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Research Article

Extreme outer connected monophonic graphs

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Abstract: For a connected graph G of order at least two, a set S of vertices in a graph G is said to be an *outer connected monophonic set* if S is a monophonic set of G and either $S = V$ or the subgraph induced by $V - S$ is connected. The minimum cardinality of an outer connected monophonic set of G is the *outer connected monophonic number* of G and is denoted by $m_{oc}(G)$. The number of extreme vertices in G is its *extreme order* $ex(G)$. A graph G is said to be an *extreme outer connected monophonic graph* if $m_{oc}(G) = ex(G)$. Extreme outer connected monophonic graphs of order p with outer connected monophonic number p and extreme outer connected monophonic graphs of order p with outer connected monophonic number $p-1$ are characterized. It is shown that for every pair a, b of integers with $0 \leq a \leq b$ and $b \geq 2$, there exists a connected graph G with $ex(G) = a$ and $m_{oc}(G) = b$. Also, it is shown that for positive integers r, d and $k \geq 2$ with $r < d$, there exists an extreme outer connected monophonic graph G with monophonic radius r, monophonic diameter d and outer connected monophonic number k.

Keywords: outer connected monophonic set, outer connected monophonic number, extreme order, extreme outer connected monophonic graph

AMS Subject classification: 05C12

1. Introduction

By a graph $G = (V, E)$ we mean a finite simple undirected connected graph. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to Harary [5, 15]. The *distance* $d(x, y)$ between two vertices x and y in G is the length of a shortest $x - y$ path in G. A $x - y$ path of length $d(x, y)$ is called $x - y$ *geodesic*. The removal of a vertex v from a graph G results

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in that subgraph $G - v$ of G consisting of all vertices of G except v and all edges not incident with v. A vertex v of G is called an *end-vertex* of G if its degree is 1. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete. The number of extreme vertices in G is its *extreme order* $ex(G)$.

The *closed interval* $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices of G is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set of G is the *geodetic number* $g(G)$ of G. The geodetic number of a graph and its variants have been studied by several authors in [1–4, 6–8, 16, 17, 19]. A set S of vertices in a graph G is said to be an *outer connected geodetic set* if S is a geodetic set of G and either $S = V$ or the subgraph induced by $V - S$ is connected. The minimum cardinality of an outer connected geodetic set of G is the *outer connected geodetic number* of G and is denoted by $g_{oc}(G)$. The outer connected geodetic number of a graph was introduced and studied in [13]. A graph G is an *extreme outer connected geodesic graph* if $g_{oc}(G) = ex(G)$. Extreme outer connected geodesic graphs was introduced and studied in [12]. A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A set S of vertices of G is a *monophonic set* of G if each vertex v of G lies on a $x - y$ monophonic path for some x and y in S. The minimum cardinality of a monophonic set of G is the *monophonic number* of G and is denoted by $m(G)$. The monophonic number of a graph, and algorithmic aspects of monophonic concepts were studied by several authors in [9–11, 18, 22]. A set S of vertices in a graph G is said to be an *outer connected monophonic set* if S is a monophonic set of G and either $S = V$ or the subgraph induced by $V - S$ is connected. The minimum cardinality of an outer connected monophonic set of G is the *outer connected monophonic number* of G and is denoted by $m_{oc}(G)$. The outer connected monophonic number of a graph was introduced and studied in [14].

For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max \{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $rad_m(G)$ of G is $rad_m(G) = \min \{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_m(G)$ of G is $diam_m(G) = \max \{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced in [20] and further studied in [21]. The following theorems will be used in the sequel.

Theorem A. [22] Each extreme vertex of a connected graph G belongs to every monophonic set of G.

Theorem B. [15] Let G be a connected graph with at least three vertices. The following statements are equivalent:

- G is a block

- Every two vertices of G lie on a common cycle.

Theorem C. [14] Each extreme vertex of a connected graph G belongs to every outer connected monophonic set of G.

Theorem D. [14] For the complete graph $K_p (p \geq 2)$, $m_{oc}(K_p) = p$.

Theorem E. [14] No cutvertex of a connected graph G belongs to any minimum outer connected monophonic set of G.

Theorem F. [14] If T is a tree with k end-vertices, then $m_{oc}(T) = k$.

Theorem G. [13] Each extreme vertex of a connected graph G belongs to every outer connected geodetic set of G.

Theorem H. [13] For the complete graph $K_p (p \geq 2)$, $g_{oc}(K_p) = p$.

Theorem I. [13] For any tree T with k end-vertices $g_{oc}(G) = k$.

Throughout this paper G denotes a connected graph with at least two vertices.

2. Main Results

Definition 1. A graph G is said to be an *extreme outer connected monophonic graph* if $m_{oc}(G) = ex(G)$.

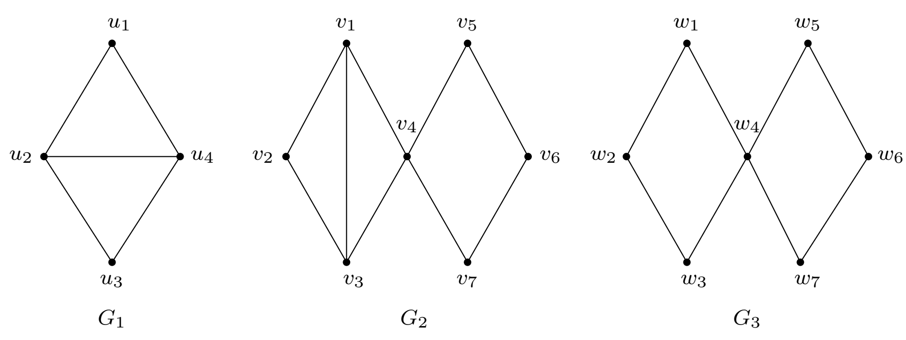

Figure 1. Extreme outer connected and non-extreme outer connected monophonic graphs

