New bounds on the energy of a graph

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Abstract: The energy of a graph G, denoted by \( E(G) \), is defined as the sum of the absolute values of all eigenvalues of G. In this paper, lower and upper bounds for energy in some of the graphs are established, in terms of graph invariants such as the number of vertices, the number of edges, and the number of closed walks.

Keywords: Eigenvalue of graph, Energy, Spectral radius, The number of closed walks

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1. Introduction

Let \( G = (V, E) \) be a simple undirected graph with vertex set \( V = V(G) = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E(G), \ | E(G) | = m \). The order and size of G are \( n = |V| \) and \( m = |E| \), respectively. A walk from a vertex \( u \) to a vertex \( v \) is a finite alternating sequence \( v_0 (= u)e_1v_1e_2\ldots e_kv_k(= v) \) of vertices and edges such that \( e_i = v_{i-1}v_i \) for \( i = 1, 2, \ldots, k \). The number \( k \) is the length of the walk. In particular, if the vertex \( v_i, i = 0, 1, \ldots, k \) in the walk are all distinct then the walk is called a path. A path of order \( n \) is denoted by \( P_n \). A closed path or cycle, is obtained from a path \( v_1, \ldots, v_k \) (where \( k \geq 3 \) ) by adding the edge \( v_1v_k \). A cycle of order \( n \) is denoted by \( C_n \). A graph is connected if each pair of vertices in a graph is joined by a walk. A bipartite graph is a graph such that its vertex set can be partitioned into two sets \( X \) and \( Y \) (called the partite sets) such that every edge meet both \( X \) and \( Y \). A complete
bipartite graph is a bipartite graph such that any vertex of a partite set is adjacent to all vertices of the other partite set. A complete bipartite graph with partite set of cardinalities \( p \) and \( q \) is denoted by \( K_{p,q} \). The graph \( K_{1,n-1} \) is also called a star of order \( n \), denoted by \( S_n \). A simple undirected graph in which every pair of distinct vertices is connected by a unique edge, is the complete graph and is denoted by \( K_n \).

For other graph theory notation and terminology we refer to [18].

The adjacency matrix \( A(G) \) of a graph \( G \) is defined by its entries as \( a_{ij} = 1 \) if \( v_i v_j \in E(G) \) and 0 otherwise. Let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n \) denote the eigenvalues of \( A(G) \). Then \( \lambda_1 \) is called the spectral radius of \( G \). When more than one graphs are under consideration, then we write \( \lambda_i(G) \) instead of \( \lambda_i \).

The energy of a graph \( G \) is defined as

\[
E(G) = \sum_{i=1}^{n} |\lambda_i|.
\]

This concept was introduced by I. Gutman and is intensively studied in chemistry, since it can be used to approximate the total \( \pi \)-electron energy of a molecule (see, e.g.[7], [10]). In 1971, McClelland [15] discovered the first upper bound for \( E(G) \) as follows:

\[
E(G) \leq \sqrt{2mn}.
\]  

(1)

Since then, numerous other bounds for \( E(G) \) were found (see, e.g. [2], [3], [12] - [14]). Here we just state some upper bounds for \( E(G) \) which were obtained recently. Koolen and Moulton [13] showed that if \( G \) is a graph with \( n \) vertices, \( m \) edges and \( 2m \geq n \) then

\[
E(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left(2m - \left(\frac{2m}{n}\right)^2\right)},
\]

(2)

with equality if and only if \( G \) is either \( \frac{n}{2} K_2, K_n \) or a non-complete connected strongly regular graph with two non-trivial eigenvalues both with absolute value \( \sqrt{\frac{(2m - \left(\frac{2m}{n}\right)^2)}{(n-1)}} \).

The same authors showed then that if \( 2m \geq n \) and \( G \) is a bipartite graph with \( n > 2 \), then

\[
E(G) \leq 2\left(\frac{2m}{n}\right) + \sqrt{(n-1)\left(2m - 2\left(\frac{2m}{n}\right)^2\right)},
\]

(3)

with equality if and only if \( G \) is either \( \frac{n}{2} K_2 \), a complete bipartite graph, or the incidence graph of a symmetric 2-(\( \nu, k, \lambda \))-design with \( k = \frac{2m}{n} \) and \( \lambda = \frac{k(k-1)}{\nu-1} \) \( (n = 2\nu) \). In [7, 13], from inequality (2), the authors get the following bound for \( E(G) \):

\[
E(G) \leq \frac{n}{2}\left(1 + \sqrt{n}\right).
\]

(4)

It is not difficult to see that inequality (4) is an improvement of the bound

\[
E(G) \leq n\sqrt{n-1}.
\]
The recent mathematical properties of this invariant can be found in e.g. [1, 8, 9, 11, 19].

The paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, we present bounds on the energy of graphs.

2. Preliminaries and known results

In this section, we list some known results that will be needed in the next sections. We first state some results on the number of closed walks of a graph.

**Proposition 1.** [16] Let $G$ be a graph with $n$ vertices, $m$ edges and $N_G(i)$ be the number of closed walks of length $i$ in $G$, then

1. $N_2 = N_2(G) = \sum_{i=1}^{n} \lambda_i^2 = 2m,$
2. $N_3 = N_3(G) = \sum_{i=1}^{n} \lambda_i^3 = 6N_G(K_3),$
3. $N_4 = N_4(G) = \sum_{i=1}^{n} \lambda_i^4 = 2m + 4N_G(P_3) + 8N_G(C_4),$
4. $N_5 = N_5(G) = \sum_{i=1}^{n} \lambda_i^5 = 30N_G(K_3) + 10N_G(C_5) + 10N_G(G_1),$
5. $N_7 = N_7(G) = \sum_{i=1}^{n} \lambda_i^7 = 126N_G(K_3) + 84N_G(G_1) + 14N_G(G_2) + 14N_G(G_3) + 14N_G(G_4) + 28N_G(G_5) + 42N_G(G_6) + 28N_G(G_7) + 112N_G(G_8) + 70N_G(C_5) + 14N_G(G_7).$

The graphs $G_i$ for $i = 1, 2, \ldots, 8$ are shown in Figure 1.

![Figure 1](image.png)

**Figure 1.** The graphs $G_i$ for $i = 1, 2, \ldots, 8$ mentioned in Proposition 1

The next lemma provides a lower bound for the energy of a graph.

**Proposition 2.** [4] If $G$ is a graph with $m$ edges, then $E(G) \geq 2\sqrt{m}$, with equality if and only if $G$ is a complete bipartite graph plus arbitrarily many isolated vertices.
Proposition 3. [5] A graph $G$ has only one eigenvalue if and only if $G$ is an empty graph. A graph $G$ has two distinct eigenvalues $\mu_1 > \mu_2$ with multiplicities $m_1$ and $m_2$ if and only if $G$ is the direct sum of $m_1$ complete graphs of order $\mu_1 + 1$. In this case, $\mu_2 = -1$ and $m_2 = m_1 \mu_1$.

We now give some known upper bounds for the spectral radius $\lambda_1(G)$.

Proposition 4. [17] Let $G$ be a connected graph with $n$ vertices and $m$ edges, then

$$\lambda_1(G) \leq n - 1.$$  

Proposition 5. [17] Let $G$ be a tree with $n$ vertices, then

$$\lambda_1(G) \leq \sqrt{n - 1}.$$  

3. Bounds for the Energy of Graphs

In this section, we obtain some new upper and lower bounds for the energy of graphs. We deal with general graphs, connected graphs, bipartite graphs, connected bipartite graphs, and trees.

We begin with the following upper bound in terms of order and size of a graph.

Theorem 1. Let $G$ be a non-empty graph with $n$ vertices and $m$ edges. Then

$$E(G) \leq \sqrt{2m^2 + \frac{n^2}{2}},$$

(5)

equality holds if and only if $G \cong \frac{n}{2}(K_2), (n = 2m)$.

Proof. Let $a_i, b_i, c_i$ and $d_i$ are sequences of real numbers and $p_i, q_i$ are non-negative for $i = 1, 2, \ldots, n$. Then the following inequality is valid (see [6] p. 7)

$$\sum_{i=1}^{n} p_i a_i^2 + \sum_{i=1}^{n} q_i b_i^2 + \sum_{i=1}^{n} p_i c_i^2 \sum_{i=1}^{n} q_i d_i^2 \geq 2 \sum_{i=1}^{n} p_i a_i c_i \sum_{i=1}^{n} q_i b_i d_i.$$  

(6)

For $a_i = b_i = p_i = q_i := 1$ and $c_i = d_i := |\lambda_i|, i = 1, 2, \ldots, n$, inequality (6) becomes

$$\sum_{i=1}^{n} 1 \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} |\lambda_i|^2 \geq 2 \sum_{i=1}^{n} |\lambda_i| \sum_{i=1}^{n} |\lambda_i|.$$

Notice that

$$\sum_{i=1}^{n} |\lambda_i|^2 = \sum_{i=1}^{n} \lambda_i^2 = 2m.$$
Proposition 3, inequality (5). If \( G \cong \frac{n}{2} K_2 \), then it is easy to check that the equality in (5) holds. Conversely, if the equality in (5) holds, since \( G \) is a non-empty graph, by Proposition 3, \( G \) has at least two distinct eigenvalues. We consider the following cases.

