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Research Article

The stress of a graph

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Abstract: Stress is an important centrality measure of graphs applicable to the study of social and biological networks. We study the stress of paths, cycles, fans and wheels. We determine the stress of a cut vertex of a graph G, when G has at most two cut vertices. We have also identified the graphs with minimum stress and maximum stress in the family of all trees of order n and in the family of all complete bipartite graphs of order n.

Keywords: Centrality measure, stress, ranking of vertices, path, tree, block-graph

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1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2].

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Let $P = (u = v_0, v_1, \ldots, v_k = v)$ be a u-v path of length k in G with *origin* $u = v_0$ and *terminus* $v = v_k$. The vertices v_i, $1 \leq i \leq k-1$ are called the *internal* vertices of the path P. The length of a shortest u-v path, denoted by $d(u,v)$ is called the *distance* between u and v. The *diameter* of G, denoted by $\text{diam}(G)$, is given by $\max\{d(u,v) : u, v \in V\}$. For $v \in V$, $e(v)$ denotes the eccentricity of the vertex v and is given by $\max\{d(u,v) : u \in V\}$. A vertex with minimum eccentricity is called a *central vertex* and the subgraph induced by the set of all central vertices is called the centre of G. The eccentricity of any vertex in the centre is called the radius of G. A graph G is called a *block graph* if every block of G is complete.

A vertex (edge) centrality measure assigns a real number to each vertex (edge) of a graph and it quantifies the importance or criticality of a vertex (edge) from a particular perspective. Different centrality measures describe the importance of a vertex from different perspectives. A few examples of vertex centrality measures are betweenness, closeness, degree, eigenvector centrality and stress. Centrality measures betweenness and closeness indicate the efficiency of a given vertex in information flow within the network. The centrality measures degree and eigen vector centrality are used to identify highly connected nodes. The stress of a vertex indicates the relevance of a vertex in holding together communicating vertices. Centrality measures in a graph are used to rank the vertices, and the vertices with higher rank are considered to be more important than the others. For further results on centrality measures, readers are referred to [1, 7, 9] and [8]. Applications of centrality measures include identification of the most influential person in a social network, proteins which play a significant role in a biological process, and key infrastructure vertices in an urban network or internet etc. For a brief survey of centrality measures with emphasis on applications in the study of biological networks, we refer to [4].

In the present paper, our focus is on the study of stress, which is a vertex centrality measure studied to some extent in [5, 6] and [10].

Definition 1. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *stress* of a vertex v_i is the number of shortest paths in G having v_i as an internal vertex and is denoted by $st(v_i)$.

Definition 2. If $(v_{i1}, v_{i2}, \ldots, v_{in})$ is an ordering of $V(G)$ such that $st(v_{i1}) \geq st(v_{i2}) \geq \cdots \geq st(v_{in})$, then $(st(v_{i1}), st(v_{i2}), \ldots, st(v_{in}))$ is called the *stress sequence* of G. The stress of a graph G is defined by $st(G) = \sum_{i=1}^{n} st(v_i)$.

We write $\text{st}(v)$ instead of $\text{st}_G(v)$, whenever the graph under discussion is clear by the context. In the present paper, we determine the stress sequence of standard graphs such as complete graphs, paths, cycles, complete bipartite graphs and wheels.

We also consider the problem of determining the stress of a cut vertex in any graph G having at most two cut vertices.

Since the stress of a graph could be interpreted in terms of *cost* or *benefit*, observing the changes in the stress when we remove or add an edge in the graph would be an interesting constrained optimization problem. Before addressing such a problem, it is necessary to understand which graphs in a given family of graphs are with minimum or maximum stress. For this purpose, we introduce the following notations. Given a family $\mathcal{F}$ of graphs with same number of vertices, we denote $G_{\max}(\mathcal{F})$ (similarly, $G_{\min}(\mathcal{F})$) to represent the set all graphs in the family $\mathcal{F}$ with maximum (minimum) stress. In other words,

$$G_{\max}(\mathcal{F}) = \{G \in \mathcal{F} : \mathrm{st}(G) \geq \mathrm{st}(F),\ \forall F \in \mathcal{F}\},$$

and

$$G_{\min}(\mathcal{F}) = \{G \in \mathcal{F} : \mathrm{st}(G) \leq \mathrm{st}(F),\ \forall F \in \mathcal{F}\}.$$

In the present paper, we consider the family of trees of order n and the family of complete bipartite graphs of order n and determine $G_{\max}(\mathcal{F})$ and $G_{\min}(\mathcal{F})$ in each of these families.