Example 1. For the graph G_1 given in Figure 1 of order 4, u_1 and u_3 are the only two extreme vertices and so $ex(G_1) = 2$. The set $S = \{u_1, u_3\}$ is the unique minimum outer connected monophonic set of G_1 so that $m_{oc}(G_1) = 2 = ex(G_1)$. Hence the graph G_1 is an extreme outer connected monophonic graph. The graph G_2 given in Figure 1 has only one extreme vertex v_2 and so $ex(G_2) = 1$. It is clear that $S_1 = \{v_2, v_6\}$ is the unique minimum outer connected monophonic set of G_2, $m_{oc}(G_2) = 2 \neq ex(G_2)$. Therefore G_2 is not an extreme outer connected monophonic graph. The graph G_3 given in Figure 1 contains no extreme vertices and so it is not an extreme outer connected monophonic graph.

For any non-trivial tree T with k end-vertices, $ex(T) = k$ and by Theorem F, $m_{oc}(T) = k = ex(T)$. Thus any non-trivial tree is an extreme outer connected monophonic graph. For the complete graph $K_p (p \geq 2)$, $ex(K_p) = p$ and by Theorem D, $m_{oc}(K_p) = p = ex(K_p)$, it follows that K_p is an extreme outer connected monophonic graph. It is easy to observe that the cycle $C_n (n \geq 4)$ and the complete bipartite graph $K_{r,s} (2 \leq r \leq s)$ are not extreme outer connected monophonic graphs. A graph G having no extreme vertices is obviously not an extreme outer connected monophonic graph.

Theorem 1. *For any connected graph G of order $p\,(p \geq 2)$, $0 \leq ex(G) \leq m(G) \leq m_{oc}(G) \leq p$.*

Proof. Any graph G may or may not contain extreme vertices and so $ex(G) \geq 0$. By Theorem A, every monophonic set of G contains all the extreme vertices of G and so $m(G) \geq ex(G)$. Note that every outer connected monophonic set of G is also a monophonic set of G, $m(G) \leq m_{oc}(G)$. Also, $V(G)$ induces an outer connected monophonic set of G, it follows that $m_{oc}(G) \leq p$. Hence, we have $0 \leq ex(G) \leq m(G) \leq m_{oc}(G) \leq p$. □

Remark 1. The bounds in Theorem 1 are sharp. For the cycle $C_n (n \geq 4)$, $ex(C_n) = 0$ and for the complete graph $K_p (p \geq 2), m_{oc}(K_p) = p$. Also, all the inequalities in Theorem 1 can be strict. For the graph G given in Figure 2 of order 7, u_6 is the only one extreme vertex of G and so $ex(G) = 1$. It is clear that no 2-element subset of $V(G)$ is a monophonic set of G. The set $S = \{u_1, u_2, u_6\}$ is a monophonic set of G and so $m(G) = 3$. Since the subgraph induced by $V - S$ is not connected, S is not an outer connected monophonic set of G. Note that, no 3-element subset of $V(G)$ is an outer connected monophonic set of G. It is easily verified that $S_1 = \{u_3, u_4, u_6, u_7\}$ is a minimum outer connected monophonic set of G and so $m_{oc}(G) = 4$. Thus, we have $0 < ex(G) < m(G) < m_{oc}(G) < p$.

The following theorem characterizes extreme outer connected monophonic graphs G of order p for which $m_{oc}(G) = p$.

Theorem 2. *Any connected graph G of order $p\,(p \geq 2)$ is an extreme outer connected monophonic graph with $m_{oc}(G) = p$ if and only if G is complete.*

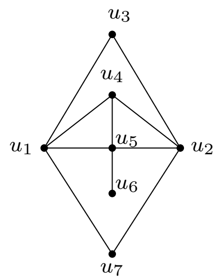

Figure 2. **Graph G satisfying the inequality** $0 < ex(G) < m(G) < m_{oc}(G) < p$

Proof. Let G be the complete graph of order p. Then every vertex of G is an extreme vertex and so $ex(G) = p$. By Theorem D, $m_{oc}(G) = p = ex(G)$. Thus G is an extreme outer connected monophonic graph with $m_{oc}(G) = p$. Conversely, let G is an extreme outer connected monophonic graph with $m_{oc}(G) = p$. Then $ex(G) = p$. Since $ex(G) = p$, every vertex of G is an extreme vertex and so G is complete. □

The next theorem characterizes extreme outer connected monophonic graphs G of order p for which $m_{oc}(G) = p - 1$.

Theorem 3. *Let G be a connected graph of order $p \geq 3$. Then G is an extreme outer connected monophonic graph with $m_{oc}(G) = p - 1$ if and only if $G = K_1 + \cup m_i K_j$, where $j \geq 1$, $\sum m_i \geq 2$.*

