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Review Article

A survey on the intersection graphs of ideals of rings

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Abstract: Let $\mathscr{L}(R)$ denote the set of all non-trivial left ideals of a ring R. The intersection graph of ideals of a ring R is an undirected simple graph denoted by $G(R)$ whose vertices are in a one-to-one correspondence with $\mathscr{L}(R)$ and two distinct vertices are joined by an edge if and only if the corresponding left ideals of R have a non-zero intersection. The ideal structure of a ring reflects many ring theoretical properties. There is so much research that has been conducted during the last decade to explore the properties of $G(R)$. This is a survey of the developments in the study on the intersection graphs of ideals of rings since its introduction in 2009.

Keywords: Ring, ideal of a ring, intersection graph, genus, chromatic number

AMS Subject classification: 05C17,05C22,05C25,05C38,05C40, 05C45, 05C69

1. Introduction

In the literature, we find many ways to associate a graph with an algebraic structure. Associating a graph with an algebraic structure allows us to obtain characterizations and representations of special classes of algebraic structures in terms of graphs and vice versa.
One of the classical topics in the theory of graphs is the intersection graph theory [43, 71, 79]. Intersection graph theory has wide range of applications in several areas of human endeavour such as, secure sensor networks, wireless communication networks, collaboration networks, common interest networks, epidemiology, social psychology, marketing etc. (refer [12, 19, 23, 109–111]). The intersection graph is defined in this manner. Let $F = \{S_i : i \in I\}$ be an arbitrary family of sets. The *intersection graph*

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$\mathscr{G}(F)$ is the graph whose vertices are S_i, $i \in I$ and in which the vertices S_i and S_j, $(i, j \in I)$ are adjacent if and only if $S_i \neq S_j$ and $S_i \bigcap S_j \neq \emptyset$. For intersection graphs we have the following result due to Marczewski [88, 89].

Theorem 1 ([88]). *Every simple graph is an intersection graph, that is, for any simple graph* $\mathbf{G}$ *there exists a family* F *of sets* S_i, $i \in I$ *such that* $\mathbf{G}$ *is isomorphic to the intersection graph* $\mathscr{G}(F)$.

Various graphs over algebraic structures were defined related to the intersection graphs by many authors starting from the year 1964. It was Bosak [22] who took the first step in that direction. Bosak [22] defined and studied intersection graph arising from a semigroup S, where he considered the proper subsemigroups of S as the vertices of that graph and two distinct vertices A, B are joined by an edge if and only if the intersection of the subsemigroups A and B is non-empty. In 1969, the intersection graph of non-trivial proper subgroups of a finite group was defined and studied by Csákány and Pollák [38]. Their work was continued by Zelinka [108], where he studied the intersection graph of non-trivial subgroups of finite Abelian groups. Many graphs related to the ring structure [33, 47] such as the zero-divisor graph [2, 3, 10, 81], the total graph and regular graph [1], the annihilating-ideal graph[14, 15], the annihilator-inclusion ideal graph [7], the toroidal annihilating-ideal graph [64], the sum-annihilating essential ideal graph [6] and the comaximal ideal graph [97]. Most of them are based on different algebraic properties of rings, mainly ideal of a ring [86].
The *intersection graph of ideals of ring* R was introduced in 2009 [26]. It has become one of the most studied algebraic graphs in last last decade. Many properties can be found in the literature that relates the ring-theoretic and the graph-theoretic concepts. The intersection graph of ideals of a ring R is defined as follows.

Definition 1 ([26]). Let $\mathscr{L}(R)$ denote the set of all non-trivial (non-zero proper) left ideals of a ring R. The intersection graph of $\mathscr{L}(R)$ is the undirected simple graph (without loops and multiple edges) whose vertices are in a one-to-one correspondence with all non-trivial left ideals of R and two distinct vertices are joined by an edge if and only if the corresponding left ideals of R have a non-trivial (non-zero) intersection.

We denote this intersection graph of ideals of ring R by $G(R)$. If I is a non-trivial left ideal of R then the corresponding vertex v_I in $G(R)$ is denoted by I. Figure 1 is the intersection graph of ideals of the ring $\mathbb{Z}_{36}$.

Remark 1. Let R be a left simple ring. Then the vertex set of $G(R)$ is empty. Thus we consider only the graphs $G(R)$, where R is not left simple.

We have mentioned that the idea of the intersection graphs of ideals of rings was introduced in [26]. In it, the authors had characterized the rings R such that $G(R)$ is connected and obtained several necessary and sufficient conditions on a ring R such that $G(R)$ is complete. In particular, they had determined the values of n for which

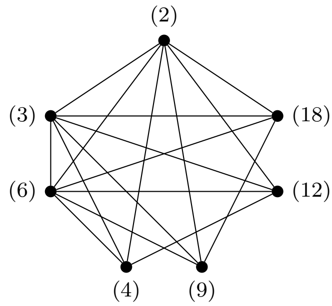

Figure 1. **The graph** $G(\mathbb{Z}_{36})$

$G(\mathbb{Z}_n)$ is connected, complete, bipartite, planar and cyclic. The formulae for counting the order, the degrees of vertices and the size of $G(\mathbb{Z}_n)$ were also obtained. Further, the number of vertices of a given degree in $G(\mathbb{Z}_n)$ was also computed. Eulerianity and Hamiltonicity were studied for $G(\mathbb{Z}_n)$. The set of all non-isomorphic graphs of $G(\mathbb{Z}_n)$ for a given number of vertices was found in [26]. Finally, it was noted in [26] that, every simple undirected graph is an induced subgraph of $G(\mathbb{Z}_n)$ for some $n \in \mathbb{N}$. Following the planarity of $G(\mathbb{Z}_n)$, Jafari and Rad [61] characterized the planar intersection graphs of ideals of a commutative ring R with identity. In 2011, the domination number of $G(R)$, where R is a commutative ring with identity was determined in [62]. Akbari et al. studied the connectedness, clique number and chromatic number of $G(R)$ for the rings R with identity [5]. They proved that if $G(R)$ is a connected graph then the diameter of $G(R)$ is at most 2. They determined the rings R such that the clique number of $G(R)$ is finite. Furthermore, they proved that for every ring R, if the clique number of $G(R)$ is finite then the chromatic number is also finite and for the reduced ring R, both are equal. In 2012, Talebi [90], had determined the conditions on ring R such that $G(R)$ is a path or a tree. It is known that for every graph G, $\omega(G) \leq \chi(G)$, where $\omega(G)$ and $\chi(G)$ represent the clique number and the chromatic number of G, respectively. Nikandish and Nikmehr continued the study of $G(\mathbb{Z}_n)$ in [72], where they proved that $\omega(G(\mathbb{Z}_n)) = \chi(G(\mathbb{Z}_n))$. In [5], the Akbari et al. were unable to find an example of a ring R such that $G(R)$ is not weakly perfect. Hence, Nikandish and Nikmerh [72] conjectured that, for every ring R, $G(R)$ is a weakly perfect graph. The equality of the edge chromatic number of $G(\mathbb{Z}_n)$ and the maximum degree of $G(\mathbb{Z}_n)$ was also discussed in [72]. Das has [39] characterized the positive integers n such that $G(\mathbb{Z}_n)$ is perfect.

Osba [73] had studied $G(R)$ where R is a finite commutative principal ideal ring with identity. The author had also extended the studies on the properties such as completeness, bipartiteness, planarity, Eulerianity and Hamiltonianicity of $G(R)$, which were generalizations of the work in [26], where the authors studied these properties for $G(\mathbb{Z}_n)$. Formula to count the number of ideals in each ring and the degree of each ideal were also computed in [73]. Moreover, in the same work, the author had studied various properties of the intersection graph of the ring of Gaussian integers modulo n. For a finite commutative principal ideal ring R with identity, Osba's [73] work on $G(R)$ was continued by Osba, Al-Addasi and Abughneim in [74], where they computed the radius, domination number, independence number, geodetic number and hull number of $G(R)$. In addition, conditions on R such that $G(R)$ is chordal and some properties of the complement of $G(R)$ were determined in [74]. For a commutative ring R with identity, the domination number of $G(R)$ was studied by Akbari and Nikandish [4] where they proved that the domination number of $G(R)$ is at most 2. They classified all rings R (not necessarily commutative) such that the domination number of $G(R)$ is at least 2. These same result was also obtained in [62]. Moreover, the authors [4] studied the graph $G(R)$, where R is the ring of $n \times n$ matrices over finite field $\mathbb{F}_q$ of order q denoted by $M_n(\mathbb{F}_q)$. They had determined the domination number, clique number and the independence number of $G(M_n(\mathbb{F}_q))$. In [4], the authors had raised a question: for two rings R and S, if $G(R) \cong G(S)$, then is it true that $R \cong S$?. They had proved that for a ring R, if $G(R) \cong G(M_n(\mathbb{F}_q))$, where $n \geq 2$ is a positive integer, then $R \cong M_n(\mathbb{F}_q)$. Further, the authors of [4] had proved that if R and S are two finite reduced rings, $m, n \geq 2$ are positive integers and $G(M_m(R)) \cong G(M_n(S))$, then $m = n$ and $R \cong S$.

In 2016, Hoseini et al. [58] studied some properties of $G(R)$, where R is a commutative ring with identity. Their main concern was to investigate all rings R such that $G(R)$ does not contain C_3 and C_4. In case of reduced ring R, they described the conditions for $G(R)$ to be C_n- free. They had observed that except few cases, all Artinian rings have Hamiltonian intersection graphs [58]. In the end of the article [58], the authors cited a necessary and sufficient condition for $G(R)$ to be pancyclic, where R is an Artinian ring.

The genus of a graph is one of the most important topological properties. The graphs of genus 0 are planar graphs, whereas the graphs of genus 1 are toroidal graphs. As we have mentioned, in [61] planarity of $G(R)$ was studied. Following this, Pucanović and Petrović [76], had focused their study on toroidality of the graph $G(R)$. They had completely characterized those toroidal graphs which may occur as intersection graphs of ideals of some commutative rings. Pucanović et al. had classified all graphs of genus 2 that are intersection graphs of ideals of some commutative rings and had obtained some lower bounds on the genus of the intersection graph of ideals of a non-local commutative rings [77], which was indeed a continuation of the works [61] and [76]. For a commutative ring R with identity, in 2019, Ramanathan [80] has determined some conditions on R such that $G(R)$ is a split graph or unicyclic or outerplanar. Moreover, Ramanathan discussed some conditions on ring R such that the graph $G(R)$ is of crosscap one and two. Very recently, in [94], the Vadhel has classified the commutative rings R with identity such that $G(R)$ is either bipartite or tripartite.

In 2014, Rad, Jafari and Ghosh [78] had determined the order and the size of the intersection graph of ideals of direct product of rings and fields. They also had studied the Eulerian and the Hamiltonian property of the intersection graph of ideals of direct product of commutative rings with identity [78].

The study of intersection graphs of ideals of rings was taken a step further in 2015. For a commutative ring R with identity, a spanning subgraph $H(R)$ of $G(R)$ was defined by Visweswaran and Vadhel [98]. They had studied the connectedness, diameter, girth and complement of $H(R)$ [98]. Motivated by the work done on planarity of $G(R)$ [61, 76], Vadhel and Visweswaran focused their study on planarity of the spanning subgraph $H(R)$ in [95] and [96]. Again in 2018, Visweswaran et al. [99] defined and studied the properties, such as connectedness, diameter, girth of a subgraph denoted by $g(R)$ of $G(R)$.

In 2017, the authors S. Sajana, D. Bharathi and K. K. Srimitra [83], had introduced the idea of signed intersection graph of ideals of a commutative ring R with identity. The values of n were determined in [83] such that the signed graph of $G(\mathbb{Z}_n)$ is balanced, homogeneous and $\mathscr{C}-$consistent. Boro, Singh and Goswami [21] have very recently studied various properties of the line graph of $G(R)$.

In Figure 2, we present the research on $G(R)$ in the chronological order.

In this survey, we collect various properties (refer Figure 3) of the graph $G(R)$.

We organise this article according to the graph theoretical properties of $G(R)$. In Section 2, we mention various results obtained for different properties of $G(R)$. Section 4, gives us the idea about the complement of the graph $G(R)$. The properties of the spanning subgraphs of $G(R)$ are discussed in Section 5. Section 6, is about the results obtained in the intersection graphs of ideals of direct product of rings. In Section 7, the relation between the complement of the graph $G(R)$ and the zero divisor graph is mentioned. Section 8, gives us the concept and study of signed graph developed for the graph $G(R)$. Finally in Section 10, we discuss some open problems that can be referred for further studies of $G(R)$. An exhaustive and extensive bibliography is included.

We use $\mathbb{N}$, $\mathbb{Z}$ to denote the set of natural numbers, set of integers respectively. Through out the paper, A denotes the set of all positive integers greater than one and which are not primes. For $n \in A$, we know that the ring of integers modulo n denoted by $\mathbb{Z}_n$ is a principal ideal ring and each of these ideals is generated by $\overline{k} \in \mathbb{Z}_n$, where k divides n. We denote the ideal $\overline{k}$ by (k) for convenience. We denote the ring of $n \times n$ matrices over R by $M_n(R)$. For details on terminology and results related to ring theory readers can refer [11, 46, 55, 60, 63, 67, 69] and for information on graph theory readers can refer [51, 68, 101]. Some definitions are given as needed.

2. Properties of G(R)

2.1. Connectedness of $G(R)$

In [26], the authors initiated their study of the graph $G(R)$ with the property connectedness.

