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Algorithmic aspects of total Roman {2}-domination in graphs

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Abstract: For a simple, undirected, connected graph G, a function $h : V \to \{0, 1, 2\}$ is called a total Roman {2}-dominating function (TR2DF) if for every vertex v in V with weight 0, either there exists a vertex u in $N_G(v)$ with weight 2, or at least two vertices x, y in $N_G(v)$ each with weight 1, and the subgraph induced by the vertices with weight more than zero has no isolated vertices. The weight of TR2DF h is $\sum_{p \in V} h(p)$. The problem of determining TR2DF of minimum weight is called minimum total Roman {2}-domination problem (MTR2DP). We show that MTR2DP is polynomial time solvable for bounded treewidth graphs, threshold graphs and chain graphs. We design a $2(\ln(\Delta - 0.5) + 1.5)$-approximation algorithm for the MTR2DP and show that the same cannot have $(1 - \delta) \ln |V|$ ratio approximation algorithm for any $\delta > 0$ unless $P = NP$. Next, we show that MTR2DP is APX-hard for graphs with $\Delta = 4$. Finally, we show that the domination and TR2DF problems are not equivalent in computational complexity aspects.

Keywords: Roman {2}-dominating function, Total Roman {2}-domination, APX-complete

AMS Subject classification: 05C69, 68Q25

1. Introduction

Let $G(V, E)$ be a undirected, simple and connected graph. For a vertex u of G, the *(open) neighborhood* denoted $N_G(u)$ is the set $\{v : (v, u) \in E(G)\}$ and its *degree* is $|N_G(u)|$. $N_G[u] = \{u\} \cup N_G(u)$ is the *closed neighborhood* of u. The *maximum degree*

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of G denoted Δ (or clearly $\Delta(G)$) is $max_{u\in V(G)}|N_G(u)|$. We refer to [19], for graph theoretic terminology.
A *dominating set* (DS) of a graph G is a set D such that $D \subseteq V$ and $\cup_{w\in D} N_G[w] = V$ and further D is called a *total dominating set* (TDS) of G if every v in V is adjacent to at least one vertex in D. The problem of finding a (T)DS of smallest cardinality is called the minimum (total) dominating set (M(T)DS) problem. The domination decision problem (DDP) is the problem of finding DS of cardinality at most k, where k is any positive integer. We refer to, [9] for the literature on domination and [11] for the literature on total domination.
In 2016, Chellali et al. in [5] introduced the concept of Roman {2}-domination (R2DOM). A function $f : V \to \{0, 1, 2\}$ is a *Roman* $\{2\}$*-dominating function* (R2DF) on G if every vertex with label zero is adjacent to at least one vertex with weight two or at least two vertices each with weight one. The concept of R2DOM has been studied for example in [4, 6, 14, 20].
Recently, Ahangar et al. in [1, 2] initiated the study of total Roman {2}-domination (TR2DOM). A function $h : V(G) \to \{0, 1, 2\}$ which satisfies the conditions below is called a *total Roman {2}-dominating function* (TR2DF).
C1). h is a R2DF and
C2). The subgraph induced by the vertices with weight more than zero has no isolated vertices.
The weight of a TR2DF is the sum of the weights of all the vertices. The minimum weight of a TR2DF of G is called the *total Roman {2}-domination number* $\gamma_{tR2}(G)$. Since a TR2DF is defined just for graphs without isolated vertices, we only consider in this paper nontrivial connected graphs. The TR2DOM problem (TR2DP) is the problem of finding a TR2DF of weight at most k, where k is any positive integer. The minimum TR2DOM problem (MTR2DP) is the problem of finding a TR2DF of minimum weight in the input graph.
It is known that TR2DP is NP-complete for bipartite and chordal graphs [2]. Here, we investigate the complexity of TR2DP in subclasses of bipartite graphs and chordal graphs. Through out this paper P refers polynomial time solvable and NPC refers NP-complete.

2. Bounded Tree-width, Threshold and Chain Graphs

We begin by giving the exact value of the total Roman {2}-domination number for bounded tree-width graphs, connected threshold graphs and connected chain graphs.

2.1. Bounded Tree-width Graphs

A *tree decomposition* (TD) of a graph H is a tree T_1 with the vertex set $V(T_1) = \{Z_1, Z_2, \ldots\}$, a subset of the power set of $V(H)$ with the following requirements.
i). $V(H) = \bigcup_{Z_t\in V(T_1)} Z_t$
ii). $\forall (u, v) \in E(H)$, there exists a vertex $Z_t \in V(T_1)$ such that $u, v \in Z_t$ and

iii). $\forall v \in V(H)$, the induced subgraph $\{Z_t : v \in Z_t \text{ and } Z_t \in V(T_1)\}$ is a subtree of T.
Then the TD T_1 of H is said to have *width* equals to $max\{|Z_t| - 1 : Z_t \in V(T_1)\}$. The *treewidth* is the smallest width of a TD of a graph.

