Complexity of the paired domination subdivision problem

Jafar Amjadi\textsuperscript{1*}, Mustapha Chellali\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran  
\textsuperscript{2}LAMDA-RO Laboratory, Department of Mathematics, University of Blida, Blida, Algeria

Abstract: The paired domination subdivision number of a graph $G$ is the minimum number of edges that must be subdivided (where each edge in $G$ can be subdivided at most once) in order to increase the paired domination number of $G$. In this note, we show that the problem of computing the paired domination subdivision number is NP-hard for bipartite graphs.

Keywords: paired domination, paired domination subdivision number, complexity

AMS Subject classification: 05C69

1. Terminology and introduction

Paired domination in graphs was introduced by Haynes and Slater \cite{5}, and is now well studied. For more details on paired domination, we refer the reader to the recent book chapter \cite{1}. Let $G = (V,E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $v$ is a vertex in $V$, then open neighborhood of $v$ is $N(v) = \{u \in V(G) | uv \in E(G)\}$. A paired dominating set, abbreviated PD-set, of a graph $G$ is a set $S$ of vertices such that every vertex is adjacent to some vertex in $S$ and the subgraph $G[S]$ induced by $S$ contains a perfect matching (not necessarily induced). If $S$ is a PD-set of $G$ with a perfect matching $M$, then two vertices $u, v \in S$ are said to be partners or paired in $S$ if the edge $uv \in M$. Since the end vertices of any maximal matching in $G$ form a PD-set, every graph $G$ without isolated vertices has a PD-set.

\textsuperscript{*} Corresponding Author

© 2022 Azarbaijan Shahid Madani University
In [3], Favaron et al. introduced the paired domination subdivision number \(sd_{\gamma pr}(G)\) of a graph \(G\) defined as the minimum number of edges that must be subdivided (where each edge in \(G\) can be subdivided at most once) in order to increase the paired domination number of \(G\). Many results have been established on this parameter (see [2–4, 7–9]). But the question of the complexity of the paired domination subdivision problem has not been addressed and therefore remained open.

The paired domination subdivision problem, to which we shall refer as PD-Subdivision problem, is the following:

**PD-Subdivision**

**Instance:** A nonempty graph \(G\) and a positive integer \(k\).

**Question:** Is \(sd_{\gamma pr}(G) \leq k\)?

Our aim in this paper is to show the NP-hardness of the PD-Subdivision problem even for bipartite graphs.

### 2. NP-hardness result

Following Garey and Johnson’s techniques for proving \(NP\)-completeness given in [6], we prove our result by providing a polynomial transformation from the well-known \(NP\)-complete 3-satisfiability problem, 3-SAT, (see Theorem 3.1 in [6]). Before stating the 3-SAT problem, we recall some terms.

Let \(U\) be a set of Boolean variables. A truth assignment for \(U\) is a mapping \(t: U \rightarrow \{T, F\}\). If \(t(u) = T\), then \(u\) is said to be “true” under \(t\); if \(t(u) = F\), then \(u\) is said to be “false” under \(t\). If \(u\) is a variable in \(U\), then \(u\) and \(\bar{u}\) are literals over \(U\). The literal \(u\) is true under \(t\) if and only if, the variable \(\bar{u}\) is false under \(t\); the literal \(\bar{u}\) is true if and only if, the variable \(u\) is false.

A clause over \(U\) is a set of literals over \(U\). It represents the disjunction of these literals and is satisfied by a truth assignment if and only if, at least one of its members is true under that assignment. A collection \(\mathcal{C}\) of clauses over \(U\) is satisfiable if and only if, there exists some truth assignment for \(U\) that simultaneously satisfies all the clauses in \(\mathcal{C}\). Such a truth assignment is called a satisfying truth assignment for \(\mathcal{C}\). The 3-SAT problem is specified as follows.