Theorem 2 ([26]). *The intersection graph $G(R)$ of a ring R is disconnected if and only if R contains at least two minimal left ideals and every nontrivial left ideal of R is minimal (as well as maximal).*

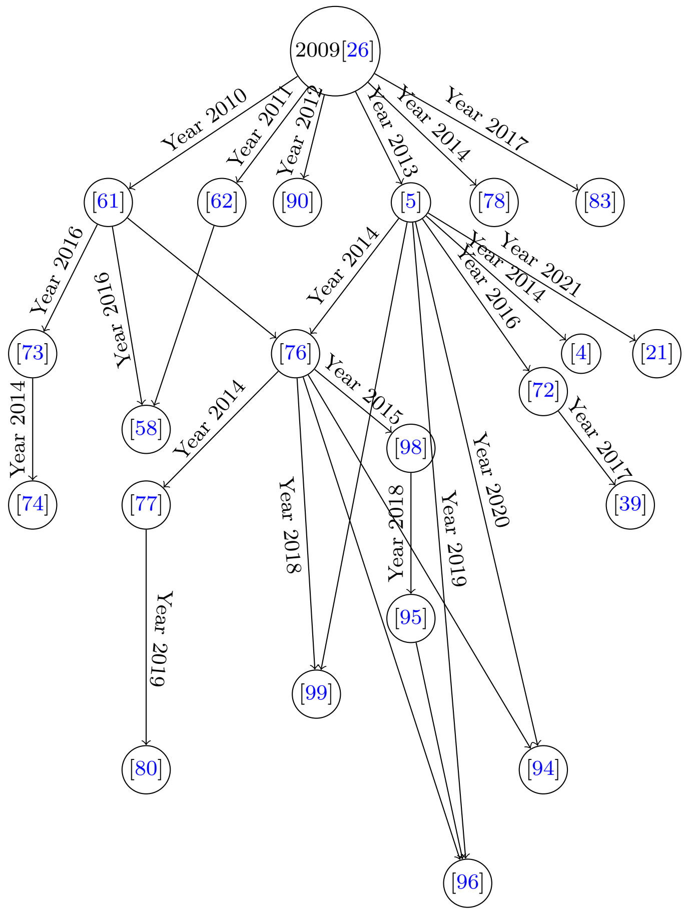

Figure 2. Development on the study of various properties of $G(R)$ in the chronological order. By $A \longrightarrow B$, we mean the article B has cited the article A.

Corollary 1 ([26]). *For a ring R, let $G(R)$ be disconnected. Then $G(R)$ has no edge.*

Example 1 ([26]). Let $n \in A$, then $G(\mathbb{Z}_n)$ is disconnected if and only if $n = pq$, where p and q are distinct primes.

Theorem 3 ([26]). *Let R be a commutative ring. Then $G(R)$ is disconnected if and only if R is a direct product of two simple commutative rings, i.e., $R = R_1 \times R_2$ where each R_i $(i = 1, 2)$ is either a field or a null ring with prime number of elements.*

Corollary 2 ([26]). *Let R be a commutative ring with identity. Then $G(R)$ is disconnected if and only if R is a direct product of two fields.*

Connectivity of $G(R)$ was also studied by Akbari et al. in [5].

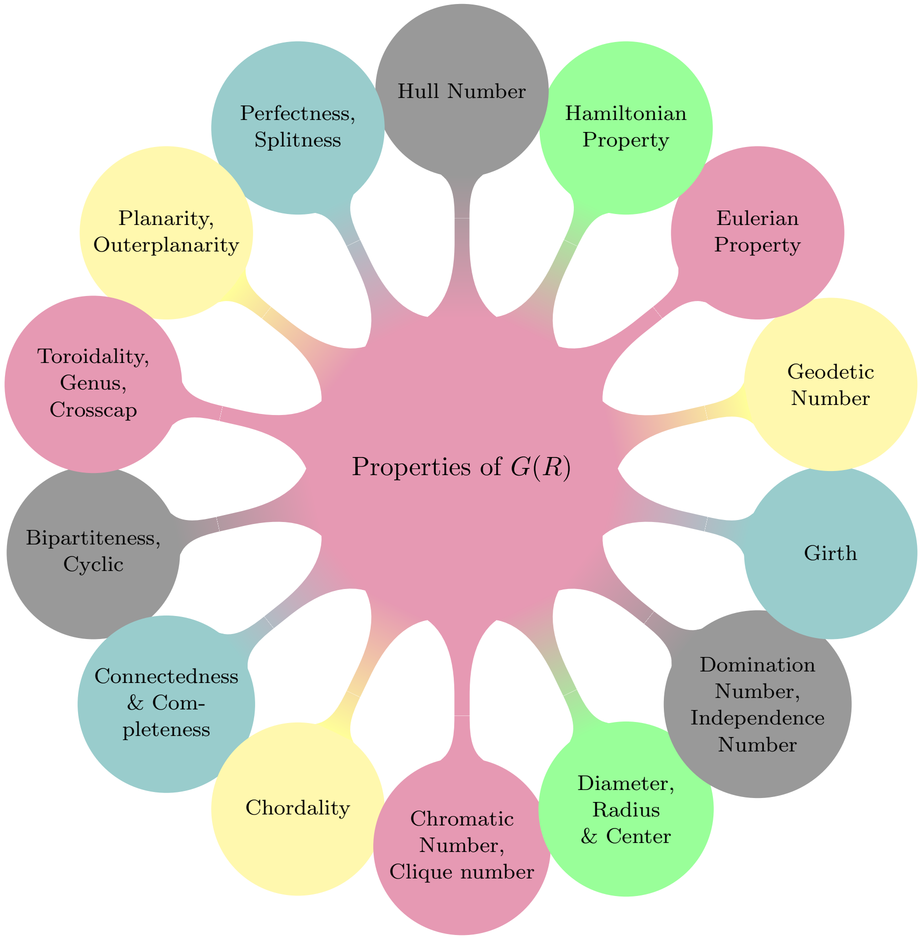

Figure 3. **Properties of** $G(R)$

Theorem 4 ([5]). *Let R be a ring with identity. Then $G(R)$ is disconnected if and only if either $R \cong D_1 \times D_2$, where D_1 and D_2 are two division rings or $R \cong M_2(D)$, where D is a division ring.*

The ring $\mathbb{Z}_n[i] = \{\bar{a} + i\bar{b} : \bar{a}, \bar{b} \in \mathbb{Z}_n\}$ is called the ring of Gaussian integers modulo n. Reader may refer [37], [41] for various results on Gaussian integers and Gaussian integers modulo n. Connectedness of $G(\mathbb{Z}_n[i])$ was discussed in [73]. As a consequence of Corollary 2 [26], Proposition 1 is obtained.

Proposition 1 ([73]). *Let p be a prime number congruent to 1 modulo 4, q_1, q_2 be prime integers congruent to 3 modulo 4. Then the graph $G(\mathbb{Z}_n[i])$ is disconnected if and only if $n = p$ or $n = q_1 q_2$.*

2.2. Diameter, radius and center of $G(R)$

The diameter, radius, center of a graph $G = (V, E)$ are denoted by $diam(G)$, $\rho(G)$ and $\mathrm{Cen}(G)$, respectively.

Theorem 5 ([5]). *Let $G(R)$ be a connected graph, where R is a ring with identity. Then $diam(G(R)) \leq 2$.*

In [90], Talebi investigated the diameter of the graph $G(\mathbb{Z}_n)$.

Example 2 ([90]). Let $n \in A$, then $diam(G(\mathbb{Z}_n)) \leq 2$.

Example 3 ([90]). Let $n \in A$, then $diam(G(\mathbb{Z}_n)) = 2$ if and only if $n \neq p^k$, where p is a prime number and $k \in \mathbb{N}$, $(k > 1)$.

A ring R is said to be *local* if it has a unique maximal left ideal. Let R be a finite local commutative principal ideal ring with identity. Let $\mathfrak{m}$ be the maximal ideal of R. To define $\mathrm{Nilpotency}(R) = n$, the author [73] used the fact that there exist $n \in \mathbb{N}$ such that $\mathfrak{m}^n = \{0\}$ but $\mathfrak{m}^{n-1} \neq \{0\}$. If R is a field, $\mathrm{Nilpotency}(R) = 1$.

Theorem 6 ([73]). *Let R be a finite commutative principal ideal ring with identity, then we have the following results on the diameter of $G(R)$:*

$$diam(G(R)) = \begin{cases} 0, & \textit{if } R \textit{ is local with Nilpotency}(R) = 2 \\ \propto, & \textit{if } R \textit{ is a product of two fields} \\ 1, & \textit{if } R \textit{ is local with Nilpotency}(R) > 2 \\ 2, & \textit{otherwise.} \end{cases}$$

We present a problem related to the above theorem. Let R_1, R_2 be two finite local commutative principal ideal rings with identity and $Nilpotency(R_i) > 2$ for $i = 1, 2$. Then, show that $G(R_1)$ is isomorphic to $G(R_2)$.

Theorem 7 ([74]). *For any finite commutative principal ideal ring R with identity which is not a field, the radius of $G(R)$ can be the following:*

$$\rho(G(R)) = \begin{cases} 0, & \textit{if } R \textit{ is local with Nilpotency}(R) = 2 \\ \propto, & \textit{if } R \textit{ is a product of two fields} \\ 2, & \textit{if } R \textit{ is a product of more than two fields} \\ 1, & \textit{otherwise.} \end{cases}$$

A graph G is said to be *self-centered* if $\mathrm{Cen}(G) = G$.

Theorem 8 ([74]). *Let R be a finite commutative principal ideal ring with identity. Then $G(R)$ is self-centered if and only if R is local ring or a product of fields.*
Further, if $R = \prod_{j=1}^{n} R_j$, where $n \geq 2$, R_j is a finite local principal ideal ring for each j and there is at least one R_j having nilpotency greater than 1, then the vertex set of $Cen(G(R))$ is $\{\prod_{j=1}^{n} I_j : I_j$ is an ideal of R_j and $I_j \neq \{0\}$ for each $j\}$ and $Cen(G(R))$ is a complete graph of size $(\prod_{j=1}^{n} n_j) - 1$, where $n_j = Nilpotency(R_j)$.

2.3. Structure of $G(R)$

Formula for counting the number of vertices, degrees of vertices and the number of edges of $G(\mathbb{Z}_n)$ were computed in [26].

Theorem 9 ([26]). *Let $n = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m} \in A$, where $m, n_j \in \mathbb{N}$, $(m, n_1) \neq (1, 1)$, p_j are distinct primes $(j = 1, 2, \ldots, m)$. Then the total number of vertices of the graph $G(\mathbb{Z}_n)$ is given by*

$$T_n = (n_1 + 1)(n_2 + 1) \cdots (n_m + 1) - 2.$$

If $I = (p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} p_{k+1}^{i_{k+1}} \cdots p_m^{i_m})$ is a nonzero proper ideal of $\mathbb{Z}_n$, where $0 \leqslant k < m$, $0 \leqslant i_j < n_j$, $i_j \in \mathbb{Z}$, $j = k+1, k+2, \ldots, m$ (provided in the case $k = 0$, not all i_j are zero) and the vertex v_I of $G(\mathbb{Z}_n)$ corresponds to the ideal I of $\mathbb{Z}_n$, then $\deg(I)$ of v_I is given by

$$\deg(I) = \begin{cases} (n_1+1)(n_2+1)\cdots(n_m+1) - (n_1+1)(n_2+1)\cdots(n_k+1) - 2, & \text{if } k \neq 0; \\ (n_1+1)(n_2+1)\cdots(n_m+1) - 3, & \text{if } k = 0. \end{cases}$$

The total number of vertices in $G(R)$ of degree d was also computed in [26]. Theorem 10, gives us the size of the graph $G(\mathbb{Z}_n)$.

Theorem 10 ([26]). *Let $n = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m} \in A$, where $m, n_j \in \mathbb{N}$, $(m, n_1) \neq (1, 1)$, p_j are distinct primes $(j = 1, 2, \ldots, m)$. Then the number of edges, e_n of the graph $G(\mathbb{Z}_n)$ is given by $e_n = \frac{1}{2}(1 + T_n + T_n^2 - T_{n^2})$, where T_n denotes the number of vertices of $G(\mathbb{Z}_n)$ and T_{n^2} is the total number of vertices of the graph $G(\mathbb{Z}_{n^2})$.*

To describe the structure of $G(\mathbb{Z}_n)$ more precisely [26], it was observed that every simple undirected graph becomes an induced subgraph of the graph $G(\mathbb{Z}_n)$ for some $n \in A$. Briefly it was described by the following example [26].

Example 4. Consider the graph G_1 with the vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ as shown in Figure 4.
The maximal cliques of G_1 are: $C_1 = \{v_1, v_2, v_3, v_4\}$ and $C_2 = \{v_2, v_3, v_4, v_5, v_6\}$. To determine the value of n such that G_1 is isomorphic to $G(\mathbb{Z}_n)$, the following method was described in [26]:

Step I: Assign distinct primes p_1 and p_2 to the cliques C_1 and C_2 respectively.

Step II: For each $v \in V$, label the vertex v with the number $\prod_{i \in \{j \in \mathbb{N} | v \in C_j\}} p_i$.

Step III: Relabel the vertices by using different powers in order to make the labeling distinct.

Vertices	v_1	v_2	v_3	v_4	v_5	v_6
Step II	p_1	p_1p_2	p_1p_2	p_1p_2	p_2	p_2
Step III	p_1	p_1p_2	$p_1^2p_2$	$p_1p_2^2$	p_2	p_2^2
Step V	$p_1p_2^2$	p_1p_2	p_2	p_1	$p_1^2p_2$	p_1^2

Table 1. **Step II, Step III and Step V of Example 4**

Step IV: Find a number n which is a common multiple of all the labels such that it is greater than each of them. We choose here $n = {p_1}^2{p_2}^2$.

Step V: Assign the final label to each vertex which is the quotient of n with the corresponding last label.

Clearly, the graph G_1 is isomorphic to the subgraph induced by the labeled vertices $\{(p_1), (p_2), ({p_1}^2), (p_1p_2), (p_1{p_2}^2), ({p_1}^2p_2)\}$ of the graph $G(\mathbb{Z}_{{p_1}^2{p_2}^2})$ as shown in the Figure 5.