2. The stress of graphs with diameter 2 or 3

In this section, we determine the stress of a graph with diameter 2 or 3.
We begin with making some interesting observations on the stress of a vertex under different conditions. The first one in the following provide a lower bound for the stress of a vertex in terms of its degree and the adjacency among neighboring vertices.

Observation 1. Let $\deg v \geq 2$. Let $m(v)$ denote the number of edges in the induced subgraph $G[N(v)]$. If $v_1, v_2 \in N(v)$ are any two non-adjacent vertices, then $P = (v_1, v, v_2)$ is a shortest path having v as an internal vertex. So, it follows that

$$\mathrm{st}_G(v) \geq \binom{\deg v}{2} - m(v), \tag{1}$$

If $\mathrm{diam}(G) = 2$ and $\deg v \geq 2$, then any shortest path P having v as an internal vertex is of the form $P = (u, v, w)$, where u, w are non-adjacent vertices in $N(v)$. In this case, equality holds in (1).

Observation 2. If $G[N(v)]$ is complete, then there is no shortest path in G having v as an internal vertex and hence $st_G(v) = 0$. In particular if $G = K_n$, then $st(v) = 0$ for all $v \in V$ and hence $st(G) = 0$.

Observation 3. Let G be a graph with diameter 2 and radius 1. Consider a vertex v at the center of G, in which case the eccentricity $e(v)$ of v is one and $\deg v = n - 1$. In this case,

$$\mathrm{st}_G(v) = \binom{n-1}{2} - m(v).$$

Also $\mathrm{st}_G(v) = \binom{n-1}{2}$ if and only if $m(v) = 0$, in which case $G = K_{1,n-1}$.

Observation 4. Let $G = K_{r,s}$ with $2 \leq r \leq s$. Let V_1, V_2 be a bipartition of G with $|V_1| = r$ and $|V_2| = s$. Since $diam(G) = 2$, we get $\mathrm{st}(K_{r,s}) = r\binom{s}{2} + s\binom{r}{2}$.

Observation 5. Let $H = G + K_1$, where G is a simple graph of order n and size m and $V(K_1) = \{v\}$. Since $\mathrm{diam}(H) = 2$, it follows from Observation 1 that

$$\mathrm{st}_H(v) = \binom{n}{2} - m. \tag{2}$$

Now let $u \in V(G)$. If $deg_G\ u = 1$, then $deg_H\ u = 2$ and the induced subgraph $H[N_H(u)]$ is complete. So from Observation 2 we have $st_G(u) = st_H(u) = 0$. If $deg_G\ u \geq 2$, then any shortest path having u as an internal vertex is a shortest path of length 2 in G. Therefore

$$\mathrm{st}_H(u) = \binom{\deg_G u}{2} - m_G(u), \tag{3}$$

where $m_G(u)$ is the number of edges in the subgraph induced by $N(u)$ in G. Since any shortest path of length two in a graph has unique internal vertex, $\sum\limits_{u \in V(G)} \mathrm{st}_H(u)$ is the number of shortest paths of length two in G, say s_2. Therefore, the stress of the graph H is given by

$$\mathrm{st}(H) = \binom{n}{2} - m + s_2. \tag{4}$$

From (3) and (4) of the present observation, we have the following:

- For any $u \in V(G)$, we have $\mathrm{st}_H(u) < \mathrm{st}_G(u)$ if and only if $\deg_G u \geq 2$ and $e(u) \geq 2$.
- If our interest is regarding the changes in the value of $\sum\limits_{u \in V(G)} \mathrm{st}_H(u)$, then we observe that
$$\sum_{u \in V(G)} \mathrm{st}_H(u) = \sum_{u \in V(G)} \mathrm{st}_G(u) \Leftrightarrow \mathrm{diam}(G) \leq 2.$$

The following theorem is a generalization of the results given in Observation 5.