**3-SAT**

**Instance:** A collection \(\mathcal{C} = \{C_1, C_2, \ldots, C_m\}\) of clauses over a finite set \(U\) of variables such that \(|C_j| = 3\) for \(j = 1, 2, \ldots, m\).

**Question:** Is there a truth assignment for \(U\) that satisfies all the clauses in \(\mathcal{C}\)?

Now we are ready to state our main result.

**Theorem 1.** PD-Subdivision problem is NP-hard for bipartite graphs.

**Proof.** The transformation is from 3-SAT to PD-Subdivision. Let \(U = \{u_1, u_2, \ldots, u_n\}\) and \(\mathcal{C} = \{C_1, C_2, \ldots, C_m\}\) be an arbitrary instance of 3SAT. We
will construct a bipartite graph $G$ and choose an integer $k$ such that $\mathcal{C}$ is satisfiable if and only if $\text{sd}_{\gamma_{pr}}(G) \leq k$. The graph $G$ is constructed as follows.

For each $i = 1, 2, \ldots, n$, corresponding to the variable $u_i \in U$, we associate a complete bipartite graph $H_i = K_{3,5}$ with bipartite sets $X = \{x_i, y_i, z_i\}$ and $Y = \{v_i, u_i, w_i, \bar{u}_i, r_i\}$. For each $j = 1, 2, \ldots, m$, corresponding to the clause $C_j = \{p_j, q_j, r_j\} \in \mathcal{C}$, we associate a single vertex $c_j$ by adding the edge $c_ju_i$ if $u_i \in C_j$ and the edge $c_j\bar{u}_i$ if $\bar{u}_i \in C_j$. Finally, add a graph $H$ with vertex set $V(H) = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ and edge set $E(H) = \{s_1s_3, s_3s_5, s_5s_7, s_2s_4, s_4s_6, s_6s_8\}$, by joining $s_1$ and $s_2$ to each vertex $c_j$ with $1 \leq j \leq m$. Set $k = 1$. Figure 1 shows an example of the graph obtained from the instance $U = \{u_1, u_2, u_3, u_4\}$ and $\mathcal{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, \bar{u}_2, \bar{u}_3\}, C_2 = \{\bar{u}_1, u_2, u_4\}, C_3 = \{u_2, u_3, u_4\}$.

To prove that this is indeed a transformation, we only need to show that $\text{sd}_{\gamma_{pr}}(G) = 1$ if and only if there is a truth assignment for $U$ that satisfies all clauses in $\mathcal{C}$. For this purpose, we need to prove the following four claims.

**Claim 1.** $\gamma_{pr}(G) \geq 2n + 4$. Moreover, if $\gamma_{pr}(G) = 2n + 4$, then for every $\gamma_{pr}(G)$-set $S$ of $G$, $|H_i \cap S| = 2$ and $|V(H) \cup \{c_1, c_2, \ldots, c_m\} \cap S = \{s_3, s_4, s_5, s_6\}$.

**Proof of Claim.** Let $S$ be a $\gamma_{pr}$-set of $G$. For each $i \in \{1, 2, \ldots, n\}$, it is clear that $|V(H_i) \cap S| \geq 2$ and $|H \cap S| \geq 4$, implying that $\gamma_{pr}(G) \geq 2n + 4$.