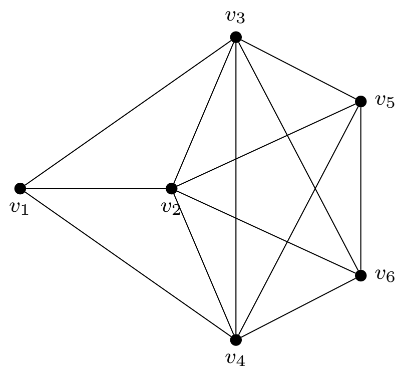

Figure 4. **The graph G_1**

Theorem 11 **([73]).** *Let R be a finite local commutative principal ideal ring with identity and Nilpotency$(R) = n$. Then the order of $G(R)$ is $n-1$ and each vertex of $G(R)$ is of degree $n-2$.*

Theorem 12 **([73]).** *Let R be a finite non-local commutative principal ideal ring with identity and $R = \prod_{j=1}^{m} R_j$, where R_j is a local principal ideal ring with Nilpotency$(R_j) = n_j$ for $j = 1, 2, \ldots, m$. Then we have the following results:*

(i) *The order of $G(R)$ is $(\prod_{j=1}^{m}(n_j + 1)) - 2$.*

(ii) *The degree of a vertex $I \in G(R)$ is $\deg(I) = \prod_{j=1}^{m}(n_j + 1) - \prod_{j=1}^{m}(n_j + 1)^{\pi_j(I)} - 2$, where $I = \prod_{j=1}^{m} I_j$, I_j is an ideal of R_j and $\pi_j(I) = 0$, if $I_j \neq \{0\}$, $\pi_j(I) = 1$, if $I_j = \{0\}$.*

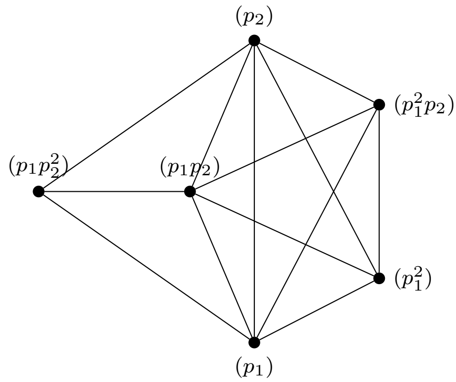

Figure 5. **A representation of G_1 as an induced subgraph of $G(\mathbb{Z}_{{p_1}^2{p_2}^2})$**

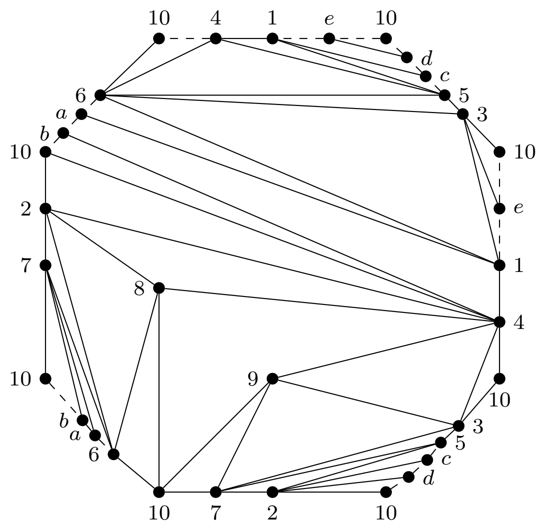

Figure 6. **The graph Γ'**

Now we describe some conditions such that $G(R)$ is a finite graph, where R is a ring with identity.

Lemma 1 ([5]). *Let R be a ring with identity and I be a left ideal of R. If in $G(R)$, the degree of the vertex v_I corresponding to the ideal I is finite, then R is a left Artinian ring.*

Theorem 13 ([5]). *Let R be a commutative ring with identity and $\mathfrak{m}$ be a maximal ideal of R. If in $G(R)$, the degree of the vertex $v_{\mathfrak{m}}$ corresponding to the ideal $\mathfrak{m}$ is finite, then $G(R)$ is a finite graph.*

The following results in this section are about the vertices of degree one in $G(R)$ [90].

Proposition 2 ([90]). *Let $G(R)$ have no cycle of length 3 and I be a minimal ideal of R. Then in $G(R)$, $\deg(v_I) = 1$, where v_I is the vertex corresponding to the ideal I.*

Lemma 2 ([90]). *Let v_I be a vertex of $G(R)$ such that $\deg(v_I) = 1$, where v_I is the vertex corresponding to the ideal I of R. Then I is a minimal ideal or a maximal ideal of R.*

Theorem 14 ([39]). *Let (i), (j) be any two distinct proper ideals of $\mathbb{Z}_n$ for $n \in A$. Then the vertices $v_{(i)}$, $v_{(j)}$ corresponding to the ideals (i) and (j) (respectively) are adjacent in $G(\mathbb{Z}_n)$ if and only if $lcm(a, b)$ is a factor of n and $1 < lcm(a, b) < n$.*

2.4. The total number of non-isomorphic graphs of $G(\mathbb{Z}_n)$ for a given number of vertices

Let $n \in A$. Let $k \in \mathbb{N}$ be the order of the graph $G(\mathbb{Z}_n)$. The total number of non-isomorphic graphs of $G(\mathbb{Z}_n)$ with k vertices were determined in [26] using the concept of unordered factorization.
Let $a > 1$ be a natural number. An *unordered factorization* of a is a representation of a as a product of positive integers greater than 1, where the order of the factors in the product is irrelevant. The total number of unordered factorizations of a is denoted by $p^*(a)$, which is described in Theorem 15.

Theorem 15 ([44]). *Let $a = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ be the prime factorization of a natural number a, where p_i are distinct primes, $n_i \in \mathbb{N}$ for all $i = 1, 2, \dots, k$, $(k \in \mathbb{N})$, with $n_1 = \min\{n_i\} i = 1, 2, \dots, k$. Then*

$$p^*(a)\ n_1 \ = \sum_{r=1}^{n_1} r \cdot \Big(\sum_{d \big| \frac{a}{p_1^{n_1}}} \lambda(p_1^r d)\ p^*\big(\frac{a}{p_1^r d}\big)\Big), \tag{1}$$

where $\lambda(m) = \sum_{i \big| c(m)} \frac{1}{i}$, $c(m) = \gcd\{r, r_2, \dots, r_k\}$ for $m = p_1^r p_2^{r_2} \dots p_k^{r_k}$, $0 < r \leqslant n_1$ and $0 \leqslant r_j \leqslant n_j$, $r, r_j \in \mathbb{N}$, $j = 2, 3, \dots, k$.

Theorem 16 ([26]). *Let $n \in A$ and T be the order of a graph $G(\mathbb{Z}_n)$. Then the number of non-isomorphic graphs of $G(\mathbb{Z}_n)$ with T vertices is equal to the number of unordered factorization of $T + 2$.*

2.5. Cycles and Completeness of $G(R)$

A close relation between regularity and completeness of $G(R)$ was established in [5].

Theorem 17 ([5]). *Let R be a ring with identity and $G(R)$ be a $r-$regular graph, for some non-negative integer r. Then either $G(R)$ is a complete graph or a null graph.*

Theorem 18 ([26]). *Let R be an integral domain which is not a field. Then the graph $G(R)$ is complete.*

Definition 2. In a ring R, an element $a \in R$ is called *nilpotent* if $a^k = 0$ for some positive integer k.

Definition 3. A ring R is said to be *reduced* if it has no non-zero nilpotent element.

Theorem 19 ([5]). *Let R be a commutative ring with identity. Then R is an integral domain if and only if R is a reduced ring and $G(R)$ is a complete graph.*

Definition 4. An ideal I of a ring R is called *decomposable* if there exist ideals J, K of R such that $I = J + K$ and $J \cap K = (0)$. The ideal I is said to be indecomposable if it is not decomposable.

Theorem 20 ([26]). *Let R be a commutative ring with identity. Then the following conditions are equivalent:*

(i) *$G(R)$ is complete;*

(ii) *for any two ideals I, J of R, $I \cap J = (0)$ implies $I = (0)$ or $J = (0)$;*

(iii) *for any $a, b \in R$, $(a) \cap (b) = (0)$ implies $a = 0$ or $b = 0$;*

(iv) *every nonzero ideal of R is indecomposable.*

It is known that if a commutative ring R with identity is an integral domain, so is the polynomial ring $R[x]$. Thus the following theorem was proved [26].

Theorem 21 ([26]). *Let R be a commutative ring with identity. Then $G(R)$ is complete if and only if $G(R[x])$ is complete.*

One can find the necessary and the sufficient condition for the intersection graph of the ideals of a left Artinian ring to be complete in Theorem 22.

Theorem 22 ([26]). *Let R be a left Artinian ring. Then $G(R)$ is complete if and only if R has a unique minimal left ideal.*

A ring R is called a *Gorenstein ring* if R is both left and right Noetherian and if R has finite self injective dimension on both the left and the right [24, 42].
Let R be a commutative ring and $A \subseteq R$. Then annihilator of A, denoted by $\mathrm{Ann}(A)$ is defined as $\mathrm{Ann}(A) = \{r \in R : rA = 0\}$.
In case of a local commutative Artinian ring we have Theorem 23.

Theorem 23 ([5]). *Let R be a local commutative Artinian ring with identity. Then the following statements are equivalent.*

(i) *R is a Gorenstein ring;*

(ii) *$I = AnnAnnI$ for all ideals I of R;*

(iii) *$G(R)$ is a complete graph;*

(iv) *R has a unique minimal ideal.*

For a commutative ring with identity, Hoseini et al. [58] had observed the coincidence of completeness and regularity of $G(R)$ for the class of Artinian rings.

Theorem 24 ([58]). *Let R be a commutative Artinian ring with identity, which is not a direct sum of two fields. If $G(R)$ is regular, then it is complete.*

Theorem 25 ([58]). *Let R be a commutative Artinian ring with identity. Then $G(R)$ is complete if and only if there exists a sequence of rings $R_1, \dots, R_n$, in which $R = R_1, (R_i, R_{i+1})$ is a local ring for all $i = 1, \dots, n-1$ and R_n is a field.*

An ideal $\mathfrak{p}$ of a ring R is called prime if for any two ideals $\mathfrak{a}$, $\mathfrak{b}$ of R, $\mathfrak{ab} \subseteq \mathfrak{p}$ implies that either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$.
A prime ideal $\mathfrak{p}$ is said to be an *associated prime ideal* of a commutative Noetherian ring R, if there exist a non-zero element x in R such that $\mathfrak{p} = Ann(x)$ and $\mathrm{Ass}(R)$ denote the set of all associated prime ideals of R.

Theorem 26 ([5]). *Let $G(R)$ be a complete graph, where R is a commutative Noetherian ring with identity. Then $Ass(R) = Min(R)$ has just one element, where $Ass(R)$ and $Min(R)$ denote the set of all associated prime and minimal prime ideals of R respectively.*

Example 5 ([26]). Let $n \in A$. The graph $G(\mathbb{Z}_n)$ is complete if and only if $n = p^k$, where p is a prime and $k \in \mathbb{N}$, $(k > 1)$.

Example 6. Let R be a finite local commutative principal ideal ring with identity and $\mathrm{Nilpotency}(R) = n$. Then $G(R)$ is complete.

Theorem 27 ([80]). *Let R be a commutative non-local Artinian ring with identity. Then $G(R)$ is unicyclic if and only if $R \cong R_1 \times F_1$, where R_1 is a local ring with the maximal ideal $\mathfrak{m}$ such that $\mathfrak{m} =< x >$, $x \in R$, $x^2 = 0$ and F_1 is a field.*

Theorem 28 ([80]). *Let R be a commutative ring with identity and $\mathfrak{m}$ is the unique maximal ideal of R such that $\mathfrak{m}$ is finitely generated. Then $G(R)$ is unicyclic if and only if $\mathfrak{m} =< x >$, where $x \in R$, $x \neq 0$ and $x^4 = 0$.*

In [26], the values of n for which $G(\mathbb{Z}_n)$ contains a cycle were determined. Completeness conditions of $G(\mathbb{Z}_n)$ for the values of n were also determined in [26].

Theorem 29 ([26]). *Let $n \in A$. The graph $G(\mathbb{Z}_n)$ contains a cycle if and only if $n = kt$, where $k = p^4, p^2q$ or pqr, p, q, r are distinct primes and $t \in \mathbb{N}$.*

In [73], the values of $n \in \mathbb{N}$ were determined such that the intersection graph of the ring of Gaussian integers modulo n, that is, $G(\mathbb{Z}_n[i])$ is complete.

Theorem 30 ([73]). *Let $k(> 1) \in \mathbb{N}$ and p be a prime number congruent to 3 modulo 4. Then $G(\mathbb{Z}_n[i])$ is complete if and only if $n = 2$ or $n = 2^k$ or $n = p^k$.*

2.6. Acyclic, bipartiteness and tripartiteness conditions of $G(R)$

Talebi [90] has characterized the ring R such that $G(R)$ is P_2 or P_3. He concluded that the star graph is the only tree that is a $G(R)$[90].

Theorem 31 ([90]). *The graph $G(R) \cong P_2$ if and only if the ring R has only two ideals, namely minimal ideal and maximal ideal.*

We pose a problem now. If the rings R_1 and R_2 have only two ideals, one is minimal and the other is maximal, then $G(R_1)$ is isomorphic to $G(R_2)$.

Theorem 32 ([90]). *Let $G(R)$ be a path, then $G(R) \cong P_2$ or $G(R) \cong P_3$. If $G(R) \cong P_3$, then R has only three ideals I_1, I_2, I_3 such that $I_2 = I_1 \oplus I_3$.*

Example 7. Let $n = p^3$, where p is a prime. Then $G(\mathbb{Z}_{p^3})$ contains a cycle.

The values of n were determined such that the graphs $G(\mathbb{Z}_n)$ are acyclic [26].

Proposition 3 ([26]). *Let $n \in A$. Then $G(\mathbb{Z}_n)$ does not contain a cycle if and only if $n = p^2, pq$ or p^3, where p and q are distinct primes. In all other cases, it contains a cycle of length 3.*

Let us describe the graphs $G(R)$ which are triangle free.

Theorem 33 ([5]). *Let R be a ring with identity and $G(R)$ be a triangle-free graph which is not null. Then R is a local ring and one of the following conditions holds:*

(i) *The maximal left ideal of R is principal and $G(R) = K_1$ or $G(R) = K_2$;*

(ii) *The minimal generating set of $\mathfrak{m}$ has size 2, $\mathfrak{m}^2 = 0$, where $\mathfrak{m}$ is the maximal left ideal of R and $G(R)$ is a star graph.*

Theorem 34 ([58]). *Let R be a commutative ring with identity, which is neither a direct sum of two fields nor a direct sum of a field with a local ring $(S, \mathfrak{m})$ such that $\mathfrak{m}$ is a field. If $G(R)$ has a pendant vertex, then $G(R)$ is a star graph.*

Theorem 35 ([58]). *Let $G(R)$ be a triangle-free graph, where R is a commutative ring with identity, then $G(R)$ is a star graph or union of two isolated vertices.*

Definition 5. The ideals $I_1, I_2, \ldots, I_k$ of a ring R are called *independent* if $I_i \cap (I_1 + I_2 + \cdots + I_{i-1} + I_{i+1} + \cdots + I_k) = 0$ for $i = 1, 2, \ldots, k$.

Theorem 36 ([58]). *Let R be a commutative ring with identity. Then $G(R)$ is C_4-free if and only if R has no set of four non-zero independent ideals. Moreover, if R is a reduced ring and $n \geq 5$, then $G(R)$ is C_n-free if and only if R has no set of n independent ideals.*

A graph $G = (V, E)$ is called *bipartite* if its vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge in G joins a vertex in V_1 with a vertex in V_2.
E. A. Osba [73] had determined the value of n such that $G(\mathbb{Z}_n[i])$ is bipartite. He also found a necessary and sufficient condition for a finite local commutative principal ideal ring with identity to be bipartite [73].

