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Research Article

A note on $\delta^{(k)}$-colouring of the Cartesian product of some graphs

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Abstract: The chromatic number, $\chi(G)$ of a graph G is the minimum number of colours used in a proper colouring of G. In an improper colouring, an edge uv is bad if the colours assigned to the end vertices of the edge is the same. Now, if the available colours are less than that of the chromatic number of graph G, then colouring the graph with the available colours lead to bad edges in G. The number of bad edges resulting from a $\delta^{(k)}$-colouring of G is denoted by $b_k(G)$. In this paper, we use the concept of $\delta^{(k)}$-colouring and determine the number of bad edges in Cartesian product of some graphs.

Keywords: Improper colouring, near proper colouring, $\delta^{(k)}$-colouring, bad edge

AMS Subject classification: 05C15

1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [1, 4, 11] and for graph products, we refer to [3, 5]. Further, for the terminology of graph colouring, see [2, 6, 9]. Unless mentioned otherwise, all graphs considered here are undirected, simple, finite and connected. In a proper vertex colouring, the vertices are coloured in such a way that no two adjacent vertices receive the same colour. A *colour class* of a graph G is a set of independent vertices assigned a same colour. In an improper colouring, an edge uv is a *bad edge* if $c(u) = c(v)$.
A colouring that restricts the number of colour classes that can have adjacency between their own elements to minimise the number of bad edges in a graph is called as

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near proper colouring[8]. From the definition of near proper colouring, the authors in [8], brought in a new concept of $\delta^{(k)}$*-colouring* which is a near proper colouring with k given colours where $1 \leq k \leq \chi(G) - 1$, that minimises the number of bad edges by restricting a colour class to have adjacency among its own elements. The number of bad edges resulting from a $\delta^{(k)}$-colouring of graph G is denoted by $b_k(G)$.
Unless mentioned otherwise, the colour class C_1 will be restricted to have adjacency between the vertices in it throughout the discussion.
The articles [7, 8, 10] give clear explanation of few engrossing studies related to this concept. Motivated by the studies mentioned above, we discuss the $\delta^{(k)}$-colouring of Cartesian products of some fundamental graph classes.

2. $\delta^{(k)}$-Colouring of Cartesian Products of Graphs

First recall the definition of the Cartesian product of two graphs as given in [5].

Definition 1. [3, 5] The *Cartesian product* of two graphs G and H is a graph, denoted as $G \square H$, whose vertex set is $V(G) \times V(H) = \{v_{ij} = (u_i, w_j) : u_i \in V(G), w_j \in V(H)$. Two vertices v_{ij} and v_{kl} are adjacent in $G \square H$ if either

(i) $u_i = u_k$ and w_j is adjacent to w_l in H; or

(ii) $w_j = w_l$ and v_i is adjacent to v_k in G.

In this discussion, we investigate the $\delta^{(k)}$-colouring of the Cartesian products of different combinations of paths, cycles and complete graphs. The Cartesian product $P_m \square P_n$, known as the grid graph, is a bipartite graph, as the Cartesian product of two bipartite graphs is bipartite (see [5]). Hence, its chromatic number is 2 and the possible number of colours for a $\delta^{(k)}$-colouring is $k = 1$. However, colouring the graph with a single colour will lead all the edges to be bad. Hence, we examine the other possible combinations.
A *prism graph*, denoted by $Y_{m,n}$, is the graph Cartesian product $C_n \square P_m$. The graph $Y_{m,n}$ has mn vertices and $m(2n-1)$ edges. Now, we discuss the $\delta^{(k)}$-colouring of prism graph.
In the Cartesian product of C_n and P_m, the number of bad edges in $C_n \square P_m$ is m times the number of bad edges in C_n (since there are m copies of C_n say $C_{n,1}, C_{n,2}, \ldots C_{n,m}$ each of order n, in $C_n \square P_m$) in addition to the number of bad edges between the m C_n's (if there is any). Since $\chi(C_n) = 2$ and $\chi(C_n) = 3$ when n is even and odd respectively, we exclude the case of C_n when n is even (as colouring a graph with $k = 1$ colour will result in all its edges to be bad) and determine the $\delta^{(k)}$-colouring of $C_n \square P_m$ when n is odd in the following theorem.

