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The Tutte polynomial of matroids constructed by a family of splitting operations

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Abstract: To extract some more information from the constructions of matroids that arise from new operations, computing the Tutte polynomial, plays an important role. In this paper, we consider applying three operations of splitting, element splitting and splitting off to a binary matroid and then introduce the Tutte polynomial of resulting matroids by these operations in terms of that of original matroids.

Keywords: Tutte polynomial, Splitting, Splitting off, Element splitting

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1. Introduction

Throughout this paper we denote a singleton set $\{x\}$ by x. A matroid $M = (E, \mathcal{I})$ is a finite ground set E together with a collection of sets $\mathcal{I} \subseteq 2^E$, known as the independent sets, satisfying the following axioms.

(I1) $\mathcal{I}$ is non-empty.

(I2) Every subset of a member of $\mathcal{I}$ is also in $\mathcal{I}$.

(I3) If X and Y are in $\mathcal{I}$ and $|X| < |Y|$, then there is an element e in $Y - X$ such that $X \cup e$ is in $\mathcal{I}$.

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A subset X of E which is not in $\mathcal{I}$ is called *dependent* set and if X is minimal, then it is a *circuit* of M. Maximal members of $\mathcal{I}$ are called *bases* of M and the matroid on $(E, \mathcal{B}^*)$ is called *dual* of M denoted by M^* where $\mathcal{B}^*$ is the collection of the complement of any basis of M. The circuits of M^* are called *cocircuits* of M. The *rank* of M denoted by $r(M)$ is the number of elements of its basis.
Let $\mathbb{F}$ be a field and let $E \subseteq \mathbb{F}^k$ be a finite set of vectors. Then a *linear matroid* is a matroid whose independent sets are the linearly independent vectors in E over $\mathbb{F}$. A *binary matroid* is a linear matroid over the finite field $GF(2)$. Let M be a binary matroid on the set $\{1, 2, \ldots, n\}$ and A be a matrix that represents M over $GF(2)$. The *cocircuit spaces* of M are the subspaces of $V(n, 2)$ that are generated by the incidence vectors of the cocircuits of M and are equal to the row space of A. Also, a non-loop element x of M is said to be *equivalent* [8] to another non-loop element y of M and denoted by $x \sim y$, if there is a row in the row space of A with entry 1 in the columns corresponding to x and y, and entry 0 elsewhere. Now, by the fact that all cocircuits of a matroid are minimal sets, we conclude that x is equivalent to y in a binary matroid M if and only if the set $\{x, y\}$ is a cocircuit of M or both of x and y are coloops of M. We refer to Oxley [4] for basic definitions not given in this paper such as minors of a matroid, matrix representations and rank functions of matroids.
Raghunathan et al. [5] extended the splitting operation and Azadi [1, 7] extended the splitting off and the element splitting operations from graphs to binary matroids. These operations are defined as follows.

Definition 1. Let M be a binary matroid on a set E and A be a matrix that represents M over $GF(2)$. Consider two elements x and y of $E(M)$. Let $A_{x,y}$ be the matrix that is obtained by adjoining an extra row to A whose entries are zero everywhere except in the columns corresponding to x and y. Let $A'_{x,y}$ be the matrix that is obtained by adjoining an extra column to $A_{x,y}$ with this column being zero everywhere except in the last row. Finally, let A_{xy} be the matrix that is obtained by adjoining an extra column to A which is the sum of the columns corresponding to x and y, and then deleting the two columns corresponding to x and y. Let $M_{x,y}$, $M'_{x,y}$ and M_{xy} be the matroids represented by the matrices $A_{x,y}$, $A'_{x,y}$ and A_{xy}, respectively. Then the transition from M to $M_{x,y}$, $M'_{x,y}$ and M_{xy} is called *the splitting operation*, *element splitting operation* and *splitting off operation* (or in short *split-off*), respectively.

Throughout this paper, for notational convenience, letting α and β be the labels of new elements that are added to an original matroid after applying split-off and element splitting operations, respectively.

2. Some preliminaries

We will use the following propositions to prove our main results in the next section. One can find the gathered parts of these propositions in [1], [5] and [8].