Theorem 1. *TR2DP can be expressed in CMSOL.*

Proof. Let $G(V, E)$ be a graph and $g : V \to \{0, 1, 2\}$ be a function defined on G, where $V_i = \{v|g(v) = i\}$ for $i \in \{0, 1, 2\}$. Then CMSOL for the TR2DP is specified as below.
$Tot\_Rom\_2\_Dom(V) = (g(V) \leq k) \wedge \exists V_0, V_1, V_2, \forall p((p \in V_0 \wedge ((\exists q, r \in V_1 \wedge adj(p, q) \wedge adj(p, r)) \vee (\exists t \in V_2 \wedge adj(p, t)))) \vee (p \in (V_1 \cup V_2)) \wedge ((p \in V_1 \wedge \exists q \in (V_1 \cup V_2) \wedge adj(p, q)) \vee (p \in V_2) \wedge \exists q \in (V_1 \cup V_2) \wedge adj(p, q))),$
where $adj(p, q)$ is the binary adjacency relation which holds iff $(p, q) \in E$.
$Tot\_Rom\_2\_Dom(V)$ ensures that 1). $\forall p \in V$, either (i) $p \in V_1$ or (ii) $p \in V_2$, or (iii) if $p \in V_0$ then either $q, r \in V_1$ such that $(p, q) \in E$ and $(p, r) \in E$ or $\exists q \in V_2$ such that $(p, q) \in E$, and 2). every vertex $p \in V_1 \cup V_2$ is adjacent to some vertex q in $V_1 \cup V_2$. □

Now, from Theorem 1 and Courcelle's result in [8], the theorem below follows.

Theorem 2. *MTR2DP for graphs with treewidth at most a constant is solvable in linear time.*

2.2. Threshold Graphs

A graph G is *threshold* iff the following conditions hold:

1. Vertex set of G is partitioned into two disjoint sets, a clique Q and an independent set R
2. There exists a permutation $(q_1, q_2, \ldots, q_p)$ of vertices of Q such that $N_G[q_1] \subseteq N_G[q_2] \subseteq \cdots \subseteq N_G[q_p]$ and
3. There exists a permutation $(r_1, r_2, \ldots, r_i)$ of vertices of R such that $N_G(r_1) \supseteq N_G(r_2) \supseteq \cdots \supseteq N_G(r_i)$.

Theorem 3. *Let G be a connected threshold graph. Then,*

$$\gamma_{tR2}(G) = \begin{cases} 2, & \text{if } G \cong K_2 \\ 3, & otherwise \end{cases} \tag{1}$$

Proof. Let G be a connected threshold graph with p clique vertices and i independent vertices as described above. Since, q_p is a universal vertex of G, clearly, this implies that $\gamma_{tR2}(G) = 3$, except when $G \cong K_2$ where $\gamma_{tR2}(G) = 2$. □

On the basis of Theorem 3 and the fact that the ordering of clique vertices of threshold graph can be found in linear time [13], the MTR2DP is solved in linear time for this class of graphs.

2.3. Chain Graphs

An ordering $\alpha = (y_1, y_2, \ldots, y_p, z_1, z_2, \ldots, z_q)$ of vertex set of a bipartite graph $G(Y, Z, E)$ is a *chain ordering* if $N_G(y_1) \subseteq N_G(y_2) \subseteq \ldots \subseteq N_G(y_p)$ and $N_G(z_1) \supseteq N_G(z_2) \supseteq \ldots \supseteq N_G(z_q)$. A bipartite graph is a *chain graph* iff it has a chain ordering [21].

Theorem 4. *Let $G(Y, Z, E)$ be a connected chain graph. Then,*

$$\gamma_{tR2}(G) = \begin{cases} 2, & \textit{if } G \textit{ is } K_2 \\ 3, & \textit{if } G \textit{ is } K_{1,s}, \textit{ where } s \geq 2 \\ 4, & \textit{otherwise} \end{cases} \tag{2}$$

Proof. Let $G(Y, Z, E)$ be a connected chain graph with $|Y| = p$ and $|Z| = q$ where $p, q \geq 1$. If $G \cong K_2$ or $G \cong K_{1,s}$, where $s \geq 2$, then $\gamma_{tR2}(G)$ can be determined directly from Theorem 3. Otherwise, define function $f : V \to \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in \{y_p, z_1\} \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Clearly, f is a TR2DF and $\gamma_{tR2}(G) \leq 4$. By contradiction, it can be easily verified that $\gamma_{tR2}(G) \geq 4$. Therefore $\gamma_{tR2}(G) = 4$. □

On the basis of Theorem 4 and the fact that chain ordering can be computed in linear time [18], the MTR2DP is solved in linear time for this class of graphs.