Suppose that $\gamma_{pr}(G) = 2n + 4$. Since $H_i = K_{m,n}$ with $m, n \geq 3$, $|V(H_i) \cap S| = 2$ and so $|(V(H) \cup \{c_1, c_2, \ldots, c_m\}) \cap S = 4$. Now we show that $|V(H) \cup \{c_1, c_2, \ldots, c_m\} \cap S = \{s_3, s_4, s_5, s_6\})$. Suppose for the sake of contradiction that $s_3 \notin S$. Then to paired...
dominate \( s_3 \) we must have either \( s_1 \in S \) paired in \( S \) with some \( c_j \) or \( s_5 \in S \) paired in \( S \) with \( s_7 \). In the former case, we must have in addition in \( S \), \( s_5 \) and \( s_7 \) as partners. But then \( |S \cap (V(H) \cup \{c_1, c_2, \ldots, c_m\})| \geq 6 \) implying that \(|S| \geq 2n + 6\), which is a contradiction. In the later case, we need that \( S \) contains some vertex \( c_j \) to paired dominate \( s_1 \). But then \( |S \cap (V(H) \cup \{c_1, c_2, \ldots, c_m\})| \geq 5 \) implying that \(|S| \geq 2n + 5\), which is a contradiction too. Therefore \( s_3 \in S \) and to paired dominate \( s_7 \), we must have \( s_5 \in S \) as a partner with \( s_3 \). Hence \( s_1 \not\in S \). Likewise, \( s_4, s_6 \in S \) and are partners and thus \( s_2 \not\in S \). From this, we deduce that \( S \cap \{c_1, c_2, \ldots, c_m\} = \emptyset \). Therefore \(|H_i \cap S| = 2\) and \((V(H) \cup \{c_1, c_2, \ldots, c_m\}) \cap S = \{s_3, s_4, s_5, s_6\} \) as desired.♦

**Claim 2.** \( \mathcal{C} \) is satisfiable if and only if \( \gamma_{pr}(G) = 2n + 4 \).

**Proof of Claim.** Suppose that \( \gamma_{pr}(G) = 2n + 4 \) and let \( S \) be a \( \gamma_{pr} \)-set of \( G \). By Claim 1, \( \{s_3, s_4, s_5, s_6\} \subset S \) and \(|V(H_i) \cap S| = 2\) for each \( i \in \{1, 2, \ldots, n\} \). Moreover, since \( N(c_j) \cap \{s_3, s_4, s_5, s_6\} = \emptyset \) for every \( j \) and \( H_i \) is a complete bipartite graph, only one of \( u_i \) and \( \bar{u}_i \) is in \( S \) and is paired with one vertex of \( \{x_i, y_i, z_i\} \), for each \( i \). Define a mapping \( t : U \rightarrow \{T, F\} \) by

\[ t(u_i) = \begin{cases} T & \text{if } u_i \in S, \\ F & \text{if } \bar{u}_i \in S, \end{cases} \]

for \( i \in \{1, \ldots, n\} \). We now show that \( t \) is a satisfying truth assignment for \( \mathcal{C} \). It is sufficient to show that every clause in \( \mathcal{C} \) is satisfied by \( t \). To this end, we arbitrarily choose a clause \( C_j \in \mathcal{C} \) for \( j \in \{1, \ldots, m\} \). By Claim 1, since \(|H_i \cap S| = 2\) and \((\{s_1, s_2\} \cup \{c_1, c_2, \ldots, c_m\}) \cap S = \emptyset \), there exists some \( i \in \{1, \ldots, n\} \) such that \( c_j \) is adjacent to \( u_i \) or \( \bar{u}_i \). Suppose that \( c_j \) is adjacent to \( u_i \) where \( u_i \in S \). Since \( u_i \) is adjacent to \( c_j \) in \( G \), the literal \( u_i \) is in the clause \( C_j \) by the construction of \( G \). Since \( u_i \in S \), it follows that \( t(u_i) = T \) by (1), which implies that the clause \( C_j \) is satisfied by \( t \). Suppose that \( c_j \) is adjacent to \( \bar{u}_i \) where \( \bar{u}_i \in S \). Since \( \bar{u}_i \) is adjacent to \( c_j \) in \( G \), the literal \( \bar{u}_i \) is in the clause \( C_j \). Since \( \bar{u}_i \in S \), it follows that \( t(u_i) = F \) by (1). Thus, \( t \) assigns \( \bar{u}_i \) the truth value \( T \), that is, \( t \) satisfies the clause \( C_j \). By the arbitrariness of \( j \), with \( 1 \leq j \leq m \), it follows that \( t \) satisfies all the clauses in \( \mathcal{C} \), so \( \mathcal{C} \) is satisfiable. Conversely, suppose that \( \mathcal{C} \) is satisfiable, and let \( t : U \rightarrow \{T, F\} \) be a satisfying truth assignment for \( \mathcal{C} \). Create a subset \( S \) of \( V(G) \) as follows: if \( t(u_i) = T \), then let \( u_i \in S \), and if \( t(u_i) = F \), then let \( \bar{u}_i \in S \). Let \( \{y_1, y_2, \ldots, y_n, s_3, s_4, s_5, s_6\} \subseteq S \) and the intersection of the set of the remaining vertices of \( G \) with \( S \) is empty. Clearly, \(|S| = 2n + 4 \). Since \( t \) is a satisfying truth assignment for \( \mathcal{C} \), for each \( j \in \{1, \ldots, m\} \), at least one of literals in \( C_j \) is true under the assignment \( t \). It follows that the corresponding vertex \( c_j \) in \( G \) is adjacent to at least one vertex \( p \) of \( S \). Since \( c_j \) is adjacent to each literal in \( C_j \) by the construction of \( G \), thus \( S \) is a paired dominating set of \( G \), and so \( \gamma_{pr}(G) \leq |S| = 2n + 4 \). By Claim 1, \( \gamma_{pr}(G) \geq 2n + 4 \), and thus \( \gamma_{pr}(G) = 2n + 4 \). ♦
Claim 3. Let $G'$ be obtained from $G$ by subdividing any edge $e$ of $E(G)$, then $\gamma_{pr}(G') \leq 2n + 6$.