Proposition 1. *Let M be a binary matroid and let $x, y \in E(M)$. Then*

i)

$$r(M_{xy}) = \begin{cases} r(M), & \text{if } x \nsim y; \\ r(M) - 1, & \text{if } x \sim y. \end{cases}$$

ii)

$$r(M_{x,y}) = \begin{cases} r(M) + 1, & \text{if } x \nsim y; \\ r(M), & \text{if } x \sim y. \end{cases}$$

iii)

$$r(M'_{x,y}) = r(M) + 1.$$

Proposition 2. *Let M be a binary matroid and let $x, y \in E(M)$. Then*

i) $M'_{x,y}/\beta = M$ *and* $M'_{x,y} \setminus \beta = M_{x,y}$*;*

ii) $M_{x,y}/x \cong M_{x,y}/y \cong M_{xy}$*;*

iii) $M'_{x,y} \setminus \{\beta, x, y\} = M_{x,y} \setminus x/y = M_{x,y} \setminus y/x = M_{x,y} \setminus \{x, y\} = M_{xy} \setminus \alpha = M \setminus \{x, y\}$*;*

iv) *y is a coloop of $M_{x,y} \setminus x$ while x is a coloop of $M_{x,y} \setminus y$;*

v) *If $x \sim y$ in M, then $M_{xy} \cong M/\{x\} \cong M/\{y\}$ and $M_{xy}/\{\alpha\} = M/\{x, y\}$;*

vi) *If at least one of x and y is a coloop of M, then α is a coloop of M_{xy} and both of x and y are coloops of $M_{x,y}$.*

Let M be a matroid and X be a subset of $E(M)$. In the following, we will use the fact that cocircuits of $M \setminus X$, for $X \subseteq E(M)$ are minimal members of

$$\{C^* - X \subseteq E(M) - X : C^* \text{ is a cocircuit of } M\}.$$

We denote by $M(X)$, the *restriction* of M with respect to X which is the matroid $M \setminus (E - X)$.

Proposition 3. *Let r and r' be the rank functions of the matroids M and M_{xy}, respectively. Let $T \subseteq E(M_{xy})$. Then*

i) *if $\alpha \notin T$, then $r'(T) = r(T)$;*

ii) *if $\alpha \in T$ and α is not a coloop of $M_{xy}(T)$, then $r'(T) = r(T - \alpha)$;*

iii) *if $\alpha \in T$ and α is a coloop of $M_{xy}(T)$, then $r'(T) = r(T - \alpha) + 1$.*

Note that, finding the subsets of a split-off matroid such that the restriction concerning to them has α as a coloop or not is not difficult. To see this, let A be the matrix that represents a binary matroid M. Let $x, y \in E(M)$ and $T \subseteq E(M_{xy})$ such that $\alpha \in T$. Recall that the cocircuit space of M_{xy} equals the row space of A_{xy} and consider the following collection:

$$\big\{C^* : C^* \text{ is a cocircuit of } M_{xy} \text{ and } C^* \cap T = \alpha\big\}.$$

If this collection is empty, then α is not a coloop of $M_{xy}(T)$, otherwise, α is a coloop of $M_{xy}(T)$. Moreover, to find the cocircuits of M_{xy}, the following theorem characterizes the collection of all cocircuits of M_{xy} in terms of the cocircuits of M.
Theorem 1 could be reworded as follows.

Theorem 1. *Let M_{xy} be the matroid obtained by split-off operation on a binary matroid M and let X be a subset of $E(M_{xy})$. Let X be minimal and be a member of one of the following collections.*

- **i)** $\mathcal{C}_1^* = \{C^* - \{x, y\} : C^*$ *is a cocircuit of* M *and* $x, y \in C^*\}$;
- **ii)** $\mathcal{C}_2^* = \{C^* : C^*$ *is a cocircuit of* M *and* $x, y \notin C^*\}$;
- **iii)** $\mathcal{C}_3^* = \{(C^* - x) \cup \alpha : C^*$ *is a cocircuit of* M *and* $x \in C^*, y \notin C^*\}$;
- **iv)** $\mathcal{C}_4^* = \{(C^* - y) \cup \alpha : C^*$ *is a cocircuit of* M *and* $y \in C^*, x \notin C^*\}$.
- **v)** $\mathcal{C}_5^* = \{(C_1^* \Delta C_2^*) - \{x, y\} : C_1^*$ *and* C_2^* *are cocircuits of* M *such that* C_1^* *contains* x *but not* y *and* C_2^* *contains* y *but not* $x\}$.

Then the set X is a cocircuit of M_{xy}.