Theorem 37 ([73]). *The graph $G(\mathbb{Z}_n[i])$ is bipartite if and only if $n = p^3$, where p is a prime number congruent to 3 modulo 4.*

Theorem 38 ([73]). *Let R be a finite local commutative principal ideal ring with identity. Then $G(R)$ is bipartite if and only if Nilpotency$(R) = 2$ or 3.*

Theorem 39 ([73]). *Let R be a finite non-local commutative principal ideal ring with identity. Then $G(R)$ is never a bipartite graph.*

Proposition 4 ([94]). *Let R be a commutative ring with identity and let $r \geq 2$. If $G(R)$ is $r-$partite, then R has at most r maximal ideals.*

Proposition 5 ([94]). *Let R be a commutative ring with identity and R has exactly r maximal ideals, where $r \geq 2$. Then $G(R)$ is $r-$partite if and only if $r \leq 3$ and $R \cong K_1 \times K_2 \times \cdots \times K_r$, where K_i is a field for each $i = 1, 2, \ldots, r$.*

Proposition 6 ([94]). *Let R be a commutative ring with identity and $\mathfrak{m}$ be the unique maximal ideal of R such that $\mathfrak{m} \neq (0)$. If $G(R)$ is bipartite and $\mathfrak{m}^2 = (0)$, then $\mathfrak{m}$ is generated by two elements.*

Example 8 ([26]). Let $n \in A$. The graph $G(\mathbb{Z}_n)$ is bipartite if and only if $n = pq$ or p^3, where p and q are distinct primes.

Theorem 40 ([94]). *Let R be a commutative ring with identity with exactly two maximal ideals. Then $G(R)$ is tripartite if and only if $R \cong K \times S$, where K is a field and S is a principal ideal ring with unique prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \neq (0)$ but $\mathfrak{p}^2 = (0)$.*

Theorem 41 ([94]). *Let R be a commutative ring with identity and $\mathfrak{m}$ be the unique maximal ideal of R. Then $G(R)$ is tripartite (but not bipartite) if and only if R is a principal ideal ring and $\mathfrak{m}$ is the unique prime ideal of R such that $\mathfrak{m}^3 \neq (0)$ but $\mathfrak{m}^4 = (0)$.*

2.7. Girth of $G(R)$

The length of a shortest cycle in a graph is known as the *girth* of a graph. The girth of a graph is infinity if it contain no cycle. For a finite local commutative principal ideal ring R with identity, the girth of $G(R)$ was found in [73]. For $n \in \mathbb{N}$, the girth of $G(\mathbb{Z}_n[i])$ also studied in [73].

Theorem 42 ([73]). *Let R be a finite local commutative principal ideal ring with identity. If Nilpotency$(R) \geq 4$ or R is a product of two finite non-trivial principal ideal rings with at least one of them is not a field, then the girth of $G(R)$ is 3.*

Theorem 43 ([73]). *Let p be a prime number such that p is congruent to 1 modulo 4 and $q, q_i, i \in \mathbb{N}$ be prime integers such that q, q_i are congruent to 3 modulo 4. Then the girth of $G(\mathbb{Z}_n[i])$ is 3 if $n \neq 1, 2, p, q, q^2, q^3, q_1q_2$.*

2.8. Eulerian property of $G(R)$

A circuit containing all the edges of a graph is an *Eulerian circuit.* A graph is *Eulerian* if it has an Eulerian circuit.
The Eulerian property of finite commutative principal ideal ring with identity in the both local and non-local cases were described in [73].

Theorem 44 ([73]). *Let R be a finite local commutative principal ideal ring with identity. Then $G(R)$ is Eulerian if and only if Nilpotency(R) is even.*

Theorem 45 ([73]). *Let R be a finite commutative non-local principal ideal ring with identity. Then $G(R)$ is Eulerian if and only if R is a product of fields or a product of local rings each of which has even nilpotency number.*

The Eulerian property of $G(\mathbb{Z}_n)$ was studied in [26].

Theorem 46 ([26]). *Let $n \in A$. Then $G(\mathbb{Z}_n)$ is Eulerian if and only if $n = p_1p_2 \dots p_k$ or $n = p_1^{n_1}p_2^{n_2} \dots p_k^{n_k}$, where each n_i is even ($n_i \in \mathbb{N},\ i = 1, 2, \dots, k$) and p_i's are distinct primes.*

Theorem 47 ([73]). *Let q be a prime integer where q is congruent to 3 modulo 4, t_j for $j = 1, 2, \dots, l, l \in \mathbb{N}$ be distinct odd primes and k be any non-negative integer. Then $G(\mathbb{Z}_n[i])$ is Eulerian if and only if $n = 2^l$ or $n = q^{2l}$ or $n = \prod_{j=1}^{l} t_j$ or $n = 2^k \prod_{j=1}^{l} {t_j}^{2l_j}$.*

2.9. Hamiltonian property of $G(R)$

A *Hamiltonian path* is a spanning path that contains all the vertices. A *Hamiltonian cycle* is a spanning cycle (a cycle through every vertex) in a graph. A simple graph is *Hamiltonian* if it has a Hamiltonian cycle. For a commutative ring R with identity, we have the following results for $G(R)$ to be Hamiltonian.

Proposition 7 ([73]). *For a finite local commutative principal ideal ring R with identity, the graph $G(R)$ is Hamiltonian if and only if Nilpotency$(R) > 3$.*

Theorem 48 ([73]). *Let R be a finite commutative principal ideal ring with identity. If $R = \prod_{j=1}^{l} R_j$, where R_j is a finite local principal ideal ring which is not a field for each j, then $G(R)$ is Hamiltonian.*

Theorem 49 ([73]). *Let R be a finite commutative principal ideal ring with identity and $R = F \times K$, where F is a finite field and K is a finite principal ideal ring. If $G(K)$ is Hamiltonian, then $G(R)$ is Hamiltonian.*

Theorem 50 ([73]). *Let R be a finite commutative non-local principal ideal ring with identity. Then $G(R)$ is Hamiltonian if and only if R has more than 4 proper non-trivial ideals.*

Theorem 51 ([73]). *Let $p,\ q, q_l, l \in \mathbb{N}$ be prime numbers, where p is congruent to 1 modulo 4 and q, q_l congruent to 3 modulo 4. Then $G(\mathbb{Z}_n[i])$ is Hamiltonian if and only if $n \neq p, q, q_1q_2, q_1q_2^2, 2q$.*

Hoseini et al. [58] proved various results on Hamiltonian property of $G(R)$, where R is an Artinian ring. Moreover they studied the pancyclic property of $G(R)$.

Theorem 52 ([58]). *Let R be a commutative Artinian ring with identity. Then $G(R)$ is Hamiltonian if and only if R is not isomorphic to the following rings:*

- **(i)** *F or $E \oplus F$;*
- **(ii)** *S or $E \oplus S$ such that (S, F) is a local ring;*
- **(iii)** *S such that (S, T) is a local ring and (T, F) is a local ring, where E and F are fields.*

A graph $G = (V, E)$ with vertex set V and edge set E is *pancyclic* if it contains cycles of all lengths l, $3 \leq l \leq |V|$ (refer [20]).

Theorem 53 ([58]). *Let R be a commutative Artinian ring with identity. Then $G(R)$ is Hamiltonian if and only if it is pancyclic.*

Lemma 3 ([26]). *Let $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \in A$, where $k, n_j \in \mathbb{N}$, $(k, n_1) \neq (1,1)$, p_j are distinct primes ($j = 1, 2, \ldots, k$). Then $G(\mathbb{Z}_n)$ is Hamiltonian if the following conditions hold:*

$$(n_1 + 1)(n_2 + 1) \cdots (n_k + 1) \geqslant 5 \quad (2)$$

$$n_1 n_2 \ldots n_k - 1 \geqslant k. \quad (3)$$

Lemma 4 ([26]). *Let $n = p\alpha$, where p is a prime number such that $p \nmid \alpha$ and $\alpha \in A$. If $G(\mathbb{Z}_\alpha)$ is Hamiltonian, then $G(\mathbb{Z}_n)$ is Hamiltonian.*

Theorem 54 ([26]). *All the graphs $G(\mathbb{Z}_n)$ ($n \in A$) are Hamiltonian except when n is one of the following forms: p^2, pq, p^2q, p^3, where p, q are distinct primes.*

2.10. The clique number, chromatic number, edge chromatic number and perfectness of $G(R)$

A complete subgraph of a graph G is called a *clique* of G. The number of vertices in a largest clique of G is called the *clique number* of G and denoted by $\omega(G)$. The *chromatic number*, $\chi(G)$ of a graph G is defined as the minimum number of colors required to color a graph G such that no two adjacent vertices receive the same color. The *edge chromatic number* of a graph G denoted by $\chi'(G)$ is defined as the minimum number of colors required to color the edges of the graph G such that no two edges incident on the same vertex receives the same color. A graph G is said to be *weakly perfect* if the clique number and the chromatic number are equal.
Akbari et al. [5] studied the clique number and the chromatic number of $G(R)$, where R is a ring with identity. In general, there are many graphs whose clique number is finite but chromatic number is not. But in the case of $G(R)$, the authors of [5] have observed that if the clique number of $G(R)$ is finite, then its chromatic number is also finite. In the case of reduced ring R, the clique number of $G(R)$ and the chromatic number of $G(R)$ both are equal.

Lemma 5 ([5]). *Let D be an infinite division ring and $n \geq 2$. Then $M_n(D)$ has an infinite number of maximal left ideals. Moreover, if $n \geq 3$, then every two distinct maximal left ideals are adjacent in $G(M_n(D))$ and $\omega(G(M_n(D)))$ is infinite.*

Lemma 6 ([5]). *Let R be a ring with identity such that $\omega(G(R)) < \propto$. Then R is a left Artinian ring.*

Theorem 55 ([5]). *Let R be a ring with identity and $\omega(G(R)) < \propto$. Then $\chi(G(R)) < \propto$.*

Theorem 56 ([5]). *Let R be a ring with identity and $\omega(G(R)) < \propto$. Then the following hold:*

(i) *If R is commutative and non-local, then $G(R)$ is finite;*

(ii) *If R is a local with the unique maximal left ideal $\mathfrak{m}$, then either $G(R)$ is finite or the size of every minimal generating set of $\mathfrak{m}$ is 2.*

Corollary 3 ([5]). *Let R be a Gorenstein ring. Then $\omega(G(R)) = n$ if and only if R has n non-trivial ideals.*

Theorem 57 ([5]). *Let R be a ring with identity which is reduced and $\omega(G(R)) <\propto$. Then $\omega(G(R)) = \chi(G(R))$.*

The clique number and the chromatic number of the intersection graph of a direct product of rings were also studied in [5].

Theorem 58 ([5]). *Let R be a ring with identity and $R \cong R_1 \times \cdots \times R_k$, where each R_i is a ring, then $\omega(G(R)) <\propto$ if and only if $\omega(G(R_i)) <\propto$, for every i, $1 \leq i \leq k$.*

Theorem 59 ([5]). *Let R be a ring with identity and $R \cong R_1 \times \cdots \times R_k$, where each R_i is a ring, then $\chi(G(R)) <\propto$ if and only if $\chi(G(R_i)) <\propto$, for every i, $1 \leq i \leq k$.*

Theorem 60 ([4]). *Let $n \geq 2$ be a positive integer and $R = M_n(\mathbb{F}_q)$, where $\mathbb{F}_q$ is a finite field of order q. Then the clique number of $G(R)$,*

$$\omega(G(R)) = \begin{cases} \begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_q + \sum_{i=\frac{n}{2}+1}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix}_q, & \textit{if } n \textit{ is even,} \\ \sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix}_q, & \textit{if } n \textit{ is odd.} \end{cases}$$

Nikandish and Nikmehr in [72], studied the clique number, chromatic number and edge chromatic number of $G(\mathbb{Z}_n)$ and proved that $G(\mathbb{Z}_n)$ is a class of weakly perfect graphs for every positive integer n.
Now we mention some concepts and notations that are going to apply in the following results in this section defined by the authors [72].
Let $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \in \mathbb{N}$, where p_i's are distinct primes, n_i's are natural numbers and $n_1 \leq n_2 \leq \cdots \leq n_k$, we know that, $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_k^{n_k}}$. Again for an ideal I of $\mathbb{Z}_n$, we have $I \in I(\mathbb{Z}_n)$ if and only if $I = I_1 \times \cdots \times I_k$, where $I_i \in I(\mathbb{Z}_{p_i{}^{n_i}})$. Let $I(\mathbb{Z}_n)^*$ denote the set of all non-trivial ideals of $\mathbb{Z}_n$. Then $|I(\mathbb{Z}_n)^*| = \prod_{i=1}^{k}(n_i+1)-2$. Each $I_i, 1 \leq i \leq k$ is called a *component* of I and I_i is a *zero component* if $I_i = 0$ for $1 \leq i \leq k$. Let t be an integer such that $1 \leq t \leq k$ and T be the set of all ideals of $\mathbb{Z}_n$ with exactly t non-zero components in which $I_{i_j} \neq 0$, $1 \leq j \leq t$. Thus T can be considered as the set of ideals with no zero component of the subring $\mathbb{Z}_{p_{i_1}^{n_{i_1}}} \times \cdots \times \mathbb{Z}_{p_{i_t}^{n_{i_t}}}$.

1. The cardinality of T is denoted by $W(T)$.
2. S be the family of ideals with no zero component of subring $\mathbb{Z}_{p_{j_1}^{n_{j_1}}} \times \cdots \times \mathbb{Z}_{p_{j_l}^{n_{j_l}}}$. Then $S \subseteq T$, if $\{p_{j_1}^{n_{j_1}}, \ldots, p_{j_l}^{n_{j_l}}\} \subseteq \{p_{i_1}^{n_{i_1}}, \ldots, p_{i_t}^{n_{i_t}}\}$.