Proof. Let $G = K_1 + \cup m_i K_j$, where $j \geq 1$, $\sum m_i \geq 2$. Since G has exactly one cutvertex and all other vertices are extreme, $ex(G) = p-1$. It follows from Theorems C and E that $m_{oc}(G) = p - 1 = ex(G)$. Thus G is an extreme outer connected monophonic graph with $m_{oc}(G) = p - 1$. Conversely, let G be an extreme outer connected monophonic graph with $m_{oc}(G) = p - 1$. Then there exists exactly one non-extreme vertex, say x, in G. If $p = 3$, then G is a path of order 3 and hence $G = K_1 + \cup 2K_1$. Let $p \geq 4$. We claim that x is the cutvertex of G. Otherwise, x lies on a smallest cycle, say $C : x, v_1, v_2, v_3, \ldots, v_n, x$, of length at least $n(n \geq 4)$. Then there exist a non-extreme vertex v_i in C, it follows that $ex(G) \leq p - 2$, which is a contradiction. Hence $G - x$ has at least $n(n \geq 2)$ components. Let $B_1, B_2, \ldots, B_n (n \geq 2)$ be the components of $G - x$. It is enough to prove that each component $B_i (1 \leq i \leq n)$ is complete and x is adjacent to every vertex of each component B_i. Suppose there exists a component B_i, which is not complete. Let u and v be two vertices in B_i such that $d(u, v) \geq 2$. Then by Theorem B, both u and v lie on a common cycle and hence u and v lie on a smallest cycle of length at least 4, let it be C. Then there exist a non-extreme vertex w in C, it follows that $ex(G) \leq p - 2$, which is a contradiction. Thus each component of $G - x$ is complete. Now, if x is not adjacent to some vertex of a component B_i, let it be t, then x lies on a $s - t$ monophonic path P of length at least three, where $s \in V(B_j)(i \neq j)$. Let

$P : s = v_0, v_1, \ldots, x, \ldots, v_{n-1}, v_n = t$. It is clear that $S = V - \{s, t\}$ is an outer connected monophonic set of G and so $m_{oc}(G) \le p - 2$, which is a contradiction. Hence $G = K_1 + \cup m_i K_j$, where $j \ge 1$ and $\sum m_i \ge 2$. □

It was shown in [22] that if G is a non-trivial connected graph of order p and monophonic diameter d_m, then $m(G) \le p - d_m + 1$. By Theorem 1, $ex(G) \le m(G) \le m_{oc}(G)$ for every non-trivial connected graph G, we have the following observation.

Observation 1. If G is a non-trivial connected graph of order p and monophonic diameter d_m, then $ex(G) \le p - d_m + 1$.

If G is a non-trivial tree of order p with k end-vertices, then $ex(G) = m_{oc}(G) = k$. For the complete graph $K_p (p \ge 2)$, $ex(K_p) = m_{oc}(K_p) = p$. Thus every pair k, p of integers with $2 \le k \le p$ is realizable as the order p and outer connected monophonic number k respectively, of an extreme outer connected monophonic graph.

Theorem 4. *For every pair k, p of integers with $2 \le k \le p$, there exits an extreme outer connected monophonic graph G of order p with outer connected monophonic number k.*

Theorem 5. *For every pair a, b of integers with $0 \le a \le b$ and $b \ge 2$, there exists a connected graph G with $ex(G) = a$ and $m_{oc}(G) = b$.*

Proof. We prove this theorem by considering two cases.

Case 1. $a = 0$ and $b \ge 2$. Let $P_3 : x, y, z$ be a path of order 3 and $C_4 : u_1, u_2, u_3, u_4, u_1$ be a cycle of order 4. Let H be the graph obtained from P_3 and C_4 by identifying the vertex y of P_3 and u_4 of C_4. Let G be the graph obtained from H by adding $b-1$ new vertices $v_1, v_2, \ldots, v_{b-1}$ and joining each $v_i (1 \le i \le b-1)$ to both the vertices x, z of H. The graph G is shown in Figure 3. Since no vertex of G is an extreme vertex, $ex(G) = 0$. It is easy to observe that any subset $S \subseteq V(G)$ with cardinality $|S| \le b-1$ is not an outer connected monophonic set of G. Let $S' = \{u_2, v_1, v_2, \ldots, v_{b-1}\}$. Since S' is a monophonic set of G and the subgraph induced by $V - S'$ is connected, S' is an outer connected monophonic set of G, it follows that $m_{oc}(G) = |S'| = b$.

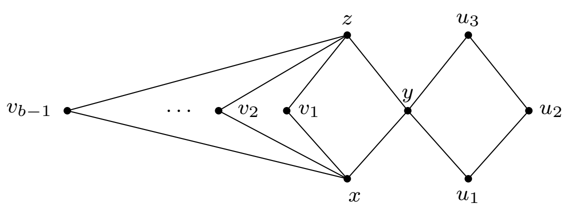

Figure 3. **A graph G with $ex(G) = 0$ and $m_{oc}(G) = b$**

Case 2. $a \geq 1$ and $b \geq 2$. If $a = b$, then the complete graph $G = K_a$ has the desired properties. If $a < b$, then we construct the required graph G as follows: let $P_3 : x, y, z$ be a path of order 3 and the graph G is obtained from P_3 by adding b new vertices $v_1, v_2, \ldots, v_{b-a}, u_1, u_2, \ldots, u_a$ and joining each $u_i (1 \leq i \leq a)$ to the vertex y of P_3; and joining each $v_i (1 \leq i \leq b-a)$ to both the vertices x, z of P_3. The graph G is shown in Figure 4. Since $S = \{u_1, u_2, \ldots, u_a\}$ is the set of all extreme vertices, $ex(G) = a$. By Theorem C, every outer connected monophonic set of G contains S. It is clear that S is not an outer connected monophonic set of G. It is easy to observe that every minimum outer connected monophonic set of G contains $\{v_1, v_2, \ldots, v_{b-a}\}$. Clearly, $S \cup \{v_1, v_2, \ldots, v_{b-a}\}$ is a minimum outer connected monophonic set of G and so $m_{oc}(G) = b$. □

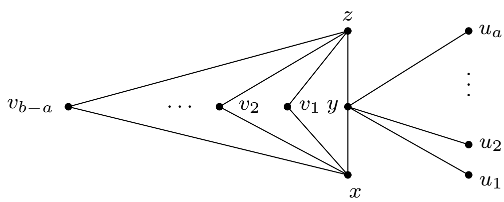

Figure 4. **A graph G with $ex(G) = a \geq 1$ and $m_{oc}(G) = b$ for $a \neq b$**

For any connected graph G, $rad_m(G) \leq diam_m(G)$. It is shown in [20] that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. This theorem can also be extended so that an extreme outer connected monophonic graph can be prescribed when $rad_m(G) < diam_m(G)$.