Proof. Let A be a matrix that represents M over $GF(2)$ and let C^* be a cocircuit of M such that $x, y \in C^*$. Then there is a row in the row space of A such that has the following form

$$\begin{array}{c}
\begin{array}{ccc} \scriptstyle E(M)-(C^*\cup\{x,y\}) & C^* & x \quad y \end{array}\\
\left[\begin{array}{c|c|cc} 0\ 0\ \dots\ 0 & 1\ 1\ \dots\ 1 & 1 & 1 \end{array}\right].
\end{array}$$

Thus, after applying the split-off operation on M, we have the following row in the row space of A_{xy}

$$\begin{array}{c}
\begin{array}{ccc} \scriptstyle E(M)-(C^*\cup\{x,y\}) & C^* & \alpha \end{array}\\
\left[\begin{array}{c|c|c} 0\ 0\ \dots\ 0 & 1\ 1\ \dots\ 1 & 0 \end{array}\right].
\end{array}$$

We conclude that $C^* - \{x, y\}$ is in the cocircuit space of M_{xy}. Similarly, one can check that if a subset X of $E(M_{xy})$ is a member of the collections in (ii)-(iv), then X is a cocircuit of M_{xy}. Finally, let $X = (C_1^* \Delta C_2^*) - \{x, y\}$ where C_1^* and C_2^* are cocircuits of M such that C_1^* contains x but not y and C_2^* contains y but not x. Then, there are two rows in the row space of A having the form,

$$\begin{array}{cccccc} {\scriptstyle E(M)-(C_1^*\cup C_2^*)-\{x,y\}} & C_1^*-C_2^*-x & C_1^*\cap C_2^* & C_2^*-C_1^*-y & x & y \end{array}$$
$$\left[\begin{array}{c|c|c|c|cc} 0\ 0\ \dots\ 0 & 1\ 1\ \dots\ 1 & 1\ 1\ \dots\ 1 & 0\ 0\ \dots\ 0 & 1 & 0 \\ 0\ 0\ \dots\ 0 & 0\ 0\ \dots\ 0 & 1\ 1\ \dots\ 1 & 1\ 1\ \dots\ 1 & 0 & 1 \end{array}\right].$$

Thus, after applying the split-off operation on M, in the row space of A_{xy}, we have the row,

$$\begin{array}{ccccc} {\scriptstyle E(M)-(C_1^*\cup C_2^*)-\{x,y\}} & C_1^*-C_2^*-x & C_1^*\cap C_2^* & C_2^*-C_1^*-y & \alpha \end{array}$$
$$\left[\begin{array}{c|c|c|c|c} 0\ 0\ \dots\ 0 & 1\ 1\ \dots\ 1 & 0\ 0\ \dots\ 0 & 1\ 1\ \dots\ 1 & 0 \end{array}\right].$$

We thus conclude that $(C_1^* \Delta C_2^*) - \{x, y\}$ is in the cocircuit space of M_{xy}. To complete the proof, we must show that if $X, Y \in \mathcal{C}_{xy}^*$ and are minimal, then $X \nsubseteq Y$ and Y$\nsubseteq$X, and $X \Delta Y$ contains at least one member of $\mathcal{C}_{xy}^*$. There are fifteen cases to check and checking them is straightforward. $\square$

3. Tutte polynomial of splitting, element splitting and split-off matroids

The Tutte polynomial is a two-variable polynomial originally defined for graphs by Tutte and Whitney and later generalized to matroids by Crapo [3]. It was first conceived as an extension of the chromatic polynomial, but nowadays it is known to have applications in many areas of combinatorics and other areas of mathematics.

Definition 2. The Tutte polynomial $T(M, u, v)$ (or briefly $T(M)$) of a matroid M on the set E is given by

$$T(M) = \sum_{A \subseteq E} (u-1)^{z(A)} (v-1)^{n(A)}$$

where $z(A) = r(M) - r(A)$ and $n(A) = |A| - r(A)$.

Computing the Tutte polynomial of a matroid is known to be #P-hard, so some formulas are presented to reduce this computation to simpler computations. The next proposition shows some of the well-known formulas for the Tutte polynomial of a given matroid.

Proposition 4. [2] *Let M be a matroid. Then the Tutte polynomial $T(M)$ has the following properties*

i) *Given a matroid N with $M \cong N$, we have $T(M) = T(N)$.*

ii) *If $e \in E(M)$ and e is neither a loop nor a coloop of M, then $T(M) = T(M \setminus e) + T(M/e)$.*

iii) *If e is a loop of M, then $T(M) = v\big(T(M \setminus e)\big)$.*

iv) *If e is a coloop of M, then $T(M) = u\big(T(M \setminus e)\big)$.*

In [6], the relation between a given binary matroid M and $M_{x,y}$ is presented as

$$T(M_{x,y}, 2, 1) \equiv T(M, 2, 1) \pmod 2$$

where $T(M, 2, 1)$ and $T(M_{x,y}, 2, 1)$ count the number of independent sets in M and $M_{x,y}$, respectively.
Now, we want to find some splitting formulas to compute the Tutte polynomial of a matroid that is obtained by splitting, element splitting and split-off operation in terms of the Tutte polynomial of original matroid or in terms of that of each other.