3. If $\{p_{j_1}^{n_{j_1}},\dots,p_{j_l}^{n_{j_l}}\}\cap\{p_{i_1}^{n_{i_1}},\dots,p_{i_t}^{n_{i_t}}\}=\phi$, then $T\cap S=\phi$.
4. S is said to be complement of T if $T\cap S=\phi$ and $\{p_{j_1}^{n_{j_1}},\dots,p_{j_l}^{n_{j_l}}\}\cup\{p_{i_1}^{n_{i_1}},\dots,p_{i_t}^{n_{i_t}}\}=\{p_1^{n_1},\dots,p_k^{n_k}\}$ and $T^c=S$.
5. Consider the set $C=\{S: W(S)>W(S^c)\}\cup A'$, where $S\in A'$ if and only if $W(S)=W(S^c)$ and $|A'\cap\{S,S^c\}|=1$.

Theorem 61 ([72]). *Let $n=p_i^{n_i}$. Then $\omega(G(\mathbb{Z}_n))=\chi(G(\mathbb{Z}_n))=n_i-1$.*

Theorem 62 ([72]). *The graph $G(\mathbb{Z}_n)$ is weakly perfect, for every $n>0$.*

Theorem 63 ([72]). *Let $n_k\geq\prod_{i=1}^{n_k-1}n_i$. Then $\omega(G(\mathbb{Z}_n))=\chi(G(\mathbb{Z}_n))=n_k\prod_{i=1}^{n_k-1}(n_i+1)-1$.*

Corollary 4 ([72]). *Let $n_1=n_2=\cdots=n_k=1$. Then $\omega(G(\mathbb{Z}_n))=\chi(G(\mathbb{Z}_n))=2^{k-1}-1$.*

Theorem 64 ([72]). *Let $k>1$ be an odd integer such that $\prod_{i=0}^{\lfloor\frac{k}{2}\rfloor-1}n_{k-i}\leq\prod_{i=1}^{k-\lfloor\frac{k}{2}\rfloor}n_i$. Then $\omega(G(\mathbb{Z}_n))=\chi(G(\mathbb{Z}_n))=|\{I\in I(\mathbb{Z}_n)^*|I$ has at most $\lfloor\frac{k}{2}\rfloor$ zero components$\}|$.*

Corollary 5 ([72]). *Let k be an odd integer and $n_1=n_2=\cdots=n_k=a$. Then $\omega(G(\mathbb{Z}_n))=\chi(G(\mathbb{Z}_n))=\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{i}a^{k-i}-1$.*

Theorem 65 ([72]). *Let $k>2$ be an even integer such that $\prod_{i=0}^{\frac{k}{2}-2}n_{k-i}\leq\prod_{i=1}^{\frac{k}{2}+1}n_i$. Then $\omega(G(\mathbb{Z}_n))=\chi(G(\mathbb{Z}_n))=|\{I\in I(\mathbb{Z}_n)^*|I$ has at most $\frac{k}{2}-1$ zero components$\}|+\sum_{T\in A'}W(T)$.*

Corollary 6 ([72]). *Let k be an even integer and $n_1=n_2=\cdots=n_k=a$. Then $\omega(G(\mathbb{Z}_n))=\chi(G(\mathbb{Z}_n))=\sum_{i=0}^{\frac{k}{2}-1}\binom{k}{i}a^{k-i}+\frac{\binom{k}{\frac{k}{2}}a^{\frac{k}{2}}}{2}-1$.*

In [72], Nikandish and Nikmehr observed (from the results obtained by Akbari et al. in [5]) that, for every ring R with identity, if $\omega(G(R))<\infty$, then $\chi(G(R))<\infty$, also could not find any example of a ring R such that $G(R)$ is not weakly perfect. These positive results motivated the authors [72] to state the following conjecture.

Conjecture 1 ([72]). For every ring R with identity, $G(R)$ is a weakly perfect graph.

The edge chromatic number of $G(\mathbb{Z}_n)$ was also computed in [72]. For various results related to chromatic number one can refer [65] and [75].

Theorem 66 ([72]). *Let n be a positive integer. Then $\chi'(G(\mathbb{Z}_n))=\Delta(G(\mathbb{Z}_n))$, unless $n=p_1p_2$, where p_1,p_2 are distinct primes or $n=p^k$, where k is an even integer.*

A graph G is said to be *perfect* if $\omega(H) = \chi(H)$ for all induced subgraphs H of G. Das has [39] characterised the values of n such that $G(\mathbb{Z}_n)$ is perfect.

Theorem 67 ([39]). *The graph $G(\mathbb{Z}_n)$ is perfect if and only if $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} p_4^{k_4}$, where p_i's are distinct primes and $k_i \in \mathbb{N} \cup \{0\}$ for $i = 1, 2, 3, 4$, that is, the number of distinct prime factors of n is less than or equal to 4.*

Coloring of a commutative ring is studied in [8] and [13].

2.11. The independence number and the domination number of $G(R)$

A set of vertices in a graph $G = (V, E)$ is said to be an *independent set* if no two vertices in the set are adjacent. The number of vertices in the largest independent set of a graph G is known as the *independence number* of the graph G and it is denoted by $\alpha(G)$. A *dominating set* D is a subset of V such that every vertex not in D is adjacent to at least one vertex of D. The *domination number* $\gamma(G)$, is the minimum cardinality among all dominating sets of G.

Theorem 68 ([5]). *Let R be a commutative ring with identity. If R is a reduced ring and $|Min(R)| <\propto$, where $Min(R)$ is the set of all minimal prime ideals of R. Then $\alpha(G(R)) = |Min(R)|$.*

The independence number of $G(R)$, where R is a finite commutative principal ideal ring with identity was studied in [74].

Theorem 69 ([74]). *If $R = \prod_{j=1}^{n} R_j$, where R_j is a finite local principal ideal ring for each j, then $\alpha(G(R)) = n$.*

Theorem 70 ([4]). *Let $n \geq 2$ be a positive integer and $R = M_n(\mathbb{F}_q)$, where $\mathbb{F}_q$ is a finite field of order q. Then the independence number of $G(M_n(\mathbb{F}_q))$ is $\frac{q^n-1}{q-1}$.*

In [62], Jafari and Rad studied the domination in $G(R)$, where R is a commutative ring with identity. One can refer [53] for various results related to domination parameters.

Remark 2 ([62]). For a field F, $\gamma(G(F)) = 0$.

Theorem 71 ([62]). *Let R_1, R_2 be two commutative rings with identity. Then $\gamma(G(R_1 \times R_2)) = 1$ if and only if $\gamma(G(R_1)) = 1$ or $\gamma(G(R_2)) = 1$.*

Theorem 72 ([4, 62]). *Let R be a commutative ring with identity. Then $\gamma(G(R)) \leq 2$.*

Theorem 73 ([62]). *Let R be a commutative Artinian ring with identity. Then $\gamma(G(R)) = 2$ if and only if $R = R_1 \times R_2 \times \cdots \times R_t$, where $t \geq 2$ and R_i is a field for $i = 1, 2, \ldots, t$.*

Osba et al. in [74] studied the domination of $G(R)$, where R is finite commutative principal ideal ring with identity. They considered both the cases when R is local and R is non-local.

Proposition 8 ([74]). *Let R be a finite local commutative principal ideal ring with identity. Then $\gamma(G(R)) = 1$.*

Theorem 74 ([74]). *Let R be a finite non-local commutative principal ideal ring with identity and $R = \prod_{j=1}^{n} R_j$, where for each j, R_j is a local ring. Then the following statements hold:*

(i) *$\gamma(G(R)) = 1$, if at least one factor R_j, $1 \leq j \leq n$ is not a field;*

(ii) *$\gamma(G(R)) = 2$, if R is a product of fields.*

In the article [4], S. Akbari et al. classified all rings R (with identity) such that the domination number of $G(R)$ is at least 2. Moreover the domination number of $G(M_n(\mathbb{F}_q))$ was also determined in [4].

Theorem 75 ([4]). *Let R be a ring with identity. Then the following statements hold:*

(i) *$\gamma(G(R)) > 2$ if and only if $R \cong M_n(D)$, where D is a division ring;*

(ii) *$\gamma(G(R)) = 2$ if and only if $R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$, where $k > 1$ and for every i, $n_i \geq 1$ and D_i is a division ring.*

Corollary 7 ([4]). *Let $G(R)$ be a disconnected graph, where R is a ring with identity. Then $\gamma(G(R)) \geq 2$ and one of the following conditions holds:*

(i) *$\gamma(G(R)) > 2$ if and only if $R \cong M_2(D)$, where D is a division ring;*

(ii) *$\gamma(G(R)) = 2$ if and only if $R \cong D_1 \times D_2$, where D_1 and D_2 are two division rings.*

Corollary 8 ([4]). *Let R be a commutative ring with identity which is not a field. Then $\gamma(G(R)) = 2$ if and only if R is a direct product of finitely many fields.*

Corollary 9 ([4]). *Let R be a commutative Artinian ring with identity which is not a field. Then one of the following holds:*

(i) *$\gamma(G(R)) = 1$ if and only if R is not reduced;*

(ii) *$\gamma(G(R)) = 2$ if and only if R is reduced.*

Theorem 76 ([4]). *Let $n \geq 2$ be a positive integer and $R \cong M_n(\mathbb{F}_q)$, where $\mathbb{F}_q$ is finite field of order q. Then $\gamma(G(R)) = q + 1$.*

2.12. Planarity, Outerplanarity, Toroidality, Genus, Crosscap of $G(R)$

In this section, we present the results associated with planarity, outerplanarity, toroidality, genus and crosscap of $G(R)$, where R is a commutative ring with identity. A graph is *planar* if it can be drawn in a plane without crossing of edges other than endpoints. From topological point of view, drawing a graph on a flat plane is equivalent to drawing the same graph on a sphere. Thus a graph is planar if and only if it can be drawn on the sphere in such a way that no edges cross. A graph G is homeomorphic from G_1 if it is possible to insert points of degree two into the lines of G_1 to produce G [28]. It follows from Kuratowski's Theorem [66] that a graph is planar if and only if it contains no subgraph homeomorphic from the complete graph K_5 or from the complete bipartite graph $K_{3,3}$. A planar graph is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the same face [51]. A graph G is outerplanar if and only if it contains no subgraph homeomorphic from K_4 or $K_{2,3}$ [28].

A study of embedding a graph into a surface is one of the important topic in topological graph theory. By a surface, we mean a connected topological space such that each point has a neighborhood homeomorphic to an open disk. It is well known that any compact surface is either homeomorphic to a sphere or to a connected sum of g tori or to a connected sum of k projective planes [70].

Let S_k denote the sphere with k handles, where k is a non-negative integer. The sphere is designated to be the surface S_0. A torus can be thought of as a sphere with one handle, so the torus is S_1. A graph is said to be *toroidal* if the vertices of the graph can be placed on a torus such that no edges cross. The double-torus S_2 is a sphere with two handles. The *genus* of graph G is the minimum integer t such that G can be embedded in S_t. More precisely, a surface is said to be of genus g if it is topologically homeomorphic to a sphere with g handles. Thus the genus of a sphere is 0 and the genus of torus is one. A graph can be drawn without crossings on the surface of genus g, but not on one of genus $g-1$, is called a graph of *genus* g. We write $g(G)$ for the genus of a graph G. For more details on genus and genus of some graphs over algebraic structures one can refer [91–93, 100, 102–104].

Inspired by the articles [76, 77], the author [80] studied the embedding of $G(R)$ into a non-orientable compact surface. We know that every non-orientable compact surface is a connected sum of finite copies of projective planes [70]. The number of the copies of projective spaces is called the crosscap number of this surface. The projective space is of crosscap one and the Klein bottle is of crosscap two. A non-planar graph is said to be projective if it can be embedded into the projective plane. For a non-negative integer k, let N_k denote the surface formed by connected sum of k projective planes, that is, N_k is an non-orientable surface of crosscap k. A simple graph which is embedded in N_k but not in N_{k-1} is called a graph of crosscap k, denoted by $\bar{\gamma}(G)$. A graph G is called *projective* if $\bar{\gamma}(G) = 1$. The reader can refer [34, 84] for crosscap of various graphs over algebraic structure.

2.12.1. Planarity of $G(R)$

Jafari and Jafari Rad in [61] had characterized the commutative ring R with identity such that $G(R)$ is planar.

Proposition 9 ([61]). *Let R be a commutative ring with identity. If $G(R)$ is planar, then*

(i) *any chain of ideals of R has length at most five;*

(ii) *R is both Noetherian and Artinian.*

Proposition 10 ([61]). *Let R_1, R_2 be commutative rings with identity. Then $G(R_1 \times R_2)$ is planar if and only if one of $G(R_1)$, $G(R_2)$ is empty, and another is empty or null with at most two vertices.*

Proposition 11 ([61]). *Let R_i, $i = 1, 2, 3$ be commutative rings with identity. Then $G(R_1 \times R_2 \times R_3)$ is planar if and only if R_i is a field for $i = 1, 2, 3$.*

Lemma 7 ([61]). *Let R be commutative ring with identity. If $G(R)$ is planar, then $|Max(R)| \leq 3$, where $Max(R)$ denote the set of all maximal ideals of R.*

Theorem 77 ([61]). *Let R is a commutative ring with identity and $|Max(R)| = 3$, where $Max(R)$ denote the set of all maximal ideals of R. Then $G(R)$ is planar if and only if $R = R_1 \times R_2 \times R_3$, where R_i is a field for $i = 1, 2, 3$.*

Theorem 78 ([61]). *Let R is a commutative ring with identity and $|Max(R)| = 2$, where $Max(R)$ denote the set of all maximal ideals of R . Then $G(R)$ is planar if and only if one of $G(R_1), G(R_2)$ is empty, and another is empty or null with one vertex.*

Theorem 79 ([61]). *Let R is a commutative ring with identity and $|Max(R)| = 1$, where $Max(R)$ denote the set of all maximal ideals of R. Then $G(R)$ is planar if and only if $G(R)$ is a star, K_1, K_3, or K_4.*

Example 9 ([26]). Let $n \in A$ and $n = p^i$, where $2 \leqslant i \leqslant 5$, p is a prime. Then $G(\mathbb{Z}_n)$ is planar.

Example 10 ([26]). Let $n \in A$ and $n = pq, p^2q, pqr$, where p, q, r are distinct primes. Then $G(\mathbb{Z}_n)$ is planar.

Continuing the work done by the authors [61] on planarity of $G(R)$, Theorem 80 was asserted by Z. S. Pucanović et al. in [76] for $G(R)$, where R is a commutative ring with identity. Theorem 80 characterizes the ring R such that $G(R)$ is a star graph.