Theorem 6. *For any three positive integers r, d and $k \geq 2$ with $r < d$, there exists an extreme outer connected monophonic graph G such that $rad_m(G) = r$, $diam_m(G) = d$ and $m_{oc}(G) = k$.*

Proof. Now, let $r = 1$ and $d \geq 2$. Let G be the graph obtained from the cycle $C_{d+2} : v_1, v_2, \ldots, v_{d+2}, v_1$ of order $d+2$ by adding $k-2$ new vertices $u_1, u_2, \ldots, u_{k-2}$ to C_{d+2} and joining each vertex $x \in \{u_1, u_2, \ldots, u_{k-2}, v_3, v_4, \ldots, v_{d+1}\}$ to the vertex v_1 of C_{d+2}. The graph G is shown in Figure 5. It is easily verified that $1 \leq e_m(u) \leq d$ for any vertex u in G, $e_m(v_1) = 1$ and $e_m(v_2) = e_m(v_{d+2}) = d$. Thus $rad_m(G) = r$ and $diam_m(G) = d$. Since $S = \{u_1, u_2, \ldots, u_{k-2}, v_2, v_{d+2}\}$ is the set of all extreme vertices of G, $ex(G) = k$. By Theorem C, every outer connected monophonic set of G contains S. It is clear that S is the unique minimum outer connected monophonic set of G and so $m_{oc}(G) = k = ex(G)$. Thus G is an extreme outer connected monophonic graph.

Now, let $r \geq 2$ and $r < d$. Let H be the graph obtained from the cycle C_{r+3} : $v_1, v_2, \ldots, v_{r+3}, v_1$ of order $r+3$ and the path $P_{d-r+1} : w_0, w_1, \ldots, w_{d-r}$ of order

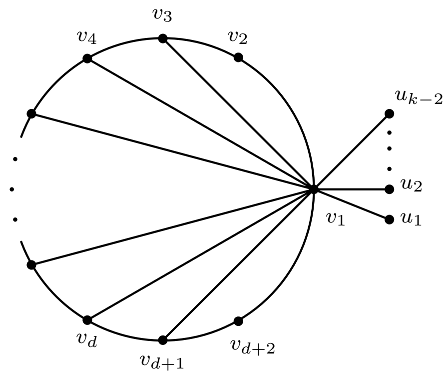

Figure 5. Extreme outer connected monophonic graph G with $rad_m(G) = 1$ and $diam_m(G) = d$

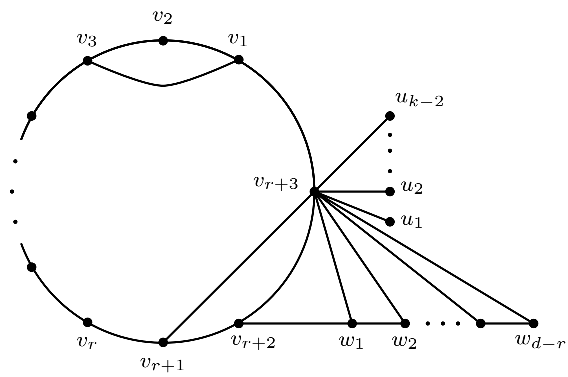

Figure 6. Extreme outer connected monophonic graph G with $rad_m(G) = r \geq 2$ and $diam_m(G) = d$

$d-r+1$ by identifying the vertex v_{r+2} in C_{r+3} and the vertex w_0 in P_{d-r+1}; and also joining each vertex $w_i (1 \leq i \leq d-r)$ in P_{d-r+1} with the vertex v_{r+3} in C_{r+3}. Now, let G be the graph obtained from H by adding $k-2$ new vertices $u_1, u_2, \ldots, u_{k-2}$ and joining each vertex $u_i (1 \leq i \leq k-2)$ to the vertex v_{r+3} in H and also joining v_1 to v_3 and v_{r+1} to v_{r+3}. The graph G is shown in Figure 6. It is easily verified that $r \leq e_m(x) \leq d$ for any vertex x in G, $e_m(v_{r+3}) = r$ and $e_m(v_1) = e_m(v_2) = e_m(w_{d-r}) = d$. Thus $rad_m(G) = r$ and $diam_m(G) = d$. Since $S = \{u_1, u_2, \ldots, u_{k-2}, w_{d-r}, v_2\}$ is the set of all extreme vertices of G, $ex(G) = k$. By Theorem C, every outer connected monophonic set of G contains S. It is clear that S is the unique minimum outer connected monophonic set of G and so $m_{oc}(G) = k = ex(G)$. Thus G is an extreme outer connected monophonic graph. $\square$

We leave the following problem as an open question.

Problem 1. For any three positive integers r, d and $k \geq 2$ with $r = d$, does there exist an extreme outer connected monophonic graph G with $rad_m(G) = r$, $diam_m(G) = d$ and $m_{oc}(G) = k$?

Theorem 7. *For each triple d, k, p of integers with $2 \leq k \leq p - d + 1$ and $d \geq 2$, there exists an extreme outer connected monophonic graph G of order p such that $diam_m(G) = d$ and $m_{oc}(G) = k$.*

Proof. Let $P_{d+1} : u_1, u_2, \ldots, u_{d+1}$ be a path of length d. Add $p-d-1$ new vertices, $v_1, v_2, \ldots, v_{k-2}, w_1, w_2, \ldots, w_{p-d-k+1}$ to P_{d+1} and join each $w_i (1 \leq i \leq p-d-k+1)$ to the vertices u_1, u_2 and u_3; and join each vertex $w_i (1 \leq i \leq p-d-k)$ to the vertex $w_j (i+1 \leq j \leq p-d-k+1)$; and also join each $v_l (1 \leq l \leq k-2)$ to the vertex u_2. The graph G is shown in Figure 7. Then G has order p and monophonic diameter d.