Theorem 6. *Let G be a graph of order n and size m. Let $H = G + \overline{K}_l$, where $l \geq 2$. Then*

$$st(H) = \left[\binom{n}{2} - m\right] l + \binom{l}{2} n + s_2, \tag{5}$$

where s_2 is the number of shortest paths of length 2 in G.

Proof. Let $V(\overline{K}_l) = \{u_1, u_2, \ldots, u_l\}$. Since $diam(H) = 2$ and $N(u_i) = V(G)$ for all i, we have

$$\mathrm{st}(u_i) = \binom{\deg u_i}{2} - m_H(u_i) = \binom{n}{2} - m. \tag{6}$$

for all i. Similarly

$$\mathrm{st}_H(v) = \binom{\deg_H\ v}{2} - m_H(v) = \binom{l}{2} + \binom{\deg_G\ v}{2} - m_G(v). \tag{7}$$

for all $v \in V(G)$. Now $\sum\limits_{v\in V(G)} \left[\binom{deg_G\ v}{2} - m_G(v)\right]$ is the number of shortest paths of length 2 in G. Hence $\sum\limits_{v\in V(G)} st_H(b) = n\binom{l}{2} + s_2$.

Therefore $st(H) = \left[\binom{n}{2} - m\right] l + \binom{l}{2}n + s_2$. $\square$

We now proceed to determine the stress of a vertex in a graph G with $diam(G) = 3$. For this purpose we introduce the following notations. For any $v \in V$, let $N_2(v) = \{u \in V : d(u,v) = 2\}$, $S(v) = \{xy : xy \in E(G), x \in N_2(v)$, and $y \in N(v)\}$, $T(xy) = \{z \in N(v) : d(x,z) = 3\}$ for $xy \in S(v)$, and $k(xy) = |T(xy)|$.

Theorem 7. *For a graph G with diameter 3 and $v \in V$, the stress of v is given by*

$$\mathrm{st}(v) = \binom{deg\ v}{2} - m(v) + \sum_{xy\in S} k(xy). \tag{8}$$

Proof. If $e(v) = 1$ and deg $v \neq 1$, then $S(v) = \emptyset$ and hence $\sum_{xy\in S} k(xy) = 0$ and the result follows from Observation 1. Now let $e(v) \geq 2$ and deg $v \geq 2$.

It follows from Observation 1 that the number of shortest paths of length 2 having v as an internal vertex is $\binom{deg\ v}{2} - m(v)$. Also, any shortest path P of length 3 having v as an internal vertex, if exists, is of the form $P = (x, y, v, z)$ where $x \in N_2(v), y, z \in N(v)$ such that $xy \in S(v)$ and $z \in T(xy)$. Therefore, the number of shortest paths of length 3 having v as an internal vertex is $\sum_{xy\in S(v)} k(xy)$. Now, (8) is immediate from the fact that $\mathrm{diam}(G) = 3$, proving the theorem. $\square$

Corollary 1. *Let $G = B(r,s)$ be the bistar with $V(G) = \{u, u_1, u_2, \ldots, u_r, v, v_1, v_2, \ldots, v_s\}$, $N(u) = \{u_1, u_2, \ldots, u_r, v\}$ and $N(v) = \{v_1, v_2, \ldots, v_s, u\}$. Then*

$$\mathrm{st}(u) = \binom{r+1}{2} + sr; \quad \mathrm{st}(v) = \binom{s+1}{2} + sr, \quad \mathrm{st}(w) = 0 \ \textit{for}\ w \neq u, v, \tag{9}$$

and

$$\mathrm{st}(G) = \binom{r+1}{2} + \binom{s+1}{2} + 2sr. \tag{10}$$

Proof. Since $diam(G) = 3, deg\ u = r+1, m(u) = 0, N_2(u) = \{v_1, v_2, \ldots, v_s\}$, $S(u) = \{v_iu : 1 \leq i \leq s\}$ and $k(uv) = r$, we get $st(u) = \binom{r+1}{2} + sr$. Similarly $st(v) = \binom{s+1}{2} + sr$. Also $deg\ w = 1$ for all $w \in V - \{u, v\}$ and hence $st(w) = 0$. Hence the result follows. $\square$