Theorem 80 (**[76]**). *Let R be a local commutative ring with identity and the maximal ideal $\mathfrak{m} =< x, y >$, where x, y are minimal generators. If $G(R)$ is planar, then $\mathfrak{m}^2 = 0$ and $G(R)$ is either an infinite star graph or it is isomorphic to K_{1,p^k+1}, for some prime number p and positive integer k.*

For a finite commutative principal ideal ring R with identity, in [73], E. A. Osba obtained the following results on planarity of $G(R)$.

Theorem 81 (**[73]**). *Let R be a finite local principal ideal ring with identity, then $G(R)$ is planar if and only if $1 < Nilpotency(R) < 6$.*

Theorem 82 (**[73]**). *Let $R = R_1 \times R_2$ be a finite non-local principal ideal ring with identity. Then $G(R)$ is planar if and only if R_1 and R_2 are fields or R_1 is a field and $Nilpotency(R_2) = 2$.*

Theorem 83 reveals the planarity of $G(\mathbb{Z}_n[i])$.

Theorem 83 (**[73]**). *The intersection graph $G(\mathbb{Z}_n[i])$ is planar if and only if $n = 2, 4, p, 2q, qp, q_1q_2, q_1q_2^2, q_1q_2q_3, q^m$ with $1 < m < 6$, where p denote the prime numbers that are congruent to 1 modulo 4 and $q, q_l, l \in \mathbb{N}$ denote the prime numbers that are congruent to 3 modulo 4.*

Remark 3. It can be observed that there are infinitely many planar graphs that are intersection graphs of ideals of some rings.

2.12.2. Outerplanarity of $G(R)$

Theorem 84 (**[80]**). *Let R be a commutative Artinian non-local ring with identity. Then $G(R)$ is outerplanar if and only if R is isomorphic to $R_1 \times F_1$ or $F_1 \times F_2$ or $F_1 \times F_2 \times F_3$, where R_1 is a local ring with maximal ideal $\mathfrak{m}_1$ such that the only non-trivial ideal of R_1 is $\mathfrak{m}_1$ and ${\mathfrak{m}_1}^2 = (0)$ and each F_i is a field for $i = 1, 2, 3$.*

Theorem 85 (**[80]**). *Let R be a commutative ring with identity with the unique maximal ideal $\mathfrak{m}$ such that $\mathfrak{m} \neq (0)$ is finitely generated. Then $G(R)$ is outerplanar if and only if the following conditions hold:*

(i) *$\mathfrak{m} =< x, y >$, where $x, y \in \mathfrak{m}$ such that $\mathfrak{m}^2 = (0)$;*

(ii) *R is a principal ideal ring with the maximal ideal $\mathfrak{m}$ of nilpotency at most four.*

Theorem 86 (**[80]**). *Let R be a commutative ring with identity such that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the only two maximal ideals of R. If $G(R)$ is outerplanar then $R \cong {\mathfrak{m}_1}^2 \times {\mathfrak{m}_2}^2$.*

2.12.3. Toroidality of $G(R)$

A toroidal graph is of genus one and can be embedded in a torus but not in a plane. The complete graphs K_5, K_6, K_7, the complete bipartite graph $K_{3,3}$ are toroidal.

Proposition 12 ([76]). *If the field $F(=\frac{R}{\mathfrak{m}})$ is infinite, then the graph $G(R)$ does not have a finite genus where R is a local commutative ring with identity and $\mathfrak{m}$ is the maximal ideal which is minimally generated by two elements such that $\mathfrak{m}^2 \neq 0$.*

Let $G = (V, E)$ be a graph and $G' = (V', E')$ be a subgraph of G. We use the notation $G[V']$ to denote the induced subgraph spanned by V' and the edge $\{x, y\}$ by xy. If $F \subseteq [V]^2$, then $G - F := (V, E \setminus F)$. Similarly, if $W \subseteq V$, then $G - W := G[V \setminus W]$.

Remark 4 ([76]). The graph Γ is a complete graph on seven vertices with some additional vertices and edges (refer Figure 7). Γ is embedded on a torus. All toroidal graphs $G(R)$ are subgraphs of Γ.

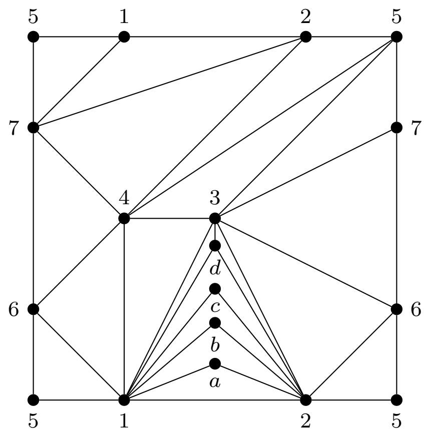

Figure 7. **The graph Γ**

Theorem 87 ([76]). *Let R be a commutative ring with identity. Then $G(R)$ is toroidal if and only if it is isomorphic to one of the following graphs: K_5, K_6, K_7, $\Gamma[1, 2, 3, 4, 5, a]$, $\Gamma[1, 2, 3, 4, 5, 6, 7] - \{36, 37, 46, 47\}$, $\Gamma[1, 2, 3, 4, 5, 6, 7, d]$, $\Gamma[1, 2, 3, 4, 5, a, b]$, $\Gamma[1, 2, 3, 4, 5, 6, a, b, c], \Gamma - \{3d\}$.*

2.12.4. The genus of $G(R)$

Genus is a topological property of a graph and finding the genus of a graph is NP-complete problem proved by Thomassen in [93]. The authors Z. S. Pucanović, M. Radovanović and A. Erić in [77] studied the genus of $G(R)$, where R is a commutative ring with identity. They wanted to study the embedding of the intersection graphs of ideals of commutative rings in double torus S_2. They had classified all graphs of genus 2 that are intersection graphs of ideals of some commutative rings. Some lower bounds on n for the genus of the intersection graph of ideals of a non-local commutative ring was obtained in [77].

Proposition 13 ([77]). *Let R be a commutative ring with identity. If $G(R)$ has finite genus, then R is an Artinian ring.*

Remark 5 ([77]). Z. S. Pucanović et al. presented the graphs Γ' and Γ'', embedded in a double torus in Figure 6 and Figure 8, respectively. All intersection graphs $G(R)$ of genus 2 are either Γ' or subgraphs of Γ''.

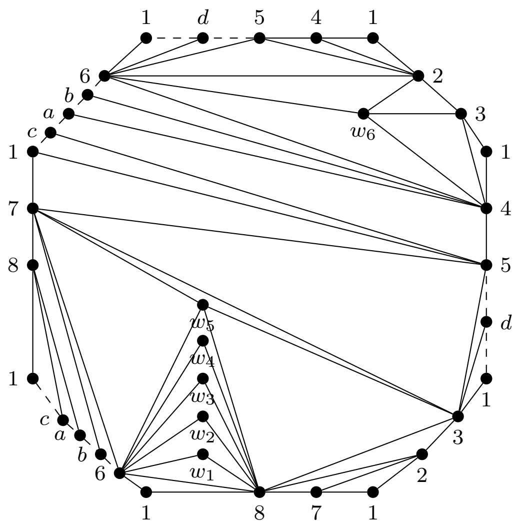

Figure 8. **The graph Γ''**

Now we investigate the graphs $G(R)$ such that the genus of $G(R)$ is 2.

Theorem 88 ([77]). *Let R be a commutative ring with identity. Then the genus of $G(R)$ is two if and only if $G(R)$ is isomorphic to one of the following graphs: $K_8, \Gamma', \Gamma''[1,\ldots,8,w_5,w_6]-\{13,17\}, \Gamma''[1,\ldots,8,w_1,\ldots,w_5]-\{7w_5,3w_5\}$.*

See Figure 6 and Figure 8 for graphs Γ' and Γ'', respectively.
For non-local ring R, Theorem 89 gives the lower bound of the genus of $G(R)$ [77].

Theorem 89 ([77]). *Let R be a commutative Artinian ring with identity and R_i, $i = 1,2,\ldots,k$ be local Artinian rings such that $R \cong R_1 \times \cdots \times$*

R_k, where $k \geq 2$. If R is non-local, then the genus of $G(R)$ is at least $\min\left\{\frac{\alpha}{8} \cdot N^{\frac{2k-2}{k}} \cdot (N^{\frac{1}{k}} - \alpha) - \frac{N}{2} + 1, \beta \cdot N^2 - \frac{N}{2} + 1, \frac{(N-6)(N-8)}{48}\right\}$, where $N = |V(G(R))|$, $\alpha = 2k(\frac{1}{3})^{\frac{k-1}{k}}$ and $\beta = \frac{3^k - 2^k - 1}{4(2\cdot 3^k - 2^{k+1} - 1)^2}$.

Remark 6. As a consequence of Theorem 89, for a non-local commutative Artinian ring R, the genus of $G(R)$ is very large.

Remark 7. For a graph $G(R)$, it is easier to find the genus of $G(R)$ if the ring R is non-local commutative ring with identity, then when the ring R is local commutative ring with identity.

As a conclusion remark of [77], the authors stated Theorem 90.

Theorem 90 ([77]). *For every positive integer g, there are only finitely many non-isomorphic graphs of genus g that are intersection graphs of some rings.*

Proposition 14 ([82]). *The genus of a complete grah K_n is given by $g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$, $n \geq 3$.*

Remark 8. Proposition 14 implies that the genus of K_n is zero for $n \leq 4$, the genus of K_n is one for $5 \leq n \leq 7$ and the genus of K_n is two if and only if $n = 8$.

2.12.5. Crosscap of $G(R)$

Theorem 91 ([80]). *Let R be a commutative Artinian ring with identity and $\mathfrak{m} \neq (0)$ be the unique maximal ideal of R. Then the crosscap of $G(R)$ is one if and only if $G(R) \cong K_5$ or $G(R) \cong K_6$.*

Theorem 92 ([80]). *Let R be a commutative Artinian ring with identity. Then the crosscap of $G(R)$ is two if and only if $R \cong R_1 \times R_2$, where each R_i for $i = 1, 2$, is local principal ideal ring with maximal ideal $\mathfrak{m}_i \neq (0)$ such that ${\mathfrak{m}_i}^2 = (0)$.*

Theorem 93 ([80]). *Let R be a commutative ring with identity and $\mathfrak{m}$ be the unique maximal ideal of R such that $\mathfrak{m}$ is generated by two elements, $\mathfrak{m}^2 \neq (0)$ and the field $\frac{R}{\mathfrak{m}}$ is finite. If $G(R)$ is projective graph, then $\mathfrak{m}^2$ is a principal ideal.*

2.13. Geodetic number and Hull number of $G(R)$

A shortest path between two vertices v_1 and v_2 in a graph $G = (V, E)$ is called a $v_1 - v_2$ *geodesic*. The set of all vertices in G that lie on a $v_1 - v_2$ geodesic is denoted by $I[v_1, v_2]$. For any subset S of G, let $I[S] = \bigcup_{v_1, v_2 \in S} I[v_1, v_2]$. If $I[S] = V(G)$, then S is called a *geodetic set* of G. The minimum cardinality of a geodetic set of G is called the *geodetic number* of G and we denote the geodetic number number by $\mathbf{g}(G)$. A subset S of $V(G)$ is *convex* if $I\,[S] = S$. If A is a subset of $V(G)$, then the *convex hull* of A (denoted by $[A]$) is the smallest convex set in G containing A. If $[A] = V(G)$,

then A is called a *hull set* of G. The smallest cardinality of a hull set of G is called the *hull number* of G and is denoted by $h(G)$. For any graph G, $h(G) \leq \mathbf{g}(G)$. A vertex v in a graph G is called an *extreme vertex* if the subgraph induced by its neighbors is complete. The set of extreme vertices of G is denoted by $Ext(G)$ [74].
For finite commutative principal ideal ring R with identity, the geodetic number and the hull number of $G(R)$ were obtained in [74]. One can refer [29, 56] for more information regarding geodetic number and hull number of a graph.

Theorem 94 ([74]). *Let R be a finite local commutative principal ideal ring with identity and Nilpotency$(R) = n$. Then $\mathbf{g}(G(R)) = h(G(R)) = n - 1$.*

Theorem 95 ([74]). *Let R be a finite non-local commutative principal ideal ring with identity such that $R = \prod_{j=1}^{n} R_j$, where R_j is a local ring with Nilpotency$(R_j) = n_j$ for $j = 1, 2, \ldots, n$ and $T = \{\prod_{j=1}^{n} I_j : I_j$ is an ideal of R_j such that $I_j \neq \{0\}$for exactly one $j\}$. Then T is the set of extreme vertices of $G(R)$. Moreover, the set T is a geodetic set as well as a hull set of $G(R)$ and $\mathbf{g}(G(R)) = h(G(R)) = \sum_{j=1}^{n} n_j$.*

2.14. Conditions on R such that $G(R)$ is chordal

A graph is said to be *chordal* if it has no induced cycle of length greater than 3. The authors E. A. Osba et al. in [74] found the conditions on a finite commutative principal ideal ring R with identity such that $G(R)$ is chordal.

Theorem 96 ([74]). *Let R be a finite commutative principal ideal ring with identity which is not a field. Then $G(R)$ is chordal if and only if R is a product of at most three local rings.*

2.15. Conditions on R such that $G(R)$ is a split graph

A simple graph $G = (V, E)$ is a *split graph* if the vertex set V can be partitioned into Q and S such that Q induces a clique and S is an independent set.

Theorem 97 ([80]). *Let R be a commutative ring with identity such that $R \cong R_1 \times R_2 \times \cdots \times R_n$, $(n \geq 3)$, where each R_i for $i = 1, 2, \ldots, n$ is a local ring. Then $G(R)$ is a split graph if and only if $R \cong F_1 \times F_2 \times F_3$, where each F_i is a field, for $i = 1, 2, 3$.*

Theorem 98 ([80]). *Let R be a commutative ring with identity. Then $G(R)$ is a split graph if and only if $R \cong F_1 \times F_2$ or $R \cong R_1 \times F_1$, where F_1, F_2 are fields and R_1 is a local principal ideal ring.*

3. Conditions on the rings R and S to be isomorphic if the corresponding graphs G(R) is isomorphic to G(S)

Let R, S be two rings with identity such that $G(R) \cong G(S)$. S. Akbari et al. [4] mentioned the following results where the rings R and S are also isomorphic.