Proof of Claim. Let $e = uv \in E(G)$ and let $G'$ be the graph obtained from $G$ by subdividing the edge $e$ with new vertex $w$. Now, consider the following cases:

Case 1. If $e = s_1s_3$, then consider the set $S = \{s_1, w, s_5, s_7, s_4, s_6\} \cup \{v_i, x_i \mid 1 \leq i \leq n\}$.

Case 2. If $e = s_3s_5$, then consider the set $S = \{s_1, s_3, s_5, s_7, s_4, s_6\} \cup \{v_i, x_i \mid 1 \leq i \leq n\}$.

Case 3. If $e = s_5s_7$, then consider the set $S = \{s_1, s_3, s_5, s_7, s_4, s_6\} \cup \{v_i, x_i \mid 1 \leq i \leq n\}$. (The case $e \in \{s_2s_4, s_4s_6, s_6s_8\}$ is similar).

Case 4. If $e = s_1c_j$ for any $j \in \{1, 2, \ldots, m\}$, then consider the set $S = \{s_1, w, s_5, s_7, s_4, s_6\} \cup \{v_i, x_i \mid 1 \leq i \leq n\}$. (The case $e = c_ju_i$ is similar).

Case 6. If $e = uv$ such that $u \in \{x_i, y_i, z_i\}$ and $v \in \{u_i, \overline{u_i}\}$, say, without loss of generality, $u = x_r, v = u_r$ and $u_r$ is adjacent to $c_j$ for some $1 \leq r \leq n$ and $1 \leq j \leq m$, then consider the set $S = \{s_1, c_j, s_5, s_7, s_4, s_6\} \cup \{x_i, u_i \mid 1 \leq i \leq n\}$.

Case 7. If $e = uv$ such that $u \in \{x_i, y_i, z_i\}$ and $v \in \{\{v_i, w_i, r_i\}\}$, say, without loss of generality, $u = x_r, v = v_r$ for some $1 \leq r \leq n$, then consider the set $S = \{s_3, s_5, s_4, s_6, w, x_r\} \cup \{y_i \mid 1 \leq i \leq n\}$ and add, as a partner of $y_i$ in $S$, the vertex $u_i$ if $u_i \in C_j$ and $\overline{u_i}$ if $\overline{u_i} \in C_j$.