3. Stress of some standard graphs

In this section we determine the stress of standard graphs such as paths, fans, cycles and wheels.

Theorem 8. *Let $P_n = (v_1, v_2, \ldots, v_n)$ be a path on n vertices. Then*

$$\mathrm{st}(v_i) = (i-1)(n-i). \tag{11}$$

Further, the ranking of the vertices in P_n is given by

$$\begin{aligned} st(v_{k+1}) \geq st(v_{k+2}) \geq st(v_k) \geq st(v_{k+3}) \geq \\ st(v_{k-1}) \geq \ldots st(v_{k+1}) \geq st(v_1) \text{ if } n = 2k+1 \end{aligned} \tag{12}$$

and

$$\begin{aligned} st(v_{k+1}) \geq st(v_k) \geq st(v_{k+2}) \geq st(v_{k-1}) \geq \\ \ldots st(v_{2k}) \geq st(v_1) \text{ if } n = 2k \end{aligned} \tag{13}$$

Proof. If $1 < i < n$, then any shortest path having v_i as an internal vertex is the v_p-v_q section of P_n where $1 \leq p < i$ and $i < q \leq n$. Hence (11) follows. To obtain the stress sequence we consider two cases.
Case 1: $n = 2k+1$.
Now for $1 \leq i \leq k$, we have

$$\begin{aligned} \mathrm{st}(v_{k+i}) - \mathrm{st}(v_{k+i+1}) &= [(k+i)-1][n-(k+i)] - \\ &\quad [(k+i+1)-1][n-(k+i+1)] \\ &= (2k-n) + 2i = 2i - 1 > 0. \end{aligned}$$

Therefore, $\mathrm{st}(v_{k+i}) > \mathrm{st}(v_{k+i+1})$. Now from (11), it is clear that $\mathrm{st}(v_{k+1+i}) = \mathrm{st}(v_{k+1-i})$ and therefore the stress sequence as given in (12) follows.
Case 2 : $n = 2k$.
Now, for $1 \leq i \leq k$, we use (11) to observe that $\mathrm{st}(v_{k-i+1}) = \mathrm{st}(v_{k+i})$ and $\mathrm{st}(v_{k+i}) - \mathrm{st}(v_{k+i+1}) = 2i > 0$. Therefore, the stress sequence in P_n is as in (13). $\square$

Since $\sum\limits_{i=1}^{n}(i-1)(n-i) = \binom{n}{3}$, we have the following corollary.

Corollary 2. *For a path graph P_n,*

$$\mathrm{st}(P_n) = \binom{n}{3}. \tag{14}$$

The following result for the fan $F_{n+1} = P_n + K_1$ follows immediately from Corollary 2 and Observation 5

Theorem 9. *For the fan graph $F_{n+1} = P_n + K_1$, we have $\mathrm{st}_{F_{n+1}}(v_1) = \mathrm{st}_{F_{n+1}}(v_n) = 0$, $\mathrm{st}_{F_{n+1}}(v_i) = 1$ for $2 \le i \le n-1$ and $\mathrm{st}_{F_{n+1}}(v) = \binom{n-1}{2}$. Further,*

$$\mathrm{st}(F_{n+1}) = \frac{(n-2)(n+1)}{2}. \tag{15}$$

Corollary 3. *Let $F_{n+1} = P_n + K_1$. Then the following statements hold.*

(i) $\mathrm{st}_{F_{n+1}}(v_i) \le \mathrm{st}_{P_n}(v_i)$, *for all $v_i \in V(P_n)$.*

(ii) $\mathrm{st}(F_{n+1}) < \mathrm{st}(P_n)$, *for $n \ge 5$.*

Observation 10. Suppose there exists an automorphism α of G such that $\alpha(v) = w$. Then P is a shortest path with v as an internal vertex if and only if $\alpha(P)$ is a shortest path with w as an internal vertex. Hence $\mathrm{st}(v) = \mathrm{st}(w)$. In particular, if G is vertex transitive, then all the vertices of G have the same stress.