Theorem 99 (**[4]**). *Let $k, l \geq 2$ be two positive integers, $\mathbb{F}_p$, $\mathbb{F}_q$ be finite fields of order p, q respectively. If $G(M_k(\mathbb{F}_p)) \cong G(M_l(\mathbb{F}_q))$, then $k = l$ and $p = q$.*

Theorem 100 (**[4]**). *Let R be a ring with identity, $\mathbb{F}_q$ be a finite field of order q and $k \geq 2$ be a positive integer. If $G(R) \cong G(M_k(\mathbb{F}_q))$, then $R \cong M_k(\mathbb{F}_q)$.*

Theorem 101 (**[4]**). *Let R and S be two finite reduced rings and $k, l \geq 2$ be positive integers. If $G(M_k(R)) \cong G(M_l(S))$, then $k = l$ and $R \cong S$.*

4. Properties of complement of G(R)

The *complement* of a graph G is the graph $\overline{G}$ with all vertices of G in which there in an edge between two vertices u_1 and u_2 if and only if there exists no edge between u_1 and u_2 in the graph G. E. A. Osba et al. in [74] studied connectedness, diameter, radius, center, chromatic number of $\overline{G(R)}$, where R is a finite commutative principal ideal ring with identity.

Theorem 102 (**[74]**). *Let R be a finite commutative principal ideal ring with identity, which is not a field. Then $\overline{G(R)}$ is connected if and only if R is either local with Nilpotency$(R) = 2$ or product of fields.*

Theorem 103 (**[74]**). *For a finite commutative principal ideal ring R with identity, which is not a field, the diameter and the radius of $\overline{G(R)}$ is as the following:*

$$diam(\overline{G(R)}) = \begin{cases} 0, & \textit{if } R \textit{ is local with Nilpotency}\,(R) = 2 \\ 1, & \textit{if } R \textit{ is a product of two fields} \\ 3, & \textit{if } R \textit{ is a product of more than two fields} \\ \propto, & \textit{otherwise.} \end{cases}$$

$$\rho(\overline{G(R)}) = \begin{cases} 0, & \textit{if } R \textit{ is local with Nilpotency}(R) = 2 \\ 1, & \textit{if } R \textit{ is a product of two fields} \\ 2, & \textit{if } R \textit{ is a product of more than two fields} \\ \propto, & \textit{otherwise.} \end{cases}$$

Theorem 104 (**[74]**). *Let R be a finite commutative principal ideal ring with identity, which is not a field. If $R = \prod_{j=1}^{n} F_j$ where $n \geq 3$ and F_j is a field for each j, then $Cen\overline{(G(R))} = K_n$ and if R is not a product of $n \geq 3$ fields, then $\overline{G(R)}$ is self-centered.*

Theorem 105 (**[74]**). *Let R be a finite local commutative principal ideal ring with identity and Nilpotency$(R) = n$. Then the chromatic number and the clique number of $\overline{G(R)}$ is 1.*

Theorem 106 ([74]). *Let R_1 be a finite local commutative principal ideal ring with identity and let S be a finite commutative principal ideal ring with identity, which is not a field. Then $\chi(\overline{G(R_1 \times S)}) = \chi(\overline{G(S)}) + 1$.*

Corollary 10 ([74]). *Let R be a product of n finite local commutative principal ideal rings with identity, then the chromatic number and the clique number of $\overline{G(R)}$ is n.*

5. Properties of some subgraphs of G(R)

A *spanning subgraph* of a graph $G = (V, E)$ is a subgraph $H = (V_H, E_H)$ with $V_H = V$, that is, H and G have exactly the same vertex set.

5.1. Properties of the subgraph $H(R)$ of $G(R)$

Consider a commutative ring R with identity which is not a field. Inspired by the work done on the graph $G(R)$ in the articles [5, 26, 61, 76], S. Visweswaran and P. Vadhel [98], defined a spanning subgraph denoted by $H(R)$, whose vertex set is the set of all non-zero proper ideals of R. Any two distinct vertices I, J of $H(R)$ are joined by an edge if and only if $IJ \neq (0)$. It is known that for any ideals I, J of R, $IJ \subseteq I \cap J$, which gives if $IJ \neq (0)$, then $I \cap J \neq (0)$. The vertex set of $G(R)$ and $H(R)$ are identical and two distinct vertices I, J are adjacent in $H(R)$, then I, J are adjacent in $G(R)$ implying $H(R)$ to be a spanning subgraph of $G(R)$.
The authors of [98] had studied the connectedness, diameter, girth of the graph $H(R)$. Let R be a commutative ring with identity, then R is said to be *quasilocal* if R has only one maximal ideal and R is not necessarily Noetherian. Let A be an ideal and $\mathfrak{P}$ be a prime ideal of R. $\mathfrak{P}$ is said to be an associated prime of A (in the Bourbaki sense) if $\mathfrak{P} = (A :_R x)$ for some $x \in R$. In this case, we say $\mathfrak{P}$ is a $B-$prime of A in R (refer [55, 98]).

Theorem 107 ([98]). *Let R be a commutative ring with identity which is not a field and R has at least two non-zero proper ideals. Then $H(R)$ is connected if and only if either $(a), (b)$ or (c) holds:*

- **(a)** *R is quasilocal with $\mathfrak{m}$ as its unique maximal ideal such that $\mathfrak{m}$ is not a $B-$prime of (0) in R;*
- **(b)** *R has exactly two maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2$ such that $\mathfrak{m}_1 \bigcap \mathfrak{m}_2 \neq (0)$;*
- **(c)** *R has at least three maximal ideals.*

As a consequence of Theorem 107, the following results regarding the diameter of $H(R)$ can be obtained.

Corollary 11 ([98]). *Let the condition (a) or (c) of Theorem 107 holds. Then diam(H(R)) ≤ 2.*

Corollary 12 ([98]). *The* $diam(H(R)) = 2$ *if and only if* R *is not an integral domain.*

Corollary 13 ([98]). *Let the condition* (b) *of Theorem 107 holds and* R *is not an integral domain, then* $2 \leq diam(H(R)) \leq 3$.

Corollary 14 ([98]). *The* $diam(H(R)) = 3$ *if and only if both* $\mathfrak{m}_1$ *and* $\mathfrak{m}_2$ *are* $B-$*primes of* (0) *in* R.

Theorem 108 ([98]). *Let* R *be a commutative ring with identity, which is not a field. Then* $girth(H(R)) = 3$ *or* $\propto$.

Let $G = (V, E)$ be an undirected graph. Referring [9] and [98], we recall, two distinct vertices u, v of G are said to be orthogonal, denoted as $u \perp v$, if u and v are adjacent in G and there is no vertex w of G which is adjacent to both u and v in G; that is, the edge $u - v$ is not the edge of any triangle in G. A vertex v of G is said to be a complement of u if $u \perp v$. Further, G is complemented if each vertex of G admits a complement in G and G is uniquely complemented, if G is complemented and whenever the vertices u, v, w of G are such that $u \perp v$ and $u \perp w$, then a vertex x of G is adjacent to v in G if and only if x is adjacent to w in G. In [9], D. F. Anderson et al. determined the rings for which the zero-divisor graphs of the corresponding rings are complemented or uniquely complemented. S. Visweswaran and P. Vadhel in [98] characterized the rings R such that $H(R)$ is complemented.

A principal ideal ring R is called a *special principal ideal ring* (SPIR) if it has a unique prime ideal. Let $\mathfrak{p}$ denote the unique prime ideal of SPIR R. Then $\mathfrak{p}$ is necessarily nilpotent. If R is a SPIR with $\mathfrak{p}$ as its unique prime ideal, then we denote it by $(R, \mathfrak{p})$.

Theorem 109 ([98]). *Let* R *be a commutative ring with identity which is not a field. Then* $H(R)$ *is complemented if and only if* $R \cong F \times S$, *where* F *is a field and* $(S, \mathfrak{a})$ *is a SPIR with* $\mathfrak{a} \neq (0)$ *but* $\mathfrak{a}^2 = (0)$.

Inspired by the results obtained on planarity of $G(R)$ in [61] and [76], P. Vadhel and S. Visweswaran focused their study on planarity of $H(R)$ in [95]. The authors [95] characterized the rings R such that $H(R)$ is planar, where R contains at least two maximal ideals.

Remark 9. For convenience, the authors [95] named the following conditions satisfied by an undirected simple graph $G = (V, E)$:

(C_1) G does not contain K_5 as a subgraph;

(C_2) G does not contain $K_{3,3}$ as a subgraph;

(C_1^*) G satisfies (C_1) and G does not contain any subgraph homeomorphic to K_5;

(C_2^*) G satisfies (C_2) and G does not contain any subgraph homeomorphic to $K_{3,3}$.

Theorem 110 ([95]). *Let R be a commutative ring with identity such that $|Max(R)| = 3$, where $Max(R)$ denote the set of all maximal ideals of R. Then the following statements are equivalent:*

(i) *$H(R)$ satisfies (C_1);*

(ii) *$R \cong F_1 \times F_2 \times F_3$ as rings, where F_i is a field for each $i \in \{1, 2, 3\}$;*

(iii) *$H(R)$ is planar;*

(iv) *$H(R)$ satisfies (C_2);*

(v) *$H(R)$ satisfies both (C_1^*) and (C_2^*).*

Theorem 111 ([95]). *Let R be a commutative ring with identity such that $|Max(R)| = 2$, where $Max(R)$ denote the set of all maximal ideals of R. Then the following statements are equivalent:*

(i) *$H(R)$ satisfies (C_1);*

(ii) *R is isomorphic to one of the rings of the following type:*

(a) *$F_1 \times F_2$, where F_i is a field for each $i \in \{1, 2\}$;*

(b) *$T_1 \times F_2$, where $(T_1, \mathbf{n_1})$ is a SPIR with $\mathbf{n_1} \neq (0)$ but ${\mathbf{n_1}}^4 = (0)$ and F_2 is a field;*

(c) *$T_1 \times T_2$, where $(T_i, \mathbf{n_i})$ is a SPIR with $\mathbf{n_i} \neq (0)$ but ${\mathbf{n_i}}^2 = (0)$ for each $i \in \{1, 2\}$.*

(iii) *$H(R)$ is planar;*

(iv) *$H(R)$ satisfies (C_2);*

(v) *$H(R)$ satisfies both (C_1^*) and (C_2^*).*

Remark 10. The ring R is not necessarily Noetherian in Theorem 112.

Theorem 112 ([96]). *Let R be a commutative ring with identity and $\mathfrak{m}$ be the unique maximal ideal of R such that $\mathfrak{m}^2 \neq (0)$. If $H(R)$ satisfies either (C_1) or (C_2), then R is an Artinian ring, $\mathfrak{m}^9 = (0)$ and $\mathfrak{m}$ can be generated by at most two elements.*

Theorem 113 ([96]). *Let R be a commutative Artinian ring with identity and $\mathfrak{m}$ be the unique maximal ideal of R such that $\mathfrak{m}$ is not principal but $\mathfrak{m} = Ra + Rb$ for some $a, b \in \mathfrak{m}$ with $a^2 = b^2 = 0$ but $ab \neq 0$. Then the following statements are equivalent:*

(i) *$H(R)$ satisfies (C_1);*

(ii) *$H(R)$ satisfies (C_2);*

(iii) *$\frac{R}{\mathfrak{m}} \in \{2, 3\}$ and $|R| \in \{16, 18\}$;*

(iv) *$H(R)$ is planar.*

Theorem 114 ([96]). *Let R be a commutative Artinian ring with identity and $\mathfrak{m}$ be the unique maximal ideal of R such that $\mathfrak{m}$ is not principal but $\mathfrak{m} = Ra + Rb$ for some $a, b \in \mathfrak{m}$ with $a^2 \neq 0$ but $b^2 = ab = 0$. Then the following statements are equivalent:*

(i) *$H(R)$ satisfies both (C_1) and (C_2);*

(ii) $|\frac{R}{\mathfrak{m}}| \leq 3$ *and* $\mathfrak{m}^3 = (0)$;

(iii) *$H(R)$ is planar.*

Theorem 115 ([96]). *Let R be a commutative Artinian ring with identity and $\mathfrak{m}$ be the unique maximal ideal of R such that $\mathfrak{m}$ is not principal but $\mathfrak{m} = Ra+Rb$ for some $a, b \in \mathfrak{m}$ for some $a, b \in \mathfrak{m}$ with $a^2 \neq 0$, $b^2 \neq 0$ and $ab = 0$. Then the following statements are equivalent:*

(i) *$H(R)$ satisfies both C_1 and C_2;*

(ii) $\mathfrak{m}^2 = Ra^2 = Rb^2$ *and* $|\frac{R}{\mathfrak{m}}| \leq 3$;

(iii) *$H(R)$ is planar.*

Theorem 116 ([96]). *Let R be a commutative Artinian ring with identity and $\mathfrak{m}$ be the unique maximal ideal of R such that $\mathfrak{m}$ is not principal but $\mathfrak{m} = Ra + Rb$ for some $a, b \in \mathfrak{m}$ with $a^2 \neq 0$, $ab \neq 0$, $a^2 + ab \neq 0$, whereas $b^2 = 0$. Then the following statements are equivalent:*

(i) *$H(R)$ satisfies both C_1 and C_2;*

(ii) *$H(R)$ satisfies C_2;*

(iii) $\mathfrak{m}^3 = (0)$, $\mathfrak{m}^2 = Rab$ *and* $|\frac{R}{\mathfrak{m}}| = 3$;

(iv) *$H(R)$ is planar.*

Theorem 117 ([96]). *Let R be a commutative Artinian ring with identity and $\mathfrak{m}$ be the unique maximal ideal of R such that $\mathfrak{m}$ is not principal but $\mathfrak{m} = Ra + Rb$ for some $a, b \in \mathfrak{m}$ such that a^2, b^2, ab, $a^2 + ab$, $b^2 + ab \in R \setminus \{0\}$. Suppose that $\mathfrak{m}^3 = (0)$, $\mathfrak{m}^2$ is not principal and $a^2 \neq b^2$. Then the following statements are equivalent:*

(i) *$H(R)$ satisfies both (C_1) and (C_2);*

(ii) *$H(R)$ satisfies (C_1);*

(iii) $\mathfrak{m}^2 = Ra^2 + Rab = Rb^2 + Rab$, $|\frac{R}{\mathfrak{m}}| = 2$, *and* $\mathfrak{m}^2 \subseteq R(a + b)$;

(iv) *$H(R)$ is planar.*

5.2. Properties of the subgraph $g(R)$ of $G(R)$

Let R be a commutative ring with identity. In [99], S. Visweswaran and P. Vadhel defined and studied a subgraph denoted by $g(R)$ of $G(R)$. The vertex set of graph $g(R)$ is the set of all non-trivial ideals of R and distinct vertices I_1, I_2 of $g(R)$ are adjacent if and only if $I_1 \cap I_2 \neq I_1 I_2$ [99]. It is clear that $g(R)$ is a spanning subgraph of $G(R)$. The authors had studied the connectedness, diameter, girth of $g(R)$.