Theorem 1. *For $n \equiv 1 \pmod 2$ and $m \geq 2$, the minimum number of bad edges in $Y_{m,n} = C_n \square P_m$ resulting from $\delta^{(k)}$-colouring is given by, $b_2(Y_{m,n}) = 2m - 1$.*

Proof. Let c_1 and c_2 be the two available colours and C_1 and C_2 be their respective colour classes. For $1 \leq i \leq n$, $v_{i,j}$ corresponds to jth vertex of ith copy of C_n. We colour $C_{n,1}$ in following manner. Let the vertex $v_{1,1}$ receive the colour c_1, $v_{1,2}$ the colour c_2, $v_{1,3}$ the colour c_1. Continuing like this, the vertex $v_{1,n}$ is assigned the colour c_1 to maintain the perquisites of $\delta^{(k)}$-colouring and this lead to one bad edge $v_{1,1}v_{1,n}$ in $C_{n,1}$. Note that, the number of bad edges in C_n, where n is odd, is always 1 (see [8]). Now, in a $\delta^{(k)}$-colouring, we try to minimise the number of bad edges with the available k colours. Hence, we colour $C_{n,2}$ in a way to minimise the bad edges between $C_{n,1}$ and $C_{n,2}$ and this is possible only if the end vertices of the bad edge in $C_{n,1}$ is not adjacent to an edge in $C_{n,2}$, whose end vertices also receive the same colour as that of the bad edge in $C_{n,1}$. Thus, since the end vertices of the bad edge $v_{1,1}v_{1,n}$ of $C_{n,1}$ is adjacent to the end vertices of $v_{2,1}v_{2,n}$ of $C_{n,2}$, colour the vertex $v_{2,1}$ of $C_{n,2}$ with c_2, the vertex $v_{2,2}$ with c_1 and continuing this, the vertices $v_{2,n-1}$ and $v_{2,n}$ are assigned the colour c_1 which leads to one bad edge in $C_{n,2}$, to maintain the requirements of $\delta^{(k)}$-colouring. Now, only the vertex $v_{1,1}$ coloured with c_1 is adjacent to $v_{2,n-1}$ coloured with c_1 leading to one bad edge between $C_{n,1}$ and $C_{n,2}$. Now, since no vertex of $C_{n,1}$ is adjacent to the vertices of $C_{n,3}$, follow the colouring pattern of $C_{n,1}$ to $C_{n,3}$, which again leads to one bad edge in $C_{n,3}$ and one in between $C_{n,2}$ and $C_{n,3}$. Again, the colouring pattern of $C_{n,2}$ can be followed to $C_{n,4}$, leading to one bad edge in $C_{n,4}$ and one in between $C_{n,3}$ and $C_{n,4}$. Thus, continuing this pattern for all the m copies of C_n, we see that there is only one bad edge in each copy of C_n and one in between two copies of C_n. Since there are m copies of C_n in $C_n \square P_m$, the total minimum number of bad edges resulting from $\delta^{(k)}$-colouring in $Y_{m,n}$ is $m + m - 1 = 2m - 1$. □

The Cartesian product of two cycles are called *torus grids*. The following theorem discusses the $\delta^{(k)}$-colouring and the minimum number of bad edges in torus grids resulting from the same.

Theorem 2. *For $C_n \square C_m$, where $n \geq m$, the minimum number of bad edges resulting from $\delta^{(k)}$-colouring is given by,*

$$b_k(C_n \square C_m) = \begin{cases} 2n, & \text{if } n \text{ is even and } m \text{ is odd,} \\ 2m, & \text{Otherwise.} \end{cases}$$

Proof. Consider a $\delta^{(k)}$-colouring for $C_n \square C_m$ with two colours c_1 and c_2 and their respective colour classes C_1 and C_2. For $C_n \square C_m$, we take m copies of C_n and colour each copy of C_n in such a way to reduce the number of bad edges in and between each copy of C_n. Let $C_{n,1}, C_{n,2}, \ldots, C_{n,m}$ be m copies of C_n and for $1 \leq i \leq n$, $v_{i,j}$ corresponds to jth vertex of ith copy of C_n. In $C_n \square C_m$, only the vertices of the consecutive copies, and the vertices of $C_{n,1}$ and $C_{n,m}$ are adjacent to each other respectively. Now, following are the three cases addressed depending on the parities of n and m in $C_n \square C_m$.