**Case 1.** The absolute value of all eigenvalues of \( G \) are equal. Then clearly \( \lambda_1 = |\lambda_i| \quad (2 \leq i \leq n) \), since \( G \) has at least two distinct eigenvalues. By Proposition 3, \( |\lambda_i| = 1(2 \leq i \leq n) \). Hence \( 2m = n \) and also, \( \lambda_1 = |\lambda_2| = \cdots = |\lambda_n| = 1 \). By applying Proposition 3 again, we obtain that \( m_2 = m_1 \lambda_1, \lambda_1 = 1 \), and therefore \( m_1 = m_2 \). Then we obtain that \( \lambda_1 = 1 \) has multiplicity \( \frac{n}{2} \), and \( \lambda_i = -1 \quad (2 \leq i \leq n) \) has multiplicity \( \frac{n}{2} \). Therefore \( G \) is the direct sum of \( m_1 = \frac{n}{2} \) complete graphs of order \( \lambda_1 + 1 = 2 \). Consequently, \( G \) is \( \frac{n}{2} K_2 \).

**Case 2.** The absolute value of all eigenvalues of \( G \) are not equal. Then \( G \) has two distinct eigenvalues with different absolute values. By Proposition 3, \( |\lambda_i| = 1 \quad (2 \leq i \leq n) \). Since, \( \sum_{i=1}^{n} \lambda_i = 0 \) and \( \lambda_2 = \lambda_3 = \cdots = \lambda_n = -1 \), we have, \( \lambda_1 = n-1 \). Hence \( \lambda_1 \) has multiplicity 1 and \( \lambda_i = -1 \) has multiplicity \( n-1 \). By Proposition 3, \( G \) is the direct sum of a complete graph of order \( \lambda_1 + 1 = n \). Consequently, \( G \) is \( K_n \). \( \square \)

**Theorem 2.** Let \( G \) be a non-empty graph with \( n \) vertices and \( m \) edges. Then

\[
\mathcal{E}(G) \leq m + \frac{n}{2},
\]

equality holds if and only if \( G \cong \frac{n}{2}(K_2), (n = 2m) \).

**Proof.** Let \( a_i, b_i, c_i \) and \( d_i \) are sequences of real numbers and \( p_i, q_i \) are non-negative for \( i = 1, 2, \ldots, n \). Then the following inequality is valid (see [6] p. 7)

\[
\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} q_i b_i^2 + \sum_{i=1}^{n} p_i c_i^2 \sum_{i=1}^{n} q_i d_i^2 \geq 2 \sum_{i=1}^{n} p_i a_i c_i \sum_{i=1}^{n} q_i b_i d_i. \quad (8)
\]

For \( a_i = b_i = d_i = p_i = q_i := 1 \) and \( c_i := |\lambda_i|, i = 1, 2, \ldots, n \), inequality (8) becomes

\[
\sum_{i=1}^{n} 1 \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} 1 \geq 2 \sum_{i=1}^{n} |\lambda_i| \sum_{i=1}^{n} 1.
\]

So

\[
n^2 + \left( \sum_{i=1}^{n} |\lambda_i|^2 \right) n \geq 2 \left( \sum_{i=1}^{n} |\lambda_i| \right) n.
\]
Then, from the above inequality directly follows the assertion of Theorem 2, i.e. the
inequality (7).
If $G \cong \frac{n}{2}K_2$, then it is easy to check that the equality in (7) holds. Conversely, if the
equality in (7) holds, since $G$ is a non-empty graph, by Proposition 3, $G$ has at least
two distinct eigenvalues. We consider the following cases.