Theorem 118 ([99]). *Let R be a commutative ring with identity which is not a field and $\mathfrak{m}$ be the unique maximal ideal of R. Then $g(R)$ is connected and $diam(g(R)) \leq 2$.*

A ring R is a *chained ring* if the set of ideals of R is linearly ordered by inclusion.

Theorem 119 ([99]). *Let R be a commutative chained ring with identity which is not an integral domain and $\mathfrak{m}$ be the unique maximal ideal of R. Then the following conditions are equivalent:*

(i) $g(R)$ *is complete;*

(ii) $g(R) = G(R)$*;*

(iii) $(R, \mathfrak{m})$ *is a SPIR.*

Theorem 120 ([99]). *Let R be a commutative ring with identity which is not an integral domain and $\mathfrak{m}$ be the unique maximal ideal of R. If R is reduced ring, then* $diam(g(R)) = 2$.

Theorem 121 ([99]). *Let R be a commutative ring with identity which is not a field and $\mathfrak{m}$ be the unique maximal ideal of R. If $g(R)$ does not contain any infinite clique, then R is Artinian. Specifically, if R is not Artinian, then* $girth(g(R)) = 3$.

Theorem 122 ([99]). *Let R be a commutative Artinian ring with identity which is not a field and $\mathfrak{m}$ be the unique maximal ideal of R. Then $g(R)$ contains a cycle if and only if* $girth(g(R)) = 3$.

Theorem 123 ([99]). *Let R be a commutative Artinian ring with identity and $\mathfrak{m}$ be the unique maximal ideal of R such that $\mathfrak{m}$ is not principal. Then $girth(g(R)) = \propto$ if and only if $g(R) = G(R)$ and $g(R)$ is a star graph.*

Theorem 124 ([99]). *Let R be a commutative ring with identity which is not a field and $\mathfrak{m}$ be the unique maximal ideal of R. Then $G(R)$ is bipartite if and only if $g(R) = G(R)$ is a star graph.*

Theorem 125 ([99]). *Let R be an integral domain such that the number of maximal ideals of R is at least two. Then $g(R)$ is connected and* $diam(g(R)) = 2$.

Theorem 126 ([99]). *Let R be an integral domain such that the number of maximal ideals of R is at least two. If $J(R) = (0)$ where $J(R)$ denote the Jacobson radical of R, then* $diam(g(R)) = 2$.

Corollary 15 ([99]). *Let R be an integral domain. Then $diam(g(R[X])) = 2$, where $R[X]$ is the polynomial ring in one variable X over R.*

Theorem 127 ([99]). *Let R be a commutative Artinian ring with identity such that the number of maximal ideals of R is at least 3. If R is not reduced, then* $girth(g(R)) = 3$.

Theorem 128 ([99]). *Let R be a commutative Artinian ring with identity such that the number of maximal ideals of R is 2. If R is not reduced, then* $girth(g(R)) \in \{3, \propto\}$.

6. The intersection graphs of ideals of direct product of rings

N. Jafari Rad, S. H. Jafari and S. Ghosh in [78] studied the intersection graph of ideals of direct product of rings. They calculated the order and the size of the intersection graphs of ideals of the direct product of rings and fields.

Theorem 129 ([78]). *Let $R = R_1 \times R_2$, where R_1, R_2 are two rings with identity, R_1 has t_1 ideals and R_2 has t_2 ideals. Then the following results are obtained:*

1. *The order of R is $t_1t_2 - 2$.*
2. *The size of R is $\binom{t_1t_2-2}{2} - 2(x+a)(y+b) - x - y + ab$, where $x = \binom{t_1-2}{2} - e(G(R_1))$, $y = \binom{t_2-2}{2} - e(G(R_2))$, $a = t_1 - 1$, $b = t_2 - 1$, $e(G(R_1))$, $e(G(R_2))$ denote the number of edges in $G(R_1)$ and $G(R_2)$ respectively.*

Corollary 16 ([78]). *Let $n \geq 2$ and $F = F_1 \times F_2 \times \cdots \times F_n$, where $F_1, F_2, \ldots, F_n$ are fields. Then the order of $G(F)$ is $2^n - 2$ and the size of $G(F)$ is $\frac{1}{2}(2^{2n} - 3 \cdot 2^n - 3^n + 5)$.*

The authors [78] studied the Eulerian and the Hamiltonian property of $G(R_1 \times R_2)$, where R_i is a commutative ring with identity.

Theorem 130 ([78]). *Let $G(R_i)$ be connected, where R_i is a commutative ring with identity for $i = 1, 2$ and $G(R_1 \times R_2)$ is Eulerian. Then both $G(R_1)$ and $G(R_2)$ are Eulerian.*

Theorem 131 ([78]). *Let R_i be a commutative ring with identity having t_i ideals such that $G(R_i)$ is Eulerian for $i = 1, 2$. Then $G(R_1 \times R_2)$ is Eulerian if and only if $t_1 + t_2$ is even.*

Proposition 15 ([78]). *Let R_1, R_2 be commutative rings with identity such that $G(R_1)$ or $G(R_2)$ is Hamiltonian. Then $G(R_1 \times R_2)$ is Hamiltonian.*

Proposition 16 ([78]). *If $\min\{t_1, t_2\} \geq 3$, where t_i is the number of ideals in commutative ring R_i with identity for $i = 1, 2$, then $G(R_1 \times R_2)$ is Hamiltonian.*

Proposition 17 ([78]). *Let R_i be a commutative ring with identity for $i = 1, 2, 3$. Then $G(R_1 \times R_2 \times R_3)$ is Hamiltonian.*

Proposition 18 ([78]). *Let R_1, R_2 be two commutative rings with identity. If R_1 or R_2 is direct product of two rings, then $G(R_1 \times R_2)$ is Hamiltonian.*

Proposition 19 ([78]). *Let t_1, t_2 be the number of ideals of the commutative rings R_1, R_2 with identity respectively, where $t_1 = 2$ and $t_2 \in \{2, 3\}$. Then $G(R_1 \times R_2)$ is not Hamiltonian.*

Proposition 20 ([78]). *Let R_1, R_2 be commutative rings with identity and R_2 has a unique minimal ideal, then $G(R_1 \times R_2)$ is Hamiltonian.*

Proposition 21 ([78]). *Let $k \geq 2$ be a positive integer. Then $G(\mathbb{Z}_{p_1}^{n_1} \times \mathbb{Z}_{p_2}^{n_2} \times \cdots \times \mathbb{Z}_{p_k}^{n_k})$ is not Hamiltonian if and only if $k = 2$ and $n_1 + n_2 \leq 3$.*

As a consequence of Proposition 21, the authors of [78] stated Corollary 17, which was obtained in Theorem 54 [26].

Corollary 17. *All the graphs of $\mathbb{Z}_n$ are Hamiltonian except when n is of the following forms: p^2, p^3, pq, p^2q, where p, q are distinct primes.*

Theorem 132 ([78]). *Let the number of ideals of a commutative ring R_1 with identity be 2 and the number of ideals in a commutative ring R_2 with identity be t_2, where $t_2 \in \{4, 5, \dots, 9\}$. Then $G(R_1 \times R_2)$ is Hamiltonian.*

In the end of the article [78], the authors raised the following problem.

Problem 1 ([78]). Let R_1, R_2 be two commutative rings with identity and t_1, t_2 be the number of ideals of R_1, R_2 respectively where $t_1 = 2$. Is $G(R_1 \times R_2)$ Hamiltonian for $t_2 \geq 10$?

7. Relation between complement of G(R) and the zero divisor graph

As mentioned by E. A. Osba et al. in [74], Professor Christian Lomp (Porto University, Portugal), suggested that there may be a relation between the graph $\overline{G(R)}$ and the zero divisor graph $\Gamma(R)$. And following the suggestion, the authors [74] proved Theorem 133. In [87], the authors S. Spiroff and C. Wickham introduced the zero divisor graph determined by equivalence classes of zero divisors of a commutative Noetherian ring R, denoted by $\Gamma_E(R)$.

Theorem 133 ([74]). *For a reduced finite commutative principal ideal ring R with identity, $\Gamma_E(R) \cong \overline{G(R)}$.*

8. Signed graph of G(R)

In [83], Saik Sajana et al. extended the study of $G(R)$ in the domain of signed graph, where R is a finite commutative ring with identity.
The signed intersection graph of ideals of a finite commutative ring R with identity is $G_\Sigma(R) = (G(R), \tau)$, where $G(R)$ is the intersection graph of ideals of the ring R and τ is a mapping defined on the edges of $G(R)$ as

$$\tau((a)(b)) = \begin{cases} +, & \text{if (a) is prime or (b) is prime and} \\ -, & \text{otherwise.} \end{cases}$$

The signed intersection graph of $G(\mathbb{Z}_{16})$ is shown in Figure 9.

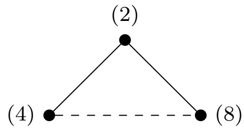

Figure 9. **Signed graph :** $G_\Sigma(\mathbb{Z}_{16})$

A signed graph is called *balanced* if every cycle in it is has an even number of negative edges. A signed graph which is not balanced is called *unbalanced*. For references on signed graphs, see [50, 85, 107].
The authors of [83] characterized the values of n such that $G_\Sigma(\mathbb{Z}_n)$ is balanced.

Theorem 134 ([83]). *Let $n \in A$ and n is square free. Then $G_\Sigma(\mathbb{Z}_n)$ is balanced if and only if n can be written as a product of at most three distinct primes.*

Theorem 135 ([83]). *Let $n \in A$ and n is not square free. Then $G_\Sigma(\mathbb{Z}_n)$ is balanced if and only if $n = p^2, p^3, p^2q$, where $p < q$ are primes.*

A signed graph is called *homogeneous* if all the edges in the signed graph are positive or negative, otherwise heterogeneous. The Homogeneous conditions of $G_\Sigma(\mathbb{Z}_n)$ for different values of n were also studied in [83].

Theorem 136 ([83]). *Let $n \in A$ and n is square free, then the graph $G_\Sigma(\mathbb{Z}_n)$ is homogeneous if and only if n can be written as a product of at most three distinct primes.*

Theorem 137 ([83]). *Let $n \in A$ and n is not square free. Then the graph $G_\Sigma(\mathbb{Z}_n)$ is homogeneous if and only if $n = p^2, p^3, p^2q$, where $p < q$ are distinct primes.*

A *marked graph* is a graph in which all the vertices are assigned with positive or negative sign. A cycle in a signed graph is said to be *cannonically consistent* or *$\mathscr{C}-$consistent* if it contains an even number of negative vertices. A signed graph is *$\mathscr{C}-$consistent* if every cycle in that graph is consistent.
The $\mathscr{C}-$consistency of $G_\Sigma(\mathbb{Z}_n)$ was studied in [83].

Theorem 138 ([83]). *Let $k, n \in \mathbb{N}$ and $n = p^k$, where $k > 1$. Then $G_\Sigma(\mathbb{Z}_n)$ is $\mathscr{C}-$consistent if and only if k is 2 or 4 or an odd positive integer.*

9. Line graph of $G(R)$

The *line graph* of a graph G, written as $L(G)$, is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G [101]. The authors [21], worked on the line graph of $G(R)$.

Theorem 139 ([21]). *For a left Artinian ring R, $L(G(R))$ is complete if and only if $R \cong \mathbb{Z}_3$ or $R \cong \mathbb{Z}_4$.*

Theorem 140 ([21]). *Let $n \in A$ and p, q be distinct primes. $L(G(\mathbb{Z}_n))$ is unicyclic graph for $n = p^4, p^2q$.*

Theorem 141 ([21]). *For $n = p^m$, p^2q, where $n, m \in A$, p, q are distinct primes and $m \leq 5$, $L(G(\mathbb{Z}_n))$ is planar.*

Theorem 142 ([21]). *For $n, m \in A$, $m \leq 4$, $L(G(\mathbb{Z}_n))$ is outerplanar, if $n = p^m$ or $n = p^2q$, where p, q are distinct primes.*

Theorem 143 ([21]). *For $n \in A$, the diameter of $L(G(\mathbb{Z}_n))$ is 0, 1 or 2.*

Theorem 144 ([21]). *For $n \in A$, the girth of $L(G(\mathbb{Z}_n))$ is 3 or 4.*

Theorem 145 ([21]). *For $n \in A$, the domination number of $L(G(\mathbb{Z}_n))$ is greater than or equal to 2 if $n = p^m$, $m \geq 5$.*

Theorem 146 ([21]). *For $n \in A$, the domination number of $L(G(\mathbb{Z}_n))$ is 2, if $n = p^2q, pqr$ where p, q, r are distinct primes.*

The *bondage number* of a graph G, denoted by $b(G)$ is the smallest number of edges whose removal from G results in a graph with domination number greater than the domination number of G [105].

Theorem 147 ([21]). *Let p, q, r be distinct primes and $n, m \in A$, then the bondage number of $L(G(\mathbb{Z}_n))$ is given by:*

$$b(L(G(\mathbb{Z}_n))) = \begin{cases} 3, & \text{if } n = p^2q \\ 4, & \text{if } n = pqr \end{cases}$$

Theorem 148 ([21]). *Let $n = p^m$, where $n, m \in A$, $m \geq 4$ and p is a prime. Then $b(L(G(\mathbb{Z}_n))) \geq 2$.*

10. Open problems on intersection graphs of ideals of rings

We now present some open problems with references for further studies of $G(R)$.

(i) [31] Study the *Hamiltonian number* of $G(R)$.

(ii) [32] Study the *convex number* of $G(R)$.

(iii) [30] Study the *detour number* of $G(R)$.

(iv) Find the *Grundy number* of $G(R)$ [35, 48, 54, 106].

(v) Find the *achromatic number* of $G(R)$ [25, 52].

(vi) [49] Find the *degree-bounded chromatic number* of $G(R)$.

(vii) [27] Find the *rainbow chromatic mean index* of $G(\mathbb{Z}_n)$ for various values of n.

(viii) Find *efficiency* of $G(\mathbb{Z}_n)$ for various values of n [16, 18].

(ix) Characterise the ring R such that $G(R)$ is *L-border energetic* [40, 45].

(x) Determine the intersection graphs of ideals of rings that can be used as a model of *interconnection networks* [36, 57, 59].

(xi) Study the book thickness of graph $G(\mathbb{Z}_n)$ for various values of n [17].

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