Theorem 2. *Let M be a binary matroid and Let $x, y \in E(M)$ such that $x \nsim y$. Then*

$$T(M'_{x,y}) = (u+1)T(M \setminus \{x, y\}) + T(M_{xy}/\alpha) + T(M).$$

Proof. Consider two non-equivalent elements x and y from a given binary matroid M. By using some parts of Proposition 2, we have the following formula.

$$\begin{aligned} T(M'_{x,y}) &= T(M'_{x,y} \setminus \beta) + T(M'_{x,y}/\beta) \\ &= T(M_{x,y}) + T(M). \end{aligned}$$

Also, y is a coloop of $M_{x,y} \setminus x$. This means $M_{x,y} \setminus x \cong M \setminus \{x, y\} \oplus U_{1,1}$. Therefore,

$$\begin{aligned} T(M'_{x,y}) &= T(M_{x,y} \setminus x) + T(M_{x,y}/x) + T(M) \\ &= uT(M \setminus \{x, y\}) + T(M_{xy}) + T(M). \end{aligned}$$

or

$$\begin{aligned} T(M'_{x,y}) &= T(M_{x,y} \setminus x) + T(M_{x,y}/x) + T(M) \\ &= uT(M \setminus \{x, y\}) + T(M_{x,y}/x \setminus y) + T(M_{x,y}/\{x, y\}) + T(M) \\ &= (u+1)T(M \setminus \{x, y\}) + T(M_{xy}/\alpha) + T(M). \end{aligned}$$

□

Corollary 1. *Let M be a binary matroid and let $x, y \in E(M)$. Then*

$$T(M_{x,y}) = (u+1)T(M \setminus \{x, y\}) + T(M_{xy}/\alpha).$$

Proof. This formula has been extracted from the proof of Theorem 2. □

To extend our formulas in Theorem 2 and Corollary 1, we must find Tutte polynomials for split-off matroids in terms of the Tutte polynomials of original matroids.

Theorem 3. *The Tutte polynomial of a split-off matroid M_{xy} from a given binary matroid M with $x, y \in E(M)$ has the following properties:*

i) *If at least one of x and y is a coloop of M, then $T(M_{xy}) = uT(M \setminus \{x, y\})$.*

ii) *If $\{x, y\}$ is a circuit of M, then $T(M_{xy}) = vT(M \setminus \{x, y\})$.*

iii) *If $\{x, y\}$ is a cocircuit of M which is not a circuit of M, then*

$$T(M_{xy}) = T(M) - uT(M \setminus \{x, y\}).$$

Proof. Suppose that at least one of x and y is a coloop of M. Then, by Proposition 2(vi), the element α is a coloop of M_{xy} and by part (iii), $M_{xy} \setminus \{\alpha\} = M \setminus \{x, y\}$. Now, by Propositions 4(v), we have

$$T(M_{xy}) = uT(M_{xy} \setminus \alpha) = uT(M \setminus \{x, y\}).$$

Suppose that $\{x, y\}$ be a circuit of M. Then α is a loop of M. By a similar argument we can establish that $T(M_{xy}) = vT(M \setminus \{x, y\})$. Finally, let $\{x, y\}$ be a cocircuit of M. Then x is a coloop of $M \setminus y$ and $M \setminus y/x = M \setminus \{x, y\}$. By Proposition 4 (iii) and (v), we have

$$\begin{aligned} T(M) &= T(M \setminus x) + T(M/x) \\ &= uT(M \setminus \{x, y\}) + T(M/x \setminus y) + \\ &\quad T(M/\{x, y\}). \end{aligned}$$

Therefore

$$T(M) = (u + 1)T(M \setminus \{x, y\}) + T(M/\{x, y\})$$

or

$$T(M) = (u)T(M \setminus \{x, y\}) + \underbrace{T(M_{xy} \setminus \alpha) + T(M_{xy}/\alpha)}_{T(M_{xy})}.$$

We conclude that

$$T(M_{xy}) = T(M) - uT(M \setminus \{x, y\}).$$

□

The following corollary is an immediate consequence of combining Theorem 3 and Proposition 2.