**Case 1.** The absolute value of all eigenvalues of $G$ are equal.
Then clearly $\lambda_1 = |\lambda_i|$ (2 $\leq$ $i$ $\leq$ $n$), since $G$ has at least two distinct eigenvalues. By
Proposition 3, $|\lambda_i| = 1(2 \leq i \leq n)$. Hence $2m = n$ and also, $\lambda_1 = |\lambda_2| = \cdots = |\lambda_n| = 1$. By applying Proposition 3 again, we obtain that $m_2 = m_1\lambda_1, \lambda_1 = 1$, and therefore $m_1 = m_2$. Then we obtain that $\lambda_1 = 1$ has multiplicity $\frac{n}{2}$, and $\lambda_i = -1$ (2 $\leq$ $i$ $\leq$ $n$)
has multiplicity $\frac{n}{2}$. Therefore $G$ is the direct sum of $m_1 = \frac{n}{2}$ complete graphs of order
$\lambda_1 + 1 = 2$. Consequently, $G$ is $\frac{n}{2}K_2$.

**Case 2.** The absolute value of all eigenvalues of $G$ are not equal. Then $G$ has two
distinct eigenvalues with different absolute values. By Proposition 3, $|\lambda_i| = 1(2 \leq
i \leq n)$. Since, $\sum_{i=1}^{n}\lambda_i = 0$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_n = -1$, we have, $\lambda_1 = n - 1$. Hence
$\lambda_1$ has multiplicity 1 and $\lambda_i = -1$ has multiplicity $n - 1$. By Proposition 3, $G$ is the
direct sum of a complete graph of order $\lambda_1 + 1 = n$. Consequently, $G$ is $K_n$. □

The next upper bound involves the number of closed walks of lengths 4 and 6 of a
graph.

**Theorem 3.** Let $G$ be a non-empty graph with $n$ vertices and $m$ edges. Then
\[
\mathcal{E}(G) \leq \frac{nN_5 + 4m^2}{2N_4},
\]

equality holds if and only if $G \cong \frac{n}{2}(K_2), (n = 2m)$.

**Proof.** Let $a_i, b_i$ are sequences of real numbers and $c_i, d_i$ are non-negative for $i =
1, 2, \ldots, n$. Then the following inequality is valid (see [6] p.8)
\[
\sum_{i=1}^{n} d_i \sum_{i=1}^{n} c_i a_i^2 + \sum_{i=1}^{n} c_i \sum_{i=1}^{n} d_i b_i^2 \geq 2 \sum_{i=1}^{n} c_i a_i \sum_{i=1}^{n} d_i b_i.
\]

For $a_i = c_i := |\lambda_i|^2$, $b_i := |\lambda_i|$ and $d_i := 1, i = 1, 2, \ldots, n$, inequality (10) becomes
\[
\sum_{i=1}^{n} \sum_{i=1}^{n} |\lambda_i|^5 + \sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} |\lambda_i|^2 \geq 2 \sum_{i=1}^{n} |\lambda_i|^4 \sum_{i=1}^{n} |\lambda_i|.
\]

Hence,
\[
n \sum_{i=1}^{n} |\lambda_i|^5 + \left(\sum_{i=1}^{n} |\lambda_i|^2\right)^2 \geq 2 \sum_{i=1}^{n} |\lambda_i|^4 \sum_{i=1}^{n} |\lambda_i|.
\]
Then, considering the Proposition 1, we demonstrate the assertion of Theorem 3, i.e. the inequality (9).

If \( G \cong \frac{n}{2}K_2 \), then it is easy to check that the equality in (9) holds. Conversely, if the equality in (9) holds, since \( G \) is a non-empty graph, by Proposition 3, \( G \) has at least two distinct eigenvalues. We consider the following cases.

**Case 1.** The absolute value of all eigenvalues of \( G \) are equal.

Then clearly \( \lambda_1 = |\lambda_i| \; (2 \leq i \leq n) \), since \( G \) has at least two distinct eigenvalues. By Proposition 3, \( |\lambda_i| = 1(2 \leq i \leq n) \). Hence \( 2m = n \) and also, \( \lambda_1 = |\lambda_2| = \cdots = |\lambda_n| = 1 \). By applying Proposition 3 again, we obtain that \( m_2 = m_1 \lambda_1, \lambda_1 = 1 \), and therefore \( m_1 = m_2 \). Then we obtain that \( \lambda_1 = 1 \) has multiplicity \( \frac{n}{2} \), and \( \lambda_i = -1 \; (2 \leq i \leq n) \) has multiplicity \( \frac{n}{2} \). Therefore \( G \) is the direct sum of \( m_1 = \frac{n}{2} \) complete graphs of order \( \lambda_1 + 1 = 2 \). Consequently, \( G \) is \( \frac{n}{2}K_2 \).