Theorem 11. *For the cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$,*

$$\mathrm{st}(v_i) = \frac{d(d-1)}{2}, \quad \textit{for all} \quad 1 \le i \le n, \tag{16}$$

where $d = \mathrm{diam}(C_n) = \lfloor \frac{n}{2} \rfloor$. Further,

$$\mathrm{st}(C_n) = n\left(\frac{d(d-1)}{2}\right). \tag{17}$$

Proof. Since C_n is vertex transitive, it follows from Observation 10 that all the vertices of C_n have the same stress. Hence it is enough to determine $st(v_1)$. Any shortest path having v_1 as an internal vertex, is of the form $(v_j, v_{j+1}, \ldots, v_n, v_1, v_2, \ldots, v_i)$, where $2 \le i \le d$ and $j \ge n-d+i$. So, for each $i \le d$, the number of choices for j is $d-i+1$. Hence $\mathrm{st}(v_1) = \sum_{i=2}^{d}(d-i+1) = \frac{d(d-1)}{2}$. Since all the vertices of C_n have the same stress, (17) follows. □

Since the wheel W_{n+1} is the graph $C_n + K_1$, Theorem 11 and Observation 5 lead to the following corollary.

Corollary 4. *For the wheel $W_{n+1} = C_n + K_1$, we have $\mathrm{st}_{W_{n+1}}(v) = \dfrac{n(n-3)}{2}$ and $\mathrm{st}_{W_{n+1}}(v_i) = 1$ for all $1 \le i \le n$. Also*

$$\mathrm{st}(W_{n+1}) = \binom{n}{2}. \tag{18}$$

Corollary 5. *Let $W_{n+1} = C_n + K_1$. Then the following statements hold.*

(i) $\mathrm{st}_{W_{n+1}}(v_i) \le \mathrm{st}_{C_n}(v_i)$, *for all $v_i \in V(C_n)$.*

(ii) $\mathrm{st}(W_{n+1}) < \mathrm{st}(C_n)$, *for $n \ge 6$.*

4. Stress of a cut vertex

Since the removal of a cut vertex v in a connected graph results in a disconnected graph, it is an important vertex which holds together several blocks of G. In this section we determine the stress of a cut vertex. To start with we consider a graph with a unique cutvertex.

Theorem 12. *Let G be a connected graph with a unique cut vertex v. Let $B_1, B_2, \ldots, B_k$ be the blocks of G. Let t_i denote the number of shortest path in B_i having v as origin. Then*

$$\mathrm{st}_G(v) = \sum_{i=1}^{k} \mathrm{st}_{B_i}(v) + \sum_{\substack{i,j=1 \\ i \neq j}}^{k} t_i t_j. \tag{19}$$

Proof. Let P be any shortest u-w path having v as an internal vertex. If $u, w \in V(B_i)$ for some i, then the number of such paths is given by $\mathrm{st}_{B_i}(v)$. If $u \in V(B_i), w \in V(B_j)$ and $i \neq j$, then the u-v section of P is a shortest path in B_i and the w-v section of P is a shortest path in B_j. The number of such shortest paths is $t_i t_j$. Hence $st_G(v) = \sum_{i=1}^{k} \mathrm{st}_{B_i}(v) + \sum_{\substack{i,j=1 \\ i \neq j}}^{k} t_i t_j$. □

Now, we shall consider graphs G with two cut vertices and determine the stress of those cut vertices.