Case 1. Let n be even and m be odd.
Since n is even, the m copies of the C_n can be properly coloured using the two colours c_1 and c_2. Now, for colouring an even cycle there are $\frac{n}{2}$ possibilities of colour c_1 and c_2 respectively. The first copy, $C_{n,1}$, is coloured in the following manner. Let the vertex $v_{1,1}$ be given the colour c_1, $v_{1,2}$ the colour c_2 and so on the $v_{1,n-1}$ the colour c_1 and the last vertex $v_{1,n}$ the colour c_2. The second copy $C_{n,2}$ is coloured as follows. The first vertex $v_{2,1}$ is assigned the colour c_2, the vertex $v_{2,2}$ the colour c_1 and the last vertex $v_{2,n}$ the colour c_1. Again, $C_{n,3}$ and $C_{n,4}$ receive the colouring pattern of $C_{n,1}$ and $C_{n,2}$ respectively and so on. Now, since m is odd, $C_{n,m}$ will receive the same colouring pattern as that of $C_{n,1}$ which thereby leads to the violation of the requirements of $\delta^{(k)}$- colouring. So, to maintain the definition, we colour the last copy in such a way that the vertices that are assigned the colour c_2 in $C_{n,1}$ and $C_{n,m-1}$, adjacent to the last copy $C_{n,m}$, will be assigned the colour c_1 . It can be observed that among the n vertices of $C_{n,1}$ and $C_{n,m-1}$, $\frac{n}{2}$ vertices of each copy is assigned the colour c_2. This thereby leads in colouring all the n vertices of $C_{n,m}$ with c_1 and this leads to all the n edges to be bad. Now, the number of bad edges between $C_{n,m}$ and $C_{n,1}$ and $C_{n,m}$ and $C_{n,m-1}$ will be $\theta(c_1) = \frac{n}{2}$ respectively. Thus the total number of bad edges resulting from $\delta^{(k)}$-colouring in $C_n \Box C_m$ when n is even and m is odd is $2n$.

Case 2. Let n be odd and m be even.
The colouring pattern for this case follows the same as that of the Theorem 1. Now, since the consecutive copies have edges between them, the number of bad edges between the m copies is $m-1$ and each odd C_n will have one bad edge in it. Now, either of the end vertices of the bad edge in $C_{n,1}$ is adjacent to a vertex assigned the colour c_1 in $C_{n,m}$. This lead to a bad edge between $C_{n,1}$ and $C_{n,m}$. Thus, the minimum number of bad edges resulting from $\delta^{(k)}$-colouring in $C_n \Box C_m$ when n is odd and m is even is $m + m - 1 + 1 = 2m$.
Note that, the Cartesian product of any two graphs is commutative and hence the number of bad edges for both the cases is the same. In the above two cases the parity of the two integers have been separately dealt with to know the $\delta^{(k)}$-colouring pattern in both the cases.

Case 3. Let both n and m be odd.
The minimum number of bad edges in an odd cycle is 1 (see [8]) and hence between any two consecutive copies, the minimum number of bad edges is one, also between the first and the last copy the minimum bad edge obtained from a $\delta^{(k)}$-colouring is again one. Hence, below given is a $\delta^{(k)}$-colouring that attains the minimality of the number of bad edges in $C_n \Box C_m$, when both n and m are odd. Assign the vertex $v_{1,1}$ the colour c_1, $v_{1,2}$ the colour c_2, $v_{1,3}$ the colour c_1 as so on the vertex $v_{1,n}$ the colour c_1 to maintain the perquisites of $\delta^{(k)}$-colouring. This lead to one bad edge $v_{1,1}v_{1,n}$ in $C_{n,1}$. Since, $v_{1,1}v_{1,n}$ is the bad edge in the $C_{n,1}$, we colour the second copy, $C_{n,2}$ in such a way that a vertex which is adjacent to either of the end vertices of the bad edge in $C_{n,1}$ is assigned colour c_2. We start with the vertex $v_{2,1}$ and assign the colour c_2 to it. The vertex $v_{2,2}$ is assigned the colour c_1 and so on the vertex $v_{2,n-1}$

the colour c_1 and $v_{2,n}$ the colour c_1, to maintain the requirements of $\delta^{(k)}$-colouring and this lead to a bad edge $(v_{2,n-1}v_{2,n})$ in $C_{n,2}$. Now, for the third copy, $C_{n,3}$, start colouring with the vertex $v_{3,n}$ which is adjacent to the end vertex of the bad edge of the previous copy, $C_{n,2}$. Colour the vertex $v_{3,n}$ with c_2, the vertex $v_{3,1}$ with c_1 and so on the vertex $v_{3,n-2}$ and $v_{3,n-1}$ will receive the colour c_1, leading to one bad edge. Thus, every copy is coloured according to its adjacency between the end vertices that lead to bad edge in the previous copy. Thus, we follow this pattern and colour the $C_{n,m}$-th copy, starting from the vertex that is adjacent to the end vertex of the previously coloured copy in clockwise direction. Now, say if $v_{m-1,n}v_{m-1,1}$ is the bad edge in $C_{n,m-1}$-th copy, we start colouring the $C_{n,m}$-th copy from the vertex $v_{m,1}$ by assigning it the colour c_2 and if $v_{m-1,j}v_{m-1,j+1}$ for any $j = 2, 3, \dots, n-1$ is the bad edge in $m-1$-th copy, then we start colouring from the vertex $v_{m,j+1}$ of the copy $C_{n,m}$ by assigning it the colour c_2, then the vertex $v_{m,j+2}$ is assigned the colour c_1 and so on. Thus, this colouring gives one bad edge in each copy of the cycle. Also, between each copy the minimum number of bad edge (one) is also maintained. Since there are m copies of C_n, the total number of bad edges between the consecutive copies and in each copy is $m-1$ and m respectively. Also, this $\delta^{(k)}$-colouring maintains the number of bad edge between the first and the last copy which is again one. Hence, the minimum total number of bad edges resulting from the mentioned $\delta^{(k)}$-colouring is $m + m - 1 + 1 = 2m$.
Note that this $\delta^{(k)}$-colouring is possible only in this particular case as both the n and m are odd.