**Case 2.** The absolute value of all eigenvalues of \( G \) are not equal. Then \( G \) has two distinct eigenvalues with different absolute values. By Proposition 3, \( |\lambda_i| = 1 \; (2 \leq i \leq n) \). Since, \( \sum_{i=1}^{n} \lambda_i = 0 \) and \( \lambda_2 = \lambda_3 = \cdots = \lambda_n = -1 \), we have, \( \lambda_1 = n - 1 \). Hence \( \lambda_1 \) has multiplicity 1 and \( \lambda_i = -1 \) has multiplicity \( n - 1 \). By Proposition 3, \( G \) is the direct sum of a complete graph of order \( \lambda_1 + 1 = n \). Consequently, \( G \) is \( K_n \).

**Theorem 4.** Let \( G \) be a non-empty graph with \( n \) vertices and \( m \) edges. Then

\[
\mathcal{E}(G) \leq \frac{2mn_4 + n^2}{4m}, \tag{11}
\]

equality holds if and only if \( G \cong \frac{n}{2}(K_2), (n = 2m) \).

**Proof.** Let \( a_i, b_i, c_i \) and \( d_i \) are sequences of real numbers and \( p_i, q_i \) are non-negative for \( i = 1, 2, \ldots, n \). Then the following inequality is valid (see [6] p.7)

\[
\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} q_i b_i^2 + \sum_{i=1}^{n} p_i c_i^2 \sum_{i=1}^{n} q_i d_i^2 \geq 2 \sum_{i=1}^{n} p_i a_i c_i \sum_{i=1}^{n} q_i b_i d_i. \tag{12}
\]

For \( a_i := |\lambda_i|^2, b_i := |\lambda_i| \) and \( c_i = d_i = p_i = q_i := 1, \; i = 1, 2, \ldots, n \), inequality (12) becomes

\[
\sum_{i=1}^{n} |\lambda_i|^4 \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i=1}^{n} \sum_{i=1}^{n} 1 \geq 2 \sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} |\lambda_i|.
\]

So,

\[
\sum_{i=1}^{n} |\lambda_i|^4 \sum_{i=1}^{n} |\lambda_i|^2 + n^2 \geq 2 \sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} |\lambda_i|.
\]

Considering the Proposition 1, we demonstrate the assertion of Theorem 4, i.e. the inequality (11).

If \( G \cong \frac{n}{2}K_2 \), then it is easy to check that the equality in (11) holds. Conversely, if
the equality in (11) holds, since G is a non-empty graph, by Proposition 3, G has at least two distinct eigenvalues. We consider the following cases.

Case 1. The absolute value of all eigenvalues of G are equal.
Then clearly $\lambda_1 = |\lambda_i|$ $(2 \leq i \leq n)$, since G has at least two distinct eigenvalues. By Proposition 3, $|\lambda_i| = 1(2 \leq i \leq n)$. Hence $2m = n$ and also, $\lambda_1 = |\lambda_2| = \cdots = |\lambda_n| = 1$. By applying Proposition 3 again, we obtain that $m_2 = m_1 \lambda_1$, $\lambda_1 = 1$, and therefore $m_1 = m_2$. Then we obtain that $\lambda_1 = 1$ has multiplicity $\frac{n}{2}$, and $\lambda_i = -1$ $(2 \leq i \leq n)$ has multiplicity $\frac{n}{2}$. Therefore G is the direct sum of $m_1 = \frac{n}{2}$ complete graphs of order $\lambda_1 + 1 = 2$. Consequently, G is $\frac{n}{2}K_2$.

Case 2. The absolute value of all eigenvalues of G are not equal. Then G has two distinct eigenvalues with different absolute values. By Proposition 3, $|\lambda_i| = 1(2 \leq i \leq n)$. Since, $\sum_{i=1}^{n} \lambda_i = 0$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_n = -1$, we have, $\lambda_1 = n - 1$. Hence $\lambda_1$ has multiplicity 1 and $\lambda_i = -1$ has multiplicity $n - 1$. By Proposition 3, G is the direct sum of a complete graph of order $\lambda_1 + 1 = n$. Consequently, G is $K_n$.

The next bounds involve lower bounds for the energy of a bipartite graph.

Theorem 5. Let G be a non-empty bipartite graph of order at least 2, with m edges and having the spectral radius $\lambda_1$. Then

$$E(G) \geq \frac{2m}{\lambda_1}$$

(13)

equality holds if and only if one of the following statements holds:
(1) $G \cong \frac{n}{2}K_2$, $(n = 2m)$;
(2) $K_{1,r-1} \cup (n - 1 - r_{n-1})K_1$.