3. Approximation Algorithm and Complexity

Here, results related to obtaining approximate solutions to MTR2DP is presented.

3.1. Approximation Bounds

An existing result obtained on lower bound of approximation ratio of MDS is given below.

Theorem 5. *([7]) For a graph $G = (V, E)$, unless $P = NP$, the MDS problem cannot have a solution with approximation ratio $(1 - \delta) \ln |V|$ for any $\delta > 0$.*

Theorem below provides a lower bound on approximation ratio of MTR2DP.

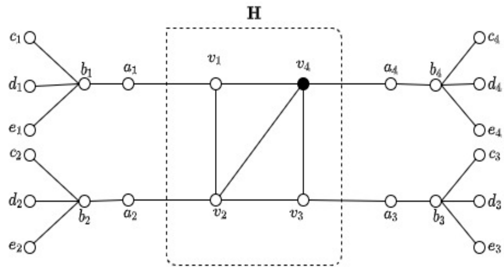

Figure 1. **Construction of H' from H**

Theorem 6. *For a graph H, unless $P = NP$, the MTR2DP cannot have a solution with approximation ratio $(1 - \delta) \ln |V|$ for any $\delta > 0$.*

Proof. We propose a reduction which preserves the approximation. Let $H(V, E)$ be an MDS problem instance, where $V(H) = \{v_1, v_2, \ldots, v_n\}$. From H, an instance H' of MTR2DP is constructed as below.
Create n copies of star graphs with b_i as the central vertex and a_i, c_i, d_i and e_i as terminal vertices. Add the edges $\{(v_1, a_1), (v_2, a_2), \ldots, (v_n, a_n)\}$. Figure 1, shows an example construction of H' from H. Next, we prove a claim.

Claim 1. $\gamma_{tR2}(H') = 3n + \gamma(H)$.

Proof of Claim 1. Let T^* be an MDS of H of size $\gamma(H)$ and let h be a function on H', defined as

$$h(v) = \begin{cases} 1, & \text{if } v \in \{a_i : 1 \leq i \leq n\} \cup T^* \\ 2, & \text{if } v \in \{b_i : 1 \leq i \leq n\} \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Clearly, h is a TR2DF and $\gamma_{tR2}(H') \leq 3n + |T^*|$.
Next, we show that $\gamma_{tR2}(H') \geq 3n + |T^*|$. Let g be a TR2DF on graph H'. Clearly, irrespective of v_i's, $g(a_i) + g(b_i) + g(c_i) + g(d_i) + g(e_i) \geq 3$. These make the size at least $3n$ and $\{v_i : g(v_i) \geq 1\}$ is a MDS since each v_i, where $g(v_i) = 0$ should have neighbors such that whose total weight is at least two. Therefore $\gamma_{tR2}(H') \geq 3n + |T^*|$. Hence $\gamma_{tR2}(H') = 3n + \gamma(H)$. ♦
Suppose that the MTR2DP has an approximation algorithm L which runs in P with approximation ratio β, where $\beta = (1 - \delta) \ln |V|$ for some fixed $\delta > 0$. Let l be a fixed positive integer. Next, we design an approximation algorithm, say DSA which runs in P to find a DS of a given graph H.
It can be noted that if T is a DS with $|T| \leq l$, then it is optimal. Otherwise, let T^* be a DS of H with minimum cardinality and g be a TR2DF of H' with $g(V') = \gamma_{tR2}(H')$. Clearly $g(V) \geq l$. If T is a DS of H obtained by the algorithm

Algorithm 1: DSA(H)

Require: A simple and undirected graph H.
Ensure: A DS T of H.
1: **if** there exists a DS T' of size at most l **then**
2: $T \leftarrow T'$
3: **else**
4: Build the graph H'
5: Calculate a TR2DF f on H' by using algorithm L
6: Find a DS T of H from TR2DF f (as illustrated in the proof of claim in Section 3.1)
7: **end if**
8: return T.