□

The $\delta^{(k)}$-colouring and the resultant number of bad edges of the Cartesian product $K_n \square P_m$, $K_n \square C_m$ and $K_n \square K_m$ are determined as explained in the following theorems:

Theorem 3. *For the integers n and m where $n \geq m$, the number of bad edges resulting from $\delta^{(k)}$-colouring is given by,*

$$b_k(K_n \square P_m) = \frac{mt(t-1)}{2} + \eta,$$

where η is the total number of bad edges between the m copies of K_n given by

$$\eta = \begin{cases} (m-1)(t-k+1), & \text{if } k \leq \frac{n+1}{2}, \\ 0, & \text{if } k > \frac{n+1}{2}, \end{cases}$$

and $t = n - k + 1$.

Proof. The chromatic number of the Cartesian product $G \square H$ of any two graphs G and H is $\max\{\chi(G), \chi(H)\}$. Since, in this case $G = K_n$, there can be $2 \leq k \leq n-1$ colours available for $\delta^{(k)}$-colouring. Let $c_1, c_2, \dots, c_k$ be the k available colours and $C_1, C_2, \dots, C_k$ be their respective colour classes. There are a total of m copies of K_n in

$K_n \square K_m$. Now, the minimum number of bad edges in a complete graph K_n is already been determined in [8] and is given by $b_k(K_n) = \frac{x(x+1)}{2}$, where $x = 1, 2, 3, \ldots, n-2$ and $k = n - x$. In short, substituting the value of x as $n - k = t - 1$ in the current theorem, we have $b_k(K_n) = \frac{t(t-1)}{2}$. This is because, among the k colours, excluding the colour c_1 (since colour class C_1 is the relaxed colour class), the $k-1$ colours can be used to colour the $k-1$ vertices properly. The remaining $n-k+1 = t$, are given the colour c_1 to maintain the perquisites of $\delta^{(k)}$-colouring and this t vertices induces a complete graph of order t. Let $K_{n,1}, K_{n,2}, \ldots, K_{n,m}$ be the m copies of K_n and for $1 \leq i \leq n$, $v_{i,j}$ corresponds to jth vertex of ith copy of K_n. As explained above each copy of K_n will have $\frac{t(t-1)}{2}$ bad edges. Now to reduce the number of bad edges between the copies, we colour each copy of K_n in the following manner. The first t vertices namely $v_{1,1}, v_{1,2}, \ldots, v_{1,t}$ of the $K_{n,1}$ are assigned the colour c_1 and the remaining $n-t$ vertices viz. $v_{1,t+1}, v_{1,t+2}, \ldots, v_{1,n}$ vertices are properly assigned the colour $c_2, c_3, \ldots, c_k$ respectively. While colouring the second copy, first start colouring from the $(t+1)$th vertex of $K_{n,2}$ corresponding to the tth vertex coloured with c_1 in the $K_{n,1}$ (so as to reduce the adjacency between the end vertices of bad edges in $K_{n,1}$ with the vertices assigned the colour c_1 in $K_{n,2}$), by following the same colouring pattern as that of the first copy $K_{n,1}$. Continue this colouring pattern to the rest of the copies. Since the structure of a complete graph is symmetric, there would always be $|E(K_t)|$ number of bad edges in each copy of K_n. Therefore, there are $\frac{mt(t-1)}{2}$ bad edges in the m copies of K_n. Now, to find the number of bad edges between the m copies of K_n's, two different cases are addressed below:

Case 1. If $k \leq \frac{n+1}{2}$, then among the $n-k+1 = t$ vertices of each copies of K_n that are assigned the colour c_1, the $k-1$ vertices can be assigned any colour other than c_1, maintaining the requirements of $\delta^{(k)}$-colouring. Thus, between any two copies of K_n there are $t - (k-1) = t - k + 1$ number of bad edges. Thus between m copies of K_n there are $(m-1)(t-k+1)$ bad edges resulting from $\delta^{(k)}$-colouring.