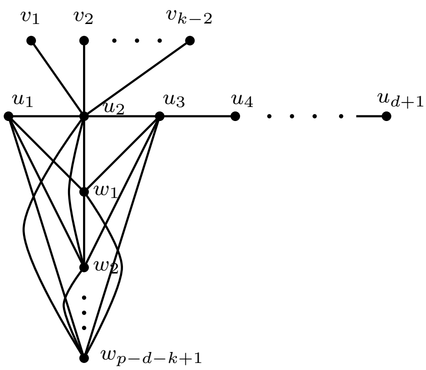

Figure 7. **An extreme outer connected monophonic graph G with $diam_m(G) = d$**

Since $S = \{v_1, v_2, \ldots, v_{k-2}, u_1, u_{d+1}\}$ is the set of all extreme vertices of G, $ex(G) = k$. By Theorem C, every outer connected monophonic set of G contains S. It is clear that S is the unique minimum outer connected monophonic set of G and so $m_{oc}(G) = k = ex(G)$. Thus G is an extreme outer connected monophonic graph of order p such that $diam_m(G) = d$ and $m_{oc}(G) = k$. □

In the following theorem we construct a non-extreme outer connected monophonic graph G of order p such that $diam_m(G) = d$ and $m_{oc}(G) = k$.

Theorem 8. *For each triple d, k, p of integers with $2 \leq k \leq p-d+1$ and $d \geq 2$, there exists a non-extreme outer connected monophonic graph G of order p such that $diam_m(G) = d$ and $m_{oc}(G) = k$.*

Proof. Let $P_{d+1} : u_1, u_2, \ldots, u_{d+1}$ be a path of length d. Add $p-d-1$ new vertices, $v_1, v_2, \ldots, v_{k-2}, w_1, w_2, \ldots, w_{p-d-k+1}$ to P_{d+1} and join each $w_i (1 \leq i \leq p-d-k+1)$ to the vertices u_1, u_2 and u_3; and also join each $v_j (1 \leq j \leq k-2)$ to the vertex u_2.

The graph G is shown in Figure 8. Then G has order p and monophonic diameter d. If $d = 2$, then $S_1 = \{v_1, v_2, \dots, v_{k-2}\}$ is the set of all extreme vertices of G, $ex(G) = k-2$. By Theorem C, every outer connected monophonic set of G contains S_1. It is clear that neither S_1 nor $S_1 \cup \{x\}$ where $x \notin S_1$, is an outer connected monophonic set of G. It is easily verified that $S_2 = S_1 \cup \{u_1, u_3\}$ is a minimum outer connected monophonic set of G and so $m_{oc}(G) = k$. If $d \geq 3$, then $S_3 = \{v_1, v_2, \dots, v_{k-2}, u_{d+1}\}$ is the set of all extreme vertices of G, $ex(G) = k-1$. By Theorem C, every outer connected monophonic set of G contains S_3. It is clear that S_3 is not an outer connected monophonic set of G. It is easily verified that $S_3 \cup \{u_1\}$ is a minimum outer connected monophonic set of G and so $m_{oc}(G) = k$. Since $m_{oc}(G) = k \neq ex(G)$, G is a non-extreme outer connected monophonic graph of order p such that $diam_m(G) = d$ and $m_{oc}(G) = k$. □

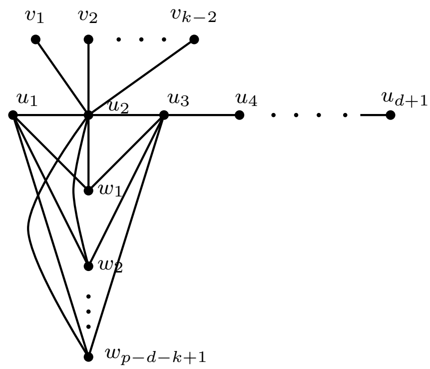

Figure 8. **A non- extreme outer connected monophonic graph G with $diam_m(G) = d$**

Next, we analyse how the extreme outer connected monophonic graphs are affected by the addition of a pendant edge.

Theorem 9. *If G' is a graph obtained by adding l pendant edges to an extreme outer connected monophonic graph G, then $ex(G) \leq ex(G') \leq ex(G) + l$ and G' is an extreme outer connected monophonic graph.*