Theorem 13. *Let G be a connected graph with exactly two cut vertices v and w. Let $B_1, B_2, \ldots, B_r, D_1, D_2, \ldots, D_s$ and H be the blocks in G such that*

(a) $v \in V(B_i)$ and $w \notin V(B_i), 1 \leq i \leq r$

(b) $w \in V(D_j)$ and $v \notin V(D_j), 1 \leq j \leq s$

(c) $v, w \in V(H)$.

Let l_i and l denote respectively the number of shortest path in B_i and in H having v as origin. Let k_i and k denote respectively the number of shortest paths in D_j having w as origin. Let t denote the number of shortest v-w path in H. Then

$$\mathrm{st}_G(v) = \mathrm{st}_H(v) + \sum_{i=1}^{r} \mathrm{st}_{B_i}(v) + \sum_{\substack{i,j=1 \\ i \neq j}}^{r} l_i l_j + l \sum_{i=1}^{r} l_i + t \sum_{i=1}^{r} \sum_{j=1}^{s} l_i k_j \tag{20}$$

and

$$\mathrm{st}_G(w) = \mathrm{st}_H(w) + \sum_{j=1}^{s} \mathrm{st}_{D_j}(w) + \sum_{\substack{i,j=1 \\ i \neq j}}^{s} k_i k_j + k \sum_{j=1}^{s} k_j + t \sum_{i=1}^{r} \sum_{j=1}^{s} l_i k_j. \tag{21}$$

Proof. Let P be any shortest x-y path in G having v as an internal vertex. Then

$$\begin{aligned}
|\{P : x, y \in V(B_i)\}| &= \mathrm{st}_{B_i}(v), \\
|\{P : x \in V(B_i), y \in V(B_j) \text{ and } i \neq j\}| &= l_i l_j, \\
|\{P : x \in V(B_i), y \in V(H)\}| &= l_i l, \\
|\{P : x \in V(B_i), y \in V(D_j)\}| &= t l_i k_j, \\
|\{P : x, y \in V(H)\}| &= \mathrm{st}_H(v), \\
|\{P : x \in V(H) \text{ and } y \in V(D_j)\}| &= 0 \text{ and} \\
|\{P : x \in D_i \text{ and } y \in D_j\}| &= 0.
\end{aligned}$$

Hence (20) follows. Proof of (21) is similar. □

In Observation 3, we have observed that the only vertex in $K_{1,n-1}$ with nonzero stress is the central vertex, which is the unique cut vertex in the graph. The next theorem characterizes all connected graphs in which a vertex with nonzero stress is unique. Surprisingly, such a graph has unique cut vertex.

Theorem 14. *Let G be a connected graph of order n. Then there exists a vertex $v \in V$ such that $st(v) > 0$ and $st(w) = 0$ for all $w \in V - \{v\}$ if and only if G is a block graph with v as its unique cut vertex.*

Proof. Let G be any graph such that there exists a vertex v with $st(v) > 0$ and $st(w) = 0$ for all $w \in V - \{v\}$. If there exists a vertex u which is not adjacent to v, then $st(w) > 0$ for any internal vertex w of a shortest u-v path in G, which is a contradiction. Hence $deg\ v = n - 1$. Since v is the unique vertex with $st(v) > 0$, any shortest path in G is of the form (u, v, w) where u and w are nonadjacent vertices in G. This implies that v is the unique cut vertex of G, v and w are in different blocks of G and each block of G is complete. The converse is obvious. □

5. Graphs with maximum and minimum stress in a given family of graphs

Let $\mathcal{F}$ be a family of graphs in the collection of all connected graphs of order n. For any graph parameter $\beta(G)$, determining the members of $\mathcal{F}$ for which $\beta(G)$ is maximum or minimum is a significant problem. Entringer et al. [3] have proved that among all trees on n vertices, star $K_{1,n-1}$ has minimum Wiener index and the path P_n has maximum Wiener index.

