacency Perron vector of $G$. From the last few years an enormous activity has been observed in the study of the spectral properties of the matrix $A_\alpha(G)$. For some recent results on $\lambda_\alpha$, we refer to [2–4, 8] and the references therein. For standard definitions, we refer to [1, 7].

2. Upper bounds for $\lambda(A_\alpha(G))$

We consider a column vector $X = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$ to be a function defined on $V(G)$ which maps vertex $v_i$ to $x_i$, that is, $X(v_i) = x_i$ for $i = 1, 2, \ldots, n$. Therefore,

$$X^T A_\alpha(G) X = \alpha \sum_{i=1}^{n} d_i x_i^2 + 2(1 - \alpha) \sum_{j \sim i} x_i x_j,$$

and $\lambda_\alpha$ is an eigenvalue of $A_\alpha(G)$ corresponding to the eigenvector $X$ if and only if $X \neq 0$ and

$$\lambda_\alpha x_i = \alpha d_i x_i + (1 - \alpha) \sum_{j \sim i} x_i x_j.$$

These equations are called as $(\lambda_\alpha, x)$-eigen equations of $G$. For a normalized column vector $X \in \mathbb{R}^n$ with at least one non-negative component, by the Rayleigh’s principle, we have

$$\lambda_\alpha \geq X^T A_\alpha(G) X,$$

with equality if and only if $X$ is the generalized adjacency Perron vector of $G$.

We have the following lemma [5].

**Lemma 1.** If $F = \{x = (x_1, x_2, \ldots, x_n)^T : x_i \geq 0, \ \sum_{i=1}^{n} x_i = 1\}$, then

$$1 - \frac{1}{\omega} = \max_{x \in F} (x, Ax)$$

(1)

Now, we obtain an upper bound for $\lambda_\alpha$ which involves the clique number and the largest degree of $G$. 

Theorem 1. If $\lambda_\alpha$ is the $A_\alpha$-spectral radius of $G$, then

$$\lambda_\alpha \leq \alpha \Delta + (1 - \alpha)\sqrt{2m \left(1 - \frac{1}{\omega}\right)},$$

(2)

where $m$, $\Delta$ and $\omega = \omega(G)$ are respectively the size, the largest degree and the clique number of $G$.

Proof. Corresponding to the eigenvalue $\lambda_\alpha$ of $A_\alpha(G)$, let $X = [x_1, x_2, \ldots, x_n]^T$ be the normalized eigenvector. Therefore,

$$\lambda_\alpha = \alpha \sum_{i=1}^{n} d_i x_i^2 + 2(1 - \alpha) \sum_{j \sim i} x_i x_j \leq \alpha \Delta \sum_{i=1}^{n} x_i^2 + 2(1 - \alpha) \sum_{j \sim i} x_i x_j = \alpha \Delta + 2(1 - \alpha) \sum_{j \sim i} x_i x_j.$$

From [6], for $0 \leq \alpha \leq \frac{1}{2}$, clearly we have $\lambda_\alpha \geq \alpha (\Delta + 1)$. Therefore, applying Cauchy’s inequality, we obtain

$$(\lambda_\alpha - \alpha \Delta)^2 \leq \left(2(1 - \alpha) \sum_{j \sim i} x_i x_j\right)^2 \leq 2(1 - \alpha)^2 m \left(2 \sum_{j \sim i} x_i^2 x_j^2\right).$$

(3)

Now, evidently $(x_1, x_2, \ldots, x_n)^T \geq 0$ and $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Thus, using (1), we get

$$2 \sum_{j \sim i} x_i^2 x_j^2 \leq 1 - \frac{1}{\omega}.$$

(4)

Using (4) in (3), we obtain

$$(\lambda_\alpha - \alpha \Delta)^2 \leq 2(1 - \alpha)^2 m \left(1 - \frac{1}{\omega}\right)$$

which implies that

$$\lambda_\alpha \leq \alpha \Delta + (1 - \alpha)\sqrt{2m \left(1 - \frac{1}{\omega}\right)}$$

and the proof is complete. $\square$
In $G$, the average 2-degree of a vertex $v_i$, denoted by $m_i$, is defined as $m_i = \sum_{v_j, v_i \in E(G)} d_j$, with $d_i$ being the degree of the vertex $v_i$. Now, we obtain an upper bound of $\lambda_\alpha$ in terms of vertex degrees and average 2-degrees.