Corollary 2. *Let M be a binary matroid and Let $x, y \in E(M)$. Then*

i) *If exactly one of x and y is a coloop of M, then* $T(M_{x,y}) = u^2 T(M \setminus \{x, y\})$ *and* $T(M'_{x,y}) = u^2 T(M \setminus \{x, y\}) + T(M)$.

ii) *If $\{x, y\}$ is a circuit of M, then* $T(M_{x,y}) = (u+v)T(M \setminus \{x, y\})$ *and* $T(M'_{x,y}) = (u+v)T(M \setminus \{x, y\}) + T(M)$.

iii) *If $x \sim y$, then $T(M_{x,y}) = T(M)$ and $T(M'_{x,y}) = uT(M)$*

Given a binary matroid M, Theorem 3 introduces Tutte polynomial of M_{xy} when $x \sim y$ or $\{x, y\}$ is a circuit of M or exactly one of x and y is a coloop of M. Now we want to provide a splitting formula for such a Tutte polynomial when $x \nsim y$.
Let M_{xy} be the split-off of a binary matroid M with $x, y \in E(M)$ and $x \nsim y$ in M. Then there is a partition of the collection of all subsets of $E(M_{xy})$ into non-empty subsets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ such that

- $\mathcal{A}_1 = \{A \subset E(M_{xy}) : \alpha \notin A\}$;
- $\mathcal{A}_2 = \{A \subseteq E(M_{xy}) : \alpha \text{ is not a coloop of } M_{xy}(A)\}$;
- $\mathcal{A}_3 = \{A \subseteq E(M_{xy}) : \alpha \text{ is a coloop of } M_{xy}(A)\}$.

Clearly, any of $\mathcal{A}_1$ and $\mathcal{B}_1 \cup \mathcal{B}_2$ where

$$\mathcal{B}_i = \{Z - \alpha : Z \in \mathcal{A}_i, \text{ for } i \in \{2, 3\}\}$$

is a collection of all subsets of $E(M \setminus \{x, y\})$.
Therefore,

$$T(M \setminus \{x, y\}) = \sum_{i=1}^{2} \sum_{A \in \mathcal{B}_i} (u-1)^{z(A)} (v-1)^{n(A)}.$$

We denote two non-zero terms of this summation formula by $T_i(M)$. Indeed,

$$T(M \setminus \{x, y\}) = T_1(M) + T_2(M).$$

The following theorem will help compute the Tutte polynomial of a matroid obtained by applying split-off operation on two non-equivalent elements of a given binary matroid. Could be deleted: We use the notation above to prove it.

Theorem 4. *Let M be a binary matroid and $x, y \in E(M)$ such that $x \nsim y$ in M. Then the Tutte polynomial of M_{xy} can be computed by the following formula.*

$$T(M_{xy}) = vT_1(M) + \frac{u}{(u-1)} T_2(M).$$

Proof. Let x and y be two elements of a given binary matroid M such that $x \nsim y$ in M. Consider the partition $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ of all subsets of $E(M_{xy})$ as above. Then

$$T(M_{xy}) = \sum_{i=1}^{3} \sum_{A \in \mathcal{A}_i} (u-1)^{z(A)}(v-1)^{n(A)}.$$

This means that

$$\begin{aligned} T(M_{xy}) = & \sum_{A \in \mathcal{A}_1} (u-1)^{z(A)}(v-1)^{n(A)} + \\ & \sum_{(X \cup \alpha) \in \mathcal{A}_2} (u-1)^{z(X \cup \alpha)}(v-1)^{n(X \cup \alpha)} + \\ & \sum_{(X' \cup \alpha) \in \mathcal{A}_3} (u-1)^{z(X' \cup \alpha)}(v-1)^{n(X' \cup \alpha)} \end{aligned}$$

such that $X \in \mathcal{B}_1$ and $X' \in \mathcal{B}_2$. Let r and r' be the rank functions of M and M_{xy}, respectively. By using Propositions 3 and 1, we have $r'(M_{xy}) = r(M)$ and the following conditions.