DSA, then $|T| \leq f(V) \leq \beta(g(V)) \leq \beta(3n + |T^*|) = \beta(1 + \frac{3n}{|T^*|})|T^*|$. Therefore, DSA approximates a MDS within a ratio $\beta(1+\frac{3n}{|T^*|})$. If $\frac{1}{|T^*|} < \delta/2$, then the approximation ratio becomes $\beta(1+\frac{3n}{|T^*|}) < (1-\delta)(1+\frac{3n\delta}{2}) \ln n = (1-\delta') \ln n$, where $\delta' = \frac{3n\delta^2}{2} - \frac{3n\delta}{2} + \delta$. By Theorem 5, if there exists an approximation algorithm for MDS problem with approximation ratio $(1-\delta)\ln|V|$ then $P = NP$. Similarly, if there exists an approximation algorithm for MTR2DP with approximation ratio $(1-\delta)\ln|V|$ then $P = NP$. For large values of n, $\ln n \approx \ln(5n)$. Hence, in a graph $H'(V', E')$, where $|V'| = 5|V|$, unless $P = NP$, the MTR2DP cannot have an approximation algorithm with a ratio of $(1-\delta)\ln|V'|$. □

3.2. Approximation Algorithm

Here, an approximation algorithm for MTR2DP is designed based on the approximation result known for MTDS problem below.

Theorem 7 ([22]). *The MTDS problem can be approximated with an approximation ratio of* $\ln(\Delta - 0.5) + 1.5$.

Let APP-TD-SET be an approximation algorithm that produces a TDS D of a graph G such that $|D| \leq (\ln(\Delta - 0.5) + 1.5)\gamma_t(G)$.
Next, we design APP-TR2DF algorithm to determine an approximate solution of MTR2DP. In our algorithm, first we determine a TDS D of G using the approximation algorithm APP-TD-SET. Next, we build a total Roman {2}-dominating triple (TR2DT) T_r such that weight 2 is assigned for all vertices in D and weight 0 is assigned for the remaining vertices.

Algorithm 2: APP-TR2DF(G)

Input: A simple, undirected graph G.
Output: A TR2DT T_r of G.
1: $D \leftarrow$ APP-TD-SET(G)
2: $T_r \leftarrow (V \setminus D, \emptyset, D)$
3: return T_r.

Now, let $T_r = (D', \emptyset, D)$ be the TR2DT obtained from the APP-TR2DF algorithm. Clearly, every vertex in G is assigned with weight either 2 or 0, T_r gives a TR2DF of G and APP-TR2DF computes a TR2DT T_r of G in P. Hence, the result follows.

Theorem 8. *The MTR2DP in a graph can be approximated with an approximation ratio of* $2(\ln(\Delta - 0.5) + 1.5)$.

Proof. Let D be the TDS from APP-TD-SET algorithm, T_r be the TR2DT produced by the APP-TR2DF algorithm and W_r be the weight of T_r. Clearly, $W_r = 2|D|$. It is known that $|D| \leq (\ln(\Delta-0.5)+1.5)\gamma_t(G)$. Therefore, $W_r \leq 2(\ln(\Delta-0.5)+1.5)\gamma_t(G)$. Since $\gamma_t(G) \leq \gamma_{tR2}(G)$ [2], it follows that $W_r \leq 2(\ln(\Delta - 0.5) + 1.5)\gamma_{tR2}(G)$. □

The corollary below follows from Theorem 8.

Corollary 1. $MTR2DP \in APX$ *for graphs with* $\Delta = O(1)$.

3.3. Approximation Completeness

Here, we prove that the MTR2DP is APX-complete for graphs with $\Delta = 4$ using the L-reduction [17]. An optimization problem X is said to be APX-complete if X belongs to APX and APX-hard classes. By providing an L-reduction from MDS problem with $\Delta = 3$ i.e., DOM-3 which is known to be APX-complete [3], we show that the MTR2DP belongs to APX-hard for graphs with $\Delta = 4$.

Theorem 9. $MTR2DP \in APX - complete$ *for graphs with* $\Delta = 4$.

Proof. From Corollary 1, clearly, MTR2DP∈APX. From the given instance $G = (V, E)$ of DOM-3, where $V = \{v_1, v_2, \ldots, v_n\}$, we construct a MTR2DP instance $G' = (V', E')$ same as in Section 3.1. Clearly, $\Delta(G') = 4$.

Claim 2. $\gamma_{tR2}(G') = 3n + \gamma(G)$, *where* $n = |V|$.

Proof of Claim 2. The proof is same as in claim in Section 3.1. ♦

Assume g be a TR2DF on G', where $g(V') = \gamma_{tR2}(G')$ and D^* be a MDS of G. For any graph H, it is known that $\gamma_(H) \geq \frac{|V(H)|}{\Delta(H)+1}$. Clearly, $|D^*| \geq \frac{n}{4}$. From the claim 2, $g(V') = |D^*| + 3n \leq |D^*| + 12|D^*| = 13|D^*|$.