In this section, with stress being the parameter of our interest, we shall consider $\mathcal{F}$ to be the family of trees and the family of complete bipartite graphs of order n to determine the corresponding $G_{\max}(\mathcal{F})$ and $G_{\min}(\mathcal{F})$.

Observation 15. Let T be a tree of order n. Note that any vertex v of T with deg $v = k \geq 2$ is a cut vertex of T. Let $T_1, T_2, \ldots, T_k$ be the components of $T - \{v\}$ and let $|V(T_i)| = n_i$. Any shortest path in T having v as an internal vertex is the unique u-w path for some $u \in V(T_i), w \in V(T_j)$ and $i \neq j$. Therefore,

$$\mathrm{st}(v) = \sum_{\substack{i,j=1 \\ i<j}}^{k} n_i n_j.$$

Theorem 16. *Let $\mathcal{F}$ be the family of all trees of order n, where $n \geq 4$. Then $G_{\max}(\mathcal{F}) = \{P_n\}$.*

Proof. Let $T \in G_{max}(\mathcal{F})$. Suppose that $T \neq P_n$, in which case, the set $S = \{v \in V(T) : deg\ v \geq 3\} \neq \emptyset$. Let $v \in S$ be such that $e(v) \geq e(x)$ for all $x \in S$. Let $e(v) = k$ and u be the eccentric vertex of v. Let $deg\ v = r$ and let $T_1, T_2, \ldots, T_r$ be the components of $T - \{v\}$. Without loss of generality we assume that $u \in V(T_1)$. Now if there exists a vertex $w \in S \cap V(T_i)$ for some $i \neq 1$, then $d(u, w) = d(v, w) + d(v, u) > e(v)$, which is a contradiction. Hence $S \subset V(T_1)$. This implies that T_i is a path for $i \neq 1$ and v is adjacent to one of the pendent vertices of T_i. Let $T_i = (v_{i1}, v_{i2}, \ldots, v_{in_i})$ and let v be adjacent to v_{i1}. Now let $T^1 = (T - vv_{21}) + v_{3n_3}v_{21}$. Clearly T^1 is a tree and $s_{T^1}(w) = s_T(w)$ for all $w \in V(T) - (\{v\} \cup V(T_3))$. It follows from Observation 15 that

$$\begin{aligned}
\mathrm{st}_T(v) &= \sum_{\substack{i,j=1 \\ i<j}}^{r} n_i n_j \\
&= \sum_{\substack{i,j \in \{1,4,\ldots,r\} \\ i<j}} n_i n_j + \sum_{i \in \{1,4,\ldots,r\}} (n_2 + n_3) n_i + n_2 n_3
\end{aligned}$$

and

$$\mathrm{st}_{T^1}(v) = \sum_{\substack{i,j \in \{1,4,\ldots,r\} \\ i<j}} n_i n_j + \sum_{i \in \{1,4,\ldots,r\}} (n_2 + n_3) n_i.$$

Further, for any $v_{3j} \in T_3,\ 1 \leq j \leq n_3$, we have

$$st_T(v_{3j}) = (n_3 - j)\left[n - (n_3 - j + 1)\right]$$

and

$$\mathrm{st}_{T^1}(v_{3j}) = (n_2 + n_3 - j)\left[n - (n_2 + n_3 - j + 1)\right].$$

Hence it follows that

$$\begin{aligned} \mathrm{st}(T^1) - \mathrm{st}(T) &= \mathrm{st}_{T^1}(v) - \mathrm{st}_T(v) + \sum_{j=1}^{n_3} [\mathrm{st}_{T^1}(v_{3j}) - \mathrm{st}_T(v_{3j})] \\ &= -n_2 n_3 + \sum_{j=1}^{n_3} [-n_2(n_3 - j) + n_2 (n - n_2 - n_3 + j - 1)] \\ &= n_2 n_3 [(n-1) - (n_3 + n_2)] > 0, \end{aligned}$$

proving that $T \notin G_{\max}(\mathcal{F})$, which is a contradiction. Therefore, $G_{\max}(\mathcal{F}) = \{P_n\}$. □