**Theorem 2.** Let $G$ be a connected graph of order $n$ and let $d_i$ and $m_i$ be respectively the degree and the average 2-degree of the vertex $v_i$. Then

$$\lambda_\alpha \leq \frac{1}{2} \max_{1 \leq i \leq n} \left[ \alpha d_i + \sqrt{\alpha^2 d_i^2 + 4m_i(1-\alpha)[\alpha + (1-\alpha)m_j]} \right]. \quad (5)$$

**Proof.** Consider the matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$, where $d_1 \geq d_2 \geq \cdots \geq d_n$ are the degrees of the vertices of $G$. Obviously $A_\alpha(G)$ and $D^{-1}A_\alpha(G)D$ are similar matrices and therefore have the same spectrum. Thus, $\lambda_\alpha$ is also the largest eigenvalue of $D^{-1}A_\alpha(G)D$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector corresponding to the eigenvalue $\lambda_\alpha$ of $D^{-1}A_\alpha(G)D$. Assume that one of the eigencomponent, say $x_i$ equals 1 and the remaining eigencomponents are less than or equal to 1.

Now, the $(i, j)$th entry of $D^{-1}A_\alpha(G)D$ is given by

$$[D^{-1}A_\alpha(G)D]_{ij} = \begin{cases} d_j, & \text{if } i = j \\ \frac{d_j}{\alpha}, & \text{if } j \sim i. \end{cases}$$

We have

$$D^{-1}A_\alpha(G)DX = A_\alpha X. \quad (6)$$

Taking the $i$th and $j$th equations of (6), we obtain

$$\lambda_\alpha x_i = \alpha d_i x_i + (1-\alpha) \left( \sum_{j \sim i} \frac{d_j}{d_i} x_j \right),$$

or

$$\lambda_\alpha = \alpha d_i + (1-\alpha) \left( \sum_{j \sim i} \frac{d_j}{d_i} x_j \right) \quad (7)$$

and

$$\lambda_\alpha x_j = \alpha d_j x_j + (1-\alpha) \left( \sum_{i \sim j} \frac{d_i}{d_j} x_i \right). \quad (8)$$

From (7), we get

$$\lambda^2_\alpha = \alpha d_i \lambda_\alpha + (1-\alpha) \left( \sum_{j \sim i} \frac{d_j}{d_i} x_j \right) \lambda_\alpha. \quad (9)$$

Now, using (8) in (9), we have
\[
\lambda_\alpha^2 = \alpha d_i \lambda + (1 - \alpha) \left( \sum_{j \sim i} \left( \frac{d_j}{d_i} \left( \alpha d_j x_j + (1 - \alpha) \sum_{t \sim j} \frac{d_t}{d_j} x_t \right) \right) \right) \\
= \alpha d_i \lambda + \alpha (1 - \alpha) \sum_{j \sim i} \frac{d^2_j}{d_i} x_j + (1 - \alpha)^2 \sum_{j \sim i} \sum_{t \sim j} \frac{d_t}{d_j} x_t \\
\leq \alpha d_i \lambda + \alpha (1 - \alpha) m_i + (1 - \alpha)^2 \sum_{j \sim i} \sum_{t \sim j} \frac{d_j}{d_i} \frac{d_t}{d_j} \\
= \alpha d_i \lambda + \alpha (1 - \alpha) m_i + (1 - \alpha)^2 m_i m_j.
\]

Therefore,
\[
\lambda_\alpha^2 - \alpha d_i \lambda - \alpha (1 - \alpha) m_i - (1 - \alpha)^2 m_i m_j \leq 0. 
\tag{10}
\]

From (10), it follows that
\[
\lambda_\alpha \leq \frac{1}{2} \max_{1 \leq i \leq n} \left[ \alpha d_i + \sqrt{\alpha^2 d_i^2 + 4 m_i (1 - \alpha) [\alpha + (1 - \alpha) m_j]} \right],
\]
completing the proof. \(\square\)

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