Proof. Let G' be the graph obtained from an extreme outer connected monophonic graph G by adding l pendant edges $u_i v_i (1 \leq i \leq l)$, where each $u_i (1 \leq i \leq l)$ is a vertex of G and each $v_i (1 \leq i \leq l)$ is not a vertex of G. Let S be a minimum outer connected monophonic set of G. Since G is an extreme outer connected monophonic graph, S is the unique minimum outer connected monophonic set of G and S is the set of all extreme vertices of G. Then it is clear that $S \cup \{v_1, v_2, \dots, v_l\}$ is an outer connected monophonic set of G'. Now, we claim that $ex(G') \leq ex(G) + l$ and G' is an extreme outer connected monophonic graph. If each $u_i (1 \leq i \leq l)$ is an extreme vertex of G then each $v_i (1 \leq i \leq l)$ is an extreme vertex of G' and each $u_i (1 \leq i \leq l)$ is not an extreme vertex of $G^{'}$. It is clear that $S' = (S - \{u_1, u_2, \ldots, u_l\}) \cup \{v_1, v_2, \ldots, v_l\}$ is the set of all extreme vertices of $G^{'}$ and so $ex(G^{'}) = |S'|$. Hence, we have $ex(G) = ex(G^{'})$. If each $u_i (1 \leq i \leq l)$ is not an extreme vertex of G then each $v_i (1 \leq i \leq l)$ is an extreme vertex of $G^{'}$. It is clear that $S' = S \cup \{v_1, v_2, \ldots, v_l\}$ is the set of all extreme vertices of $G^{'}$ and so $ex(G^{'}) = |S'|$. Hence, we have $ex(G^{'}) = ex(G) + l$. Without loss of generality, if each $u_i (1 \leq i \leq k, k < l)$ is an extreme vertex of G and each $u_j (k+1 \leq j \leq l)$ is not an extreme vertex of G, then $\{u_1, u_2, \ldots, u_k\} \subseteq S$. It is clear that $S' = (S - \{u_1, u_2, \ldots, u_k\} \cup \{v_1, v_2, \ldots, v_k\}) \cup \{v_{k+1}, v_{k+2}, \ldots, v_l\}$ is the set of all extreme vertices of $G^{'}$ and so $ex(G') = |S'|$. Hence, we have $ex(G^{'}) < ex(G) + l$. Note that in all the above cases, it is easily verified that S' is the unique minimum outer connected monophonic set of $G^{'}$, $m_{oc}(G^{'}) = |S'| = ex(G^{'})$. Thus $G^{'}$ is an extreme outer connected monophonic graph.

Next, we show that $ex(G) \leq ex(G^{'})$. Suppose that $ex(G) > ex(G^{'})$. Let S_1 be a minimum outer connected monophonic set of $G^{'}$. Since $G^{'}$ is an extreme outer connected monophonic graph, S_1 is the unique minimum outer connected monophonic set of $G^{'}$ and S_1 is the set of all extreme vertices of $G^{'}$. Then with $|S_1| = ex(G^{'}) < ex(G)$. Since each $v_i (1 \leq i \leq l)$ is an extreme vertex of $G^{'}$, it follows from Theorem C that $\{v_1, v_2, ..., v_l\} \subseteq S_1$. Let $S_2 = (S_1 - \{v_1, v_2, \ldots, v_l\}) \cup \{u_1, u_2, \ldots, u_l\}$. Then S_2 is a subset of $V(G)$ and $|S_2| = |S_1| < ex(G)$. Now, we show that S_2 is a monophonic set of G. Let $w \in V(G) - S_2$. Since S_1 is an outer connected monophonic set of $G^{'}$, w lies on an $x - y$ monophonic path P in $G^{'}$ for some vertices $x, y \in S_1$. If neither x nor y is $v_i (1 \leq i \leq l)$, then $x, y \in S_2$. If exactly one of x, y is $v_i (1 \leq i \leq l)$, say $x = v_i$. Then w lies on the $u_i - y$ monophonic path in G obtained from P by removing v_i. If both $x, y \in \{v_1, v_2, \ldots, v_l\}$, then let $x = v_i$ and $y = v_j$ where $i \neq j$. Hence w lies on the $u_i - u_j$ monophonic path in G obtained from P by removing v_i and v_j. Thus S_2 is a monophonic set of G. By Theorem A, every monophonic set of G contains all the extreme vertices of G, $ex(G) \leq |S_2|$. Also, since G is an extreme outer connected monophonic graph, $ex(G) = m_{oc}(G)$. Hence $ex(G) = m_{oc}(G) \leq |S_2| < ex(G)$, which is a contradiction. □

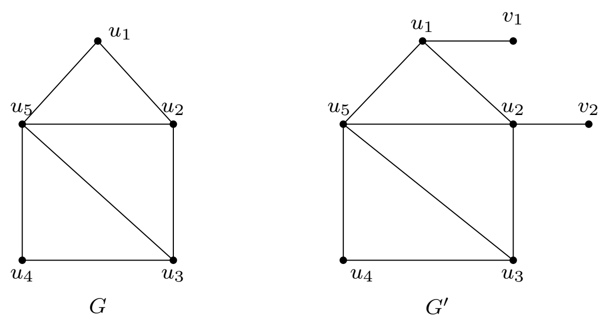

Figure 9. **Graphs G and G' with $ex(G) < ex(G') < ex(G) + l$**

Remark 2. The bounds in Theorem 9 are sharp. Consider a tree T with number of end-vertices $k \geq 2$. Let $S = \{v_1, v_2, \ldots, v_k\}$ be the set of all end-vertices of T. Then by Theorem F, $m_{oc}(T) = k = ex(T)$ and hence T is an extreme outer connected monophonic graph. If we add a pendant edge to an end-vertex of T, then we obtain another tree $T^{'}$ with k end-vertices. Then by Theorem F, $m_{oc}(T^{'}) = k = ex(T^{'})$. Hence $ex(T) = ex(T^{'})$. On the otherhand, if we add l pendant edges to a cutvertex of T, then we obtain another tree $T^{'}$ with $k + l$ end-vertices. Then by Theorem F, $ex(T^{'}) = k + l = ex(T) + l$. In both cases, $T^{'}$ is an extreme outer connected monophonic graph. Also, all the inequalities in Theorem 9 can be strict. For the graph G given in Figure 9, it is clear that $S = \{u_1, u_4\}$ is the set of all extreme vertices of G and so $ex(G) = 2$. Since S is the unique minimum outer connected monophonic set of G, $m_{oc}(G) = 2 = ex(G)$. Thus G is an extreme outer connected monophonic graph. The graph $G^{'}$ given in Figure 9 is obtained from the graph G by adding $l = 2$ pendant edges $u_iv_i (1 \leq i \leq 2)$. Since $S_1 = \{v_1, v_2, u_4\}$ is the set of all extreme vertices of $G^{'}$, $ex(G^{'}) = 3$. It is easy to see that S_1 is the unique minimum outer connected monophonic set of $G^{'}$ and so $m_{oc}(G^{'}) = 3 = ex(G^{'})$. Thus $G^{'}$ is an extreme outer connected monophonic graph. Hence we have $ex(G) < ex(G^{'}) < ex(G) + l$.