Let $h : V' \to \{0, 1, 2\}$ be a TR2DF of G'. Then, clearly, $D = \{v_i : h(v_i) \geq 1\}$ is a DS of G. Hence, $|D| \leq h(V')-3n$. Therefore, $|D|-|D^*| \leq h(V')-3n-|D^*| \leq h(V')-g(V')$. This infers that there exists an L-reduction with $\beta = 1$ and $\alpha = 13$. □

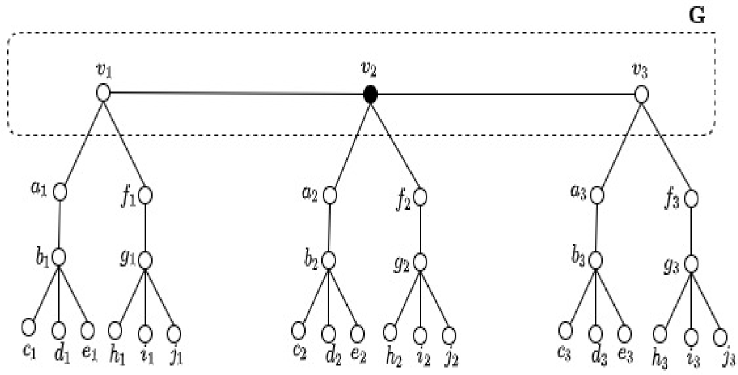

Figure 2. **An illustration to the construction of GT graph from G**

4. Computational Complexity Contrast between Total Roman {2}-domination and Domination Problems

MTR2DP and domination problem vary in complexity aspects i.e., there are some graph classes for which the MTR2DP is in P and DDP is NPC and vice versa. We refer to [10, 15, 16] for the similar kind of study.
We build a new graph class in which the DDP is NPC, whereas the MTR2DP can be solved trivially.
Definition 2. (GT graph). Let $G = (V, E)$, where $|V| = n$ and $V = \{v_1, v_2, \ldots, v_n\}$ be a connected graph. A *GT graph* can be constructed from graph G in the following way :

1. Create two copies of P_2 graphs such as $a_i - b_i$ and $f_i - g_i$, for each i.

2. Consider six additional vertices $\{c_i, d_i, e_i, h_i, i_i, j_i\}$, for each i.

3. Add edges $\{(v_i, a_i), (v_i, f_i), (b_i, c_i), (b_i, d_i), (b_i, e_i), (g_i, h_i), (g_i, i_i), (g_i, j_i) : 1 \leq i \leq n\}$.

General GT graph construction is shown in Figure 2.

Theorem 10. $\gamma_{tR2}(G') = 6n$.

Proof. Let $G' = (V', E')$ be a GT graph constructed from G. Let g be a function defined on G' as follows.

$$g(x) = \begin{cases} 1, & \text{if } x \in \{a_i, f_i : 1 \leq i \leq n\} \\ 2, & \text{if } x \in \{b_i, g_i : 1 \leq i \leq n\} \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

Clearly, g is a TR2DF, $\gamma_{tR2}(G') \leq 6n$.
Next, we show that $\gamma_{tR2}(G') \geq 6n$. Let h be a TR2DF defined on G'. It can be easily verified that, the sum of the weights of the vertices in each set $\{a_i, b_i, c_i, d_i, e_i\}$, $\{f_i, g_i, h_i, i_i, j_i\}$, where $1 \leq i \leq n$ is greater than 3.
Hence $h(V) \geq 6n$. Therefore $h(V) = 6n$. □

Lemma 1. *G has a DS D such that $|D| \leq k$ iff G' has a DS D' such that $|D'| \leq k + 2n$.*

Proof. Suppose D be DS of G with $|D| \leq k$, then, clearly, $D' = D \cup \{b_i, g_i : 1 \leq i \leq n\}$ is a DS of G', where $|D'| \leq k + 2n$.
Let D' is a DS of G' with $|D'| \leq k + 2n$. Clearly, D' should contain at least one vertex from each set $\{b_i, c_i, d_i, e_i\}$ and $\{g_i, h_i, i_i, j_i\}$. Let D'' be the set formed by replacing all a_i's (f_i's) in D' by the corresponding v_i's. Clearly, D'' is a DS of G, where $|D| \leq k$. Hence the lemma. □

The following theorem follows from the fact DDP is NPC for general graphs [12] and above lemma.

Theorem 11. *The DDP for GT graphs is NPC.*

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