Proof. Let $a_i, b_i$ are decreasing non-negative sequences with $a_i, b_i \neq 0$ and $w_i$ a non-negative sequence, for $i = 1, 2, \ldots, n$. Then the following inequality is valid (see [6] p.85)

$$\sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2 \leq \max \left\{ b_1 \sum_{i=1}^{n} w_i a_i, a_1 \sum_{i=1}^{n} w_i b_i \right\} \sum_{i=1}^{n} w_i a_i b_i.$$  

(14)

For $a_i = b_i := |\lambda_i|$, and $w_i := 1$, $i = 1, 2, \ldots, n$, inequality (14) becomes

$$\sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} |\lambda_i|^2 \leq \max \left\{ \lambda_1 \sum_{i=1}^{n} |\lambda_i|, \lambda_1 \sum_{i=1}^{n} |\lambda_i| \right\} \sum_{i=1}^{n} |\lambda_i|^2.$$  

Then

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \lambda_1 \sum_{i=1}^{n} |\lambda_i|$$  

from the above inequality directly follows the assertion of Theorem 5, i.e. the inequality (13).
If \( G \cong \frac{n}{2}K_2 \), then it is easy to check that the equality in (13) holds. Conversely, if the equality in (13) holds, since \( G \) is a non-empty graph, by Proposition 3, \( G \) has at least two distinct eigenvalues. We consider the following cases.

**Case 1.** The absolute value of all eigenvalues of \( G \) are equal. Then clearly \( \lambda_1 = |\lambda_i| \quad (2 \leq i \leq n) \), since \( G \) has at least two distinct eigenvalues. By Proposition 3, \( |\lambda_i| = 1 \) (2 \( \leq i \leq n \)). Hence \( 2m = n \) and also, \( \lambda_1 = |\lambda_2| = \cdots = |\lambda_n| = 1 \). By applying Proposition 3 again, we obtain that \( m_2 = m_1\lambda_1, \lambda_1 = 1 \), and therefore \( m_1 = m_2 \). Then we obtain that \( \lambda_1 = 1 \) has multiplicity \( \frac{n}{2} \), and \( \lambda_i = -1 \quad (2 \leq i \leq n) \) has multiplicity \( \frac{n}{2} \). Therefore \( G \) is the direct sum of \( m_1 = \frac{n}{2} \) complete graphs of order \( \lambda_1 + 1 = 2 \). Consequently, \( G \) is \( \frac{n}{2}K_2 \).

**Case 2.** The absolute value of all eigenvalues of \( G \) are not equal. If two eigenvalues of \( G \) have different absolute values, then by Proposition 3, \( |\lambda_i| = -1 \) (2 \( \leq i \leq n \)). Noting that \( G \) is a bipartite graph, we have \( \lambda_1 = -\lambda_n \), that is a contradiction, since the two eigenvalues of \( G \) have different absolute values. Thus assume that \( G \) has three distinct eigenvalues. Since \( G \) is a bipartite graph, we have that \( \lambda_1 = -\lambda_n \neq 0 \) and \( \sum_{i=1}^{n} \lambda_i = 0 \), and therefore, \( \lambda_i = 0 \) (2 \( \leq i \leq n-1 \)). Thus \( \mathcal{E}(G) = 2\lambda_1 \), and by Proposition 2, we have that \( 2\lambda_1 \geq 2\sqrt{m} \), and so \( 2\lambda_1^2 \geq 2m \). Notice that \( 2m = \sum_{i=1}^{n} \lambda_i^2 = 2\lambda_1^2 \). Therefore \( \lambda_1 = \sqrt{m} \) and \( \mathcal{E}(G) = 2\sqrt{m} \). Hence by Proposition 2, \( G \) is a complete bipartite graph plus arbitrarily many isolated vertices. Thus, there exist integers \( r_1 \geq 1 \) and \( r_2 \geq 2 \) such that \( G \) is \( K_{r_1, r_2} \cup (n - r_1 - r_2)K_1 \).

Finally, considering the Theorem 5, we present two new lower bounds for connected bipartite graphs and tree graph.

**Corollary 1.** Let \( G \) be a connected bipartite graph with \( n \) vertices and \( m \) edges, then

\[ \mathcal{E}(G) \geq \frac{2m}{n-1}. \]

**Corollary 2.** Let \( G \) be a tree with \( n \) vertices and \( m \) edges, then

\[ \mathcal{E}(G) \geq \frac{2m}{\sqrt{n-1}}. \]

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