Now, we proceed to characterize graphs G for which $ex(G) = ex(G^{'})$, where $G^{'}$ is an outer connected monophonic graph obtained from an outer connected monophonic graph G by adding l pendant edges.

Theorem 10. *Let $G^{'}$ be a graph obtained from an outer connected monophonic graph G by adding l pendant edges $u_iv_i (1 \leq i \leq l)$, where $u_i \in V(G)$ and $v_i \notin V(G)$. Then $ex(G) = ex(G^{'})$ and G' is an extreme outer connected monophonic graph if and only if $l \leq ex(G)$ and $\{u_1, u_2, \ldots, u_l\}$ is a subset of the minimum outer connected monophonic set of G.*

Proof. Let S be a minimum outer connected monophonic set of G. Since G is an extreme outer connected monophonic graph, S is the unique minimum outer connected monophonic set of G and S is the set of all extreme vertices of G. Hence $m_{oc}(G) = |S| = ex(G)$. Let $l \leq ex(G)$ and let $\{u_1, u_2, \ldots, u_l\}$ be a subset of S of G. Let $S^{'} = (S - \{u_1, u_2, \ldots, u_l\}) \bigcup \{v_1, v_2, \ldots, v_l\}$. Then $|S'| = |S|$ and S' is the set of all extreme vertices of $G^{'}$. We show that $S^{'}$ is an outer connected monophonic set of $G^{'}$. Let $z \in V(G^{'}) - S^{'}$. If $z = u_i$ $(1 \leq i \leq l)$, then z lies on every $v_i - w$ monophonic path in $G^{'}$, where $w \in S^{'}$, since u_i is the only vertex adjacent to v_i. So we may assume that $z \neq u_i (1 \leq i \leq l)$. Since z is a vertex of G and S is the unique minimum outer connected monophonic set of G, it follows that z lies on some $x - y$ monophonic path P in G for some $x, y \in S$. Then by an argument similar to the one used in the proof of Theorem 9, we can show that $S^{'}$ is the unique minimum outer connected monophonic set of $G^{'}$. Hence $ex(G^{'}) = m_{oc}(G^{'}) = |S'| = |S| = ex(G) = m_{oc}(G)$ and so $G^{'}$ is an outer connected monophonic graph.
Conversely, let $ex(G) = ex(G^{'})$ and $G^{'}$ is an outer connected monophonic graph. Suppose that $l > ex(G)$. Since G is an extreme outer connected monophonic graph, $m_{oc}(G) = ex(G)$. Also, since each $v_i (1 \leq i \leq l)$ is an end-vertex of $G^{'}$ and $G^{'}$ is an extreme outer connected monophonic graph, $m_{oc}(G^{'}) = ex(G^{'}) \geq l$. Hence

$m_{oc}(G') = ex(G') > m_{oc}(G) = ex(G)$, which is a contradiction. Thus $l \leq ex(G')$. Now, let S' be a minimum outer connected monophonic set of G'. Since each u_i $(1 \leq i \leq l)$ is a cutvertex of G', it follows from Theorem E that $u_i \notin S'$ for $1 \leq i \leq l$. Since each $v_i (1 \leq i \leq l)$ is an end-vertex of G', it follows from Theorem C that $v_i \in S'$ for $1 \leq i \leq l$. Let $S = (S' - \{v_1, v_2, ..., v_l\}) \bigcup \{u_1, u_2, \ldots, u_l\}$. Then S is a subset of $V(G)$ and $|S| = |S'|$. Then, as in the proof of Theorem 9, S is an outer connected monophonic set of G. Since $|S| = |S'| = ex(G') = ex(G)$, it follows that S is the unique minimum outer connected monophonic set of G that contains $\{u_1, u_2, ..., u_l\}$. □

Theorem 11. *For each triple a, b and l of integers with $2 \leq a \leq b$, $1 \leq l \leq b$, and $a + l - b \geq 0$, there exists a connected graph G with $m_{oc}(G) = a$ and $m_{oc}(G') = b$, where G' is an extreme outer connected monophonic graph obtained by adding l pendant edges to an extreme outer connected monophonic graph G.*

Proof. Let G be a tree with number of end-vertices a. Let G' be a graph obtained by adding $b-a$ pendant edges to a cutvertex of G and also adding $l+a-b$ pendant edges each with different end-vertices of G. Then G' is another tree with b end-vertices. By Theorem F, $m_{oc}(G) = a$ and $m_{oc}(G') = b$. □

3. Extreme outer connected monophonic graphs and extreme outer connected geodesic graphs

It is easy to observe that every outer connected geodetic set of G is an outer connected monophonic set of G and by Theorem G, each extreme vertex belongs to every outer connected geodetic set, every extreme outer connected geodesic graph is an extreme outer connected monophonic graph. However, the converse need not be true. For example the graph G given in Figure 10, it is clear that $S = \{v_2, v_4\}$ is the set of all extreme vertices of G and so $ex(G) = 2$. Since S is the unique minimum outer connected monophonic set of G, $m_{oc}(G) = 2 = ex(G)$. Hence G is an extreme outer connected monophonic graph. It is easily verified that S is not a geodetic set of G. Since $S_1 = S \cup \{v_6\}$ is a geodetic set and the subgraph induced by $V - S_1$ is connected, S_1 is an outer connected geodetic set of G so that $g_{oc}(G) = 3 \neq ex(G)$. Therefore G is not an extreme outer connected geodesic graph.