Clearly, in either of the above cases, $S$ is a paired dominating set of $G'$ with $|S| = 2n + 6$, and therefore $\gamma_{pr}(G') \leq 2n + 6$. ♦

Claim 4. $\gamma_{pr}(G) = 2n + 4$ if and only if, $sd_{\gamma_{pr}}(G) = 1$.

Proof of Claim. Assume $\gamma_{pr}(G) = 2n + 4$. Let $G'$ be the graph obtained from $G$ by subdividing the edge $e = s_3s_5$ with new vertex $w$. Suppose, for a contradiction, that $\gamma_{pr}(G) = \gamma_{pr}(G')$, and let $S'$ be a $\gamma_{pr}(G')$-set whose subgraph has a perfect matching $M$.

First assume that $w \notin S'$. Then to paired dominate $s_7$, we have $\{s_5, s_7\} \subseteq S'$. Now, if $s_3 \notin S'$, then $s_1 \in S'$ and is paired in $S'$ with some vertex $c_j$. Hence $S'$ is a $\gamma_{pr}(G)$-set containing $c_j$, contradicting Claim 1. Assume now that $s_3 \in S'$. Then $s_1 \in S'$ and since $|S' \cap \{s_2, s_4, s_6, s_8\}| \geq 2$ and $|V(H_i) \cap S'| \geq 2$ for each $i \in \{1, \ldots, n\}$, we deduce that $|S'| \geq 2n + 6$, contradicting the fact that $|S'| = 2n + 4$.

Secondly, assume that $w \in S'$. Then to paired dominate $s_7$, we must have $s_5 \in S'$. Now, if $s_1 \notin S'$, then $s_3 \notin S'$ and to paired dominate $s_1$, we must have $c_j \in S'$ for some $j \in \{1, \ldots, m\}$. In this case, let $S = (S' \setminus \{w\}) \cup \{s_3\}$, where $s_3$ and $s_5$ are partners in $S$. Clearly, $S$ is a paired dominating set of $G$, and since $|S| = 2n + 4$, we deduce that $|S'| \leq 2n + 4$, contradicting the assumption that $|S'| = 2n + 4$. Therefore, $w \notin S'$, and we can proceed as before to obtain a contradiction.
we obtain a contradiction with Claim 1. Therefore we assume that \( s_1 \in S' \). Then \( S = (S' \setminus \{w\}) \cup \{s_7\} \) is a paired dominating set of \( G \) yielding a contradiction with Claim 1, as above. Hence, \( \gamma_{\text{pr}}(G) < \gamma_{\text{pr}}(G') \), and therefore \( \text{sd}_{\gamma_{\text{pr}}}(G) = 1 \).

Conversely, assume that \( \text{sd}_{\gamma_{\text{pr}}}(G) = 1 \). By Claim 1, we have \( \gamma_{\text{pr}}(G) \geq 2n + 4 \). Let \( G' \) be the graph obtained from \( G \) by subdividing an edge \( e \) such that \( \gamma_{\text{pr}}(G) < \gamma_{\text{pr}}(G') \).

By Claim 3, we have \( \gamma_{\text{pr}}(G') \leq 2n + 6 \). Thus, \( 2n + 4 \leq \gamma_{\text{pr}}(G) < \gamma_{\text{pr}}(G') \leq 4n + 6 \), which yields \( \gamma_{\text{pr}}(G) = 2n + 4 \), as desired.

By Claims 2 and 4, we prove that \( \text{sd}_{\gamma_{\text{pr}}}(G) = 1 \) if and only if there is a truth assignment for \( U \) that satisfies all clauses in \( \mathcal{C} \). Since the construction of the paired subdivision number instance is straightforward from a 3-satisfiability instance, the size of the paired domination subdivision number instance is bounded above by a polynomial function of the size of 3-satisfiability instance. It follows that this is a polynomial reduction and the proof is complete.

References