Theorem 17. *Let $\mathcal{F}$ be the family of all trees of order n, where $n \geq 4$. Then, $G_{min}(\mathcal{F}) = \{K_{1,n-1}\}$.*

Proof. Let $T \in G_{\min}(\mathcal{F})$. Suppose $T \neq K_{1,n-1}$. Let $S = \{v \in V(T) : deg\ v \geq 2\}$. Then $|S| \geq 2$. Let $v \in S$ be such that $e(v) \geq e(x)$ for all $x \in S$. Let u be an eccentric vertex of v. Let $deg\ v = r, T_1, T_2, \ldots, T_r$ be the components of $T - \{v\}$ and $u \in V(T_1)$. Now if $|V(T_i)| \geq 2$ for some $i \neq 1$ then $S \cap V(T_i) \neq \emptyset$. Let $w \in S \cap V(T_i)$. Then $d(w,u) = d(w,v) + d(v,u) > e(u)$, which is a contradiction to the fact then $e(v)$ is a vertex of maximum eccentricity in S. Hence $|V(T_i)| = 1$ for all $i \neq 1$. Let $V(T_i) = \{w_i\}$. Let v' be a vertex in T_1 which is adjacent to v. Then $T^1 = T - \{vw_2, vw_3, \ldots, vw_r\} \cup \{v'w_2, v'w_3, \ldots, v'w_r\}$ is a tree, $s_{T^1}(w) = st_T(w)$ for all $w \neq v, v'$ and $s_{T'}(v') = s_T(v') + \binom{r}{2}$. Hence

$$\begin{aligned} st(T) - st(T') &= st_T(v) - st_{T'}(v) + st_T(u') - st_{T'}(u') \\ &= \binom{r-1}{2} + (n-r)(r-1) - \binom{r}{2} \\ &= (r-1)[(n-r) - 1] > 0. \end{aligned}$$

Thus $\mathrm{st}(T) > \mathrm{st}(T')$, proving that $T \notin G_{\min}(\mathcal{F})$, a contradiction. Therefore $G_{\min}(\mathcal{F}) = \{K_{1,n-1}\}$. □

We now proceed to determine $G_{\min}(\mathcal{F})$ and $G_{\max}(\mathcal{F})$, where $\mathcal{F}$ is the family of all complete bipartite graphs $K_{r,s}$ of order n.

Theorem 18. *Let $\mathcal{F}$ be the family of all complete bipartite graphs of order n. Then*

(i) $G_{\max}(\mathcal{F}) = \{K_{r,s}\}$ *where* $|r - s| = 0$ *or* 1

(ii) $G_{\min}(\mathcal{F}) = \{K_{n-1,1}\}$.

Proof. Let $G = K_{r,s}$ and $r \geq s$.

Now, from the Observation 4, we have that $\mathrm{st}(K_{r,s}) = \frac{rs(r+s-2)}{2}$ and $\mathrm{st}(K_{r-1,s+1}) = \frac{(r-1)(s+1)(r+s-2)}{2}$. Therefore,

$$\mathrm{st}(K_{r,s}) - \mathrm{st}(K_{r-1,s+1}) = \left(\frac{r+s-2}{2}\right)(s-r+1) < 0. \tag{22}$$

Now, substituting different values for r, starting from $r = n-1$ we get that

$$\mathrm{st}(K_{n-1,1}) < \mathrm{st}(K_{n-2,2}) < \ldots < \mathrm{st}(K_{r_0,s_0}),$$

where $r_0 = \dfrac{n}{2}$, if n is even and $r_0 = \dfrac{n+1}{2}$, if n is odd. Hence the result follows. □

6. Conclusion and Scope

In this paper we have determined the stress of graphs with diameter two or three. The problem remains open for graphs with diameter greater than or equal to four. Investigation of stress of various graph products is a significant direction for further research. Also, determination of $G_{\min}(\mathcal{F})$ and $G_{\max}(\mathcal{F})$ for other families of graphs such as unicyclic graphs and split graphs of order n is open.

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