Case 2. If $k > \frac{n+1}{2}$, then all the t vertices in each copy of K_n that are assigned the colour c_1 can be coloured properly with $k-1$ colours other than c_1, leading to no bad edges between any two copies of K_n. Thus, there are no number of bad edges, resulting from $\delta^{(k)}$-colouring, between the K_n's when $k > \frac{n+1}{2}$. □

Theorem 4. *For any two integers n and m, the number of bad edges is given by,*

$$b_k(K_n \square C_m) = \begin{cases} \frac{mt(t-1)}{2} + m(t-k+1), & \text{if } k \leq \lceil \frac{n}{2} \rceil, \\ \frac{mt(t-1)}{2}, & \text{if } k > \lceil \frac{n}{2} \rceil. \end{cases}$$

Proof. In this case, all the m copies of K_n are coloured as per the colouring pattern followed in the Theorem 3. Thus, each copy of K_n induces a clique of order t. Hence, there are $\frac{mt(t-1)}{2}$ bad edges in the m copies of K_n. Similarly, as discussed in the cases of Theorem 3, either there are $(m-1)(t-k+1)$ number of bad edges in each m copies when $k \leq \lceil \frac{n}{2} \rceil$ or no bad edges between the m copies when $k > \lceil \frac{n}{2} \rceil$. Now, in this case of $K_n \square C_n$, the first copy is also adjacent to the last copy and there are a minimum of $t-k+1$ number of bad edges between these copies. Thus, the total number of bad edges between the m copies of K_n is, $m(n-2k+2)$ or $m(t-k+1)$. Hence, for any two integers n and m the total number of bad edges in $K_n \square C_m$ resulting from $\delta^{(k)}$-colouring is $\frac{mt(t-1)}{2}+m(t-k+1)$ when $k \leq \lceil \frac{n}{2} \rceil$ and $\frac{mt(t-1)}{2}$ when $k > \lceil \frac{n}{2} \rceil$. □

Theorem 5. *For $K_n \square K_m$, where $n \geq m$, the number of bad edges is given by,*

$$b_k(K_n \square K_m) = \frac{mt(t-1)}{2} + \eta(G),$$

where $\eta(G)$ is the total number of bad edges between the m copies of K_n and $\eta(G) = \frac{1}{2}(tm \pmod n)\lfloor \frac{tm}{n} \rfloor \lfloor \frac{tm-n}{n} \rfloor + \frac{1}{2}(n - tm \pmod n)\lfloor \frac{tm-n}{n} \rfloor \lfloor \frac{tm-2n}{n} \rfloor$ and $t = n-k+1$.

Proof. The m copies of K_n are coloured as per the colouring pattern followed in the Theorem 3. Thus there are $\frac{mt(t-1)}{2}$ bad edges in the m copies of K_n. Now, after colouring the m copies of K_n, it can be observed that there are $tm \pmod n$ number of $K_{\lfloor \frac{tm}{n} \rfloor}$ and $n - tm \pmod n$ number of $K_{\lfloor \frac{tm-n}{n} \rfloor}$. Thus the number of bad edges between the m copies of K_n is $\frac{1}{2}(tm \pmod n)\lfloor \frac{tm}{n} \rfloor \lfloor \frac{tm-n}{n} \rfloor + \frac{1}{2}(n - tm \pmod n)\lfloor \frac{tm-n}{n} \rfloor \lfloor \frac{tm-2n}{n} \rfloor$. Hence, the total number of bad edges in $K_n \square K_m$ resulting from $\delta^{(k)}$-colouring is, $\frac{mt(t-1)}{2} + \eta(G)$, where $\eta(G) = \frac{1}{2}(tm \pmod n)\lfloor \frac{tm}{n} \rfloor \lfloor \frac{tm-n}{n} \rfloor + \frac{1}{2}(n - tm \pmod n)\lfloor \frac{tm-n}{n} \rfloor \lfloor \frac{tm-2n}{n} \rfloor$. □

3. Conclusion

In this paper, we have determined the number of bad edges of the Cartesian product of some graphs by discussing all the possible cases for each of the product. Moreover, the $\delta^{(k)}$-colouring of Cartesian product of few other graphs and other graph products viz. direct products, Lexicographical product, rooted product etc., of any two graphs can also be studied. Finding the minimum number of bad edges by adding few more conditions to the concept of $\delta^{(k)}$-colouring can be a ground for further research work.

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