- If $A \in \mathcal{A}_1$, then $z(A) = r'(M_{xy}) - r'(A) = r(M) - r(A)$ and $n(A) = |A| - r'(A) = |A| - r(A)$.
- If $A \in \mathcal{A}_2$, then α is not a coloop of $M_{xy}(A)$. Suppose that $A = X \cup \alpha$. Then $r'(A) = r(A - \alpha) = r(X)$. and so $z(A) = r'(M_{xy}) - r'(A) = r(M) - r(X)$ and $n(A) = |A| - r'(A) = |X| - r(X) + 1$.
- If $A \in \mathcal{A}_3$, then α is a coloop of $M_{xy}(A)$. Suppose that $A = X' \cup \alpha$. Then $r'(A) = r(A - \alpha) + 1 = r(X') + 1$. and so $z(A) = r'(M_{xy}) - r'(A) = r(M) - r(X') - 1$ and $n(A) = |A| - r'(A) = |X'| - r(X')$.

Therefore,

$$\begin{aligned} T(M_{xy}) = & \sum_{A \in \mathcal{A}_1} (u-1)^{z(A)}(v-1)^{n(A)} + \\ & (v-1) \sum_{X \in \mathcal{B}_1} (u-1)^{z(X)}(v-1)^{n(X)} + \\ & \frac{1}{u-1} \sum_{X' \in \mathcal{B}_2} (u-1)^{z(X')}(v-1)^{n(X')}. \end{aligned}$$

Thus we conclude that

$$\begin{aligned} T(M_{xy}) &= T(M \setminus \{x, y\}) + (v-1)T_1(M) + \frac{T_2(M)}{u-1} \\ &= vT_1(M) + \frac{u}{(u-1)} T_2(M). \end{aligned}$$

□

The following corollaries are immediate consequences of the Theorem 4.

Corollary 3. *Let M be a binary matroid and $x, y \in E(M)$ such that $x \nsim y$ in M. Then the Tutte polynomial of $M_{x,y}$ can be computed by the following formula*

$$T(M_{x,y}) = (u+v)T_1(M) + \frac{u^2}{(u-1)}T_2(M).$$

Corollary 4. *Let M be a binary matroid and $x, y \in E(M)$ such that $x \nsim y$ in M. Then the Tutte polynomial of $M'_{x,y}$ can be computed by the following formula*

$$T(M'_{x,y}) = (u+v)T_1(M) + \frac{u^2}{(u-1)}T_2(M) + T(M).$$

Example 1. Consider the Fano matroid F_7. Let $M = F_7$ and A be a matrix that represents M over $GF(2)$ that is given as,

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

Let $x = 6$ and $y = 7$. Clearly, x is not equivalent to y in M. Moreover, $T(M)$ is equal to

$$v^4 + u^3 + 3v^3 + 4u^2 + 7uv + 6v^2 + 3u + 3v.$$

To compute $T(M_{xy})$, by using Theorem 1, we first determine the collection of all cocircuits of M_{xy} which are containing the new elements α in terms of cocircuits of M. So by Theorem 1, there are four such cocircuits, $C_1^* = \{1, 2, 3, \alpha\}$, $C_2^* = \{1, 3, 4, \alpha\}$, $C_3^* = \{2, 3, 5, \alpha\}$ and $C_4^* = \{3, 4, 5, \alpha\}$. Now, we find those subsets of $E(M_{xy})$ such that α is a coloop in the restriction of M_{xy} to them. We need to check that if $T \subseteq E(M_{xy})$ with $\alpha \in T$, then there is a C_i^*, for $i \in \{1, 2, 3, 4\}$ such that $T \cap C_i^* = \alpha$. Therefore, the collection $\mathcal{B}_2$ in Theorem 4 is $\{\emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{1, 4\}, \{2, 5\}, \{4, 5\}\}$ and so $\mathcal{B}_1 = 2^{E(M \setminus \{x,y\})} - \mathcal{B}_2$ and hence the two polynomials $T_1(M)$ and $T_2(M)$ are as follows.

$$T_1(M) = u^2 + v^2 + 2uv + 2u + v + 1$$

and

$$T_2(M) = u^3 + u^2 - u - 1.$$

Therefore, by Theorem 4 and Corollaries 3 and 4, we have

$T(M_{xy}) = u^3 + v^3 + u^2v + 2uv^2 + 2u^2 + v^2 + 2uv + u + v.$
$T(M_{x,y}) = u^4 + 3u^3 + 3u^2 + 3u^2v + 3uv^2 + 3uv + v^3 + v^2 + u + v.$
$T(M'_{xy}) = u^4 + v^4 + 3u^2v + 3uv^2 + 4u^3 + 4v^3 + 10uv + 7u^2 + 7v^2 + 4u + 4v.$

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