Remark 3. If $m_{oc}(G) = g_{oc}(G) = ex(G)$, then the graph G is both extreme outer connected monophonic graph and extreme outer connected geodesic graph. By Theorems F and I, any non-trivial tree T with a end-vertices, $m_{oc}(T) = g_{oc}(T) = ex(T) = a$. Hence any non-trivial tree T is both extreme outer connected monophonic graph and extreme outer connected geodesic graph. By Theorems D and H, any complete graph $K_p (p \geq 2)$, $m_{oc}(K_p) = g_{oc}(K_p) = ex(K_p) = p$. Hence any complete graph is both extreme outer connected monophonic graph and extreme outer connected geodesic graph.

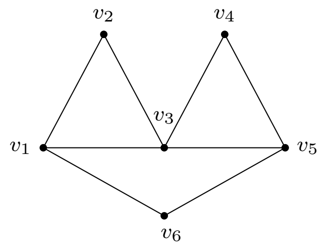

Figure 10. **A graph G with $m_{oc}(G) = 2$ and $g_{oc}(G) = 3$**

Theorem 12. *A graph G is both extreme outer connected monophonic graph and extreme outer connected geodesic graph if and only if every non-extreme vertex of G lies on a $x - y$ geodesic for some extreme vertices x and y in G.*

Proof. Let S be the set of all extreme vertices of G. If every non-extreme vertex of G lies on a $x-y$ geodesic for some vertices x and y in S, then it follows from Theorems C and G, S is the unique minimum outer connected monophonic set and the unique minimum outer connected geodetic set of G and so $m_{oc}(G) = g_{oc}(G) = |S|$. Hence G is both extreme outer connected monophonic graph and extreme outer connected geodesic graph.
Conversely, let G be both extreme outer connected monophonic graph and extreme outer connected geodesic graph. Then $m_{oc}(G) = g_{oc}(G) = ex(G)$. Since $ex(G) = g_{oc}(G)$, S is the unique minimum outer connected geodetic set of G. Hence every non-extreme vertex of G lies on a $x - y$ geodesic for some extreme vertices $x, y \in S$. $\square$

Theorem 13. *For every pair a, b of integers with $2 \leq a < b$, there exists a connected graph G which is an extreme outer connected monophonic graph and not an extreme outer connected geodesic graph such that $m_{oc}(G) = a$ and $g_{oc}(G) = b$.*

Proof. Let $P_i : u_i, v_i$ $(1 \leq i \leq b - a)$ be a $b - a$ copies of a path of length one and $P : x_1, x_2, x_3, x_4$ be a path of length 3. The graph H is obtained from P and P_i by joining each $u_i(1 \leq i \leq b - a)$ of P_i to the vertex x_2 in P; and also joining each $v_i(1 \leq i \leq b - a)$ of P_i to the vertex x_4 in P. The required graph G is obtained from H by adding $a - 1$ new vertices $w_1, w_2, \ldots, w_{a-1}$ and joining each $w_i(1 \leq i \leq a - 1)$ to x_4 in H. The graph G is shown in Figure 11. Since $S = \{x_1, w_1, w_2, \ldots, w_{a-1}\}$ is the set of all extreme vertices of G, $ex(G) = a$. By Theorem C, every outer connected monophonic set of G contains S. It is clear that S is the unique minimum outer connected monophonic set of G and so $m_{oc}(G) = |S| = a = ex(G)$. Hence G is an extreme outer connected monophonic graph. By Theorem G, every outer connected geodetic set of G contains S. Clearly, S is not an outer connected geodetic set of G. It is easily observed that at least one of the vertex of each $P_i : u_i, v_i(1 \leq i \leq b-a)$ must belongs to every outer connected geodetic set of G. Since $S_1 = S \cup \{u_1, u_2, \ldots, u_{b-a}\}$

is a minimum outer connected geodetic set of G, $g_{oc}(G) = b \neq ex(G)$. Thus G is not an extreme outer connected geodesic graph. □

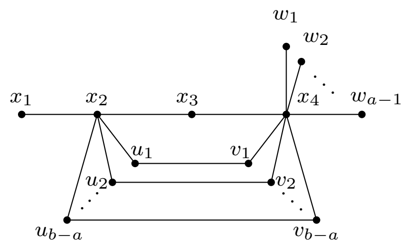

Figure 11. Example of an extreme outer connected monophonic graph and not an extreme outer connected geodesic graph

Theorem 14. *For every pair a, b of integers with $2 \leq a < b$, there exists a connected graph G which is neither an extreme outer connected monophonic graph nor an extreme outer connected geodesic graph such that $m_{oc}(G) = a$ and $g_{oc}(G) = b$.*

Proof. The required graph G_1 is obtained from the graph G given in Figure 11 of Theorem 13 by removal of a vertex x_1. Since $S = \{w_1, w_2, \ldots, w_{a-1}\}$ is the set of all extreme vertices of G_1, $ex(G_1) = a - 1$. By Theorem C, every outer connected monophonic set of G_1 contains S. It is clear that S is not an outer connected monophonic set of G_1. Since $S \cup \{x_2\}$ is a minimum outer connected monophonic set of G_1, $m_{oc}(G_1) = a \neq ex(G_1)$. Hence G_1 is not an extreme outer connected monophonic graph. By Theorem G, every outer connected geodetic set of G_1 contains S. Clearly, S is not an outer connected geodetic set of G_1. It is easy to observe that $S_1 = S \cup \{x_2, u_1, u_2, \ldots, u_{b-a}\}$ is a minimum outer connected geodetic set of G_1, $g_{oc}(G_1) = b \neq ex(G_1)$. Thus G_1 is not an extreme outer connected geodesic graph. □

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