Communications in Combinatorics and Optimization
Vol. 8, No. 1 (2023), pp. 103-113
DOI: 10.22049/CCO.2021.27202.1212

CCO
Commun. Comb. Optim.

Research Article

On Randić spectrum of zero divisor graphs of commutative ring $\mathbb{Z}_n$

Bilal A. Rather[1,†], S. Pirzada[1,*], M. Imran Bhat[2,‡], T. A. Chishti[3,§]

[1]Department of Mathematics, University of Kashmir, Srinagar, India
†bilalahmadrr@gmail.com
*pirzadasd@kashmiruniversity.ac.in

[2]Department of Mathematics, Central University of Kashmir, Kashmir, India
‡imran_bhat@yahoo.com

[3]DDE, Mathematics Section, University of Kashmir, Srinagar, India
§tachishti@uok.edu.in

Received: 4 April 2021; Accepted: 18 October 2021
Published Online: 21 October 2021

Abstract: For a finite commutative ring $\mathbb{Z}_n$ with identity $1 \neq 0$, the zero divisor graph $\Gamma(\mathbb{Z}_n)$ is a simple connected graph having vertex set as the set of non-zero zero divisors, where two vertices x and y are adjacent if and only if $xy = 0$. We find the Randić spectrum of the zero divisor graphs $\Gamma(\mathbb{Z}_n)$, for various values of n and characterize n for which $\Gamma(\mathbb{Z}_n)$ is Randić integral.

Keywords: Randić matrix; Randić spectrum; zero divisor graph; commutative rings

AMS Subject classification: 05C50, 05C12, 15A18

1. Introduction

A graph $G(V(G), E(G))$ consists of the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E(G)$. We consider connected and finite graphs. $|V(G)| = n$ is the *order* and $|E(G)| = m$ is the *size* of G. The neighborhood $N(v)$ of a vertex v is the set of vertices adjacent to $v \in V(G)$. The *degree* of v, denoted by $d_G(v)$ (or simply d_v), is the cardinality of $N(v)$. A graph is called *regular* if each of its vertices has the same degree. The adjacency matrix $A = (a_{ij})$ of G is a $(0,1)$-square matrix of order n whose (i,j)-entry is equal to 1, if v_i is adjacent to v_j and equal to 0, otherwise.

* *Corresponding Author*

The *Randić* matrix of graph G is defined as

$$\mathbf{R}_G = (r_{ij}) = \begin{cases} \frac{1}{\sqrt{d_G(v_i)d_G(v_j)}}, & v_iv_j \in E(G) \\ \\ 0, & otherwise. \end{cases}$$

This matrix is real symmetric and all its eigenvalues are real. So, we can order its eigenvalues as $\rho_1 \geq \rho_2 \cdots \geq \rho_n$. More about the Randić matrix can be found in [3, 4] and the references therein. Randić matrix has background from Randić index and the recent work on Randić index can be seen in [8].

Let R be a commutative ring with multiplicative identity $1 \neq 0$. A non-zero element $x \in R$ is called a zero divisor of R if there exists a non-zero $y \in R$ such that $xy = 0$. The zero divisor graphs of commutative rings were first introduced by Beck [1]. In the definition he included the additive identity and was interested mainly in coloring of commutative rings. Later Anderson and Livingston [2] modified the definition of zero divisor graphs and excluded the additive identity of the ring in the zero divisor set. Zero divisor graphs are simple, connected and undirected graphs having vertex set as the set of non-zero zero divisors, in which two vertices x and y are joined by an edge if and only if $xy = 0$. The zero divisor graph $\mathbb{Z}_n$ is of order $N = n - \phi(n) - 1$ and size M, where ϕ is Euler's totient function. These graphs help us to study algebraic properties of rings using graph theory. Recent work on the spectra of zero divisor graphs can be seen in [11–17].

For any graph G, we write $Spec(G)$ for the spectrum of G which contains its eigenvalues including multiplicities. K_n denotes the complete graph and $K_{a,b}$ denotes the complete bipartite graph. Other undefined notations and terminology from algebraic graph theory, algebra and matrix theory can be found in [7, 9, 10].

In Section 2, we obtain the Randić spectrum of the zero divisor graph $\Gamma(\mathbb{Z}_n)$ for general n and show that $\Gamma(\mathbb{Z}_{pq})$ is Randić integral.

2. Randić spectrum of $\Gamma(\mathbb{Z}_n)$

Definition 1. Let $G(V, E)$ be a graph of order n having vertex set $\{1, 2, \ldots, n\}$ and $G_i(V_i, E_i)$ be disjoint graphs of order n_i, $1 \leq i \leq n$. The graph $G[G_1, G_2, \ldots, G_n]$ is formed by taking the graphs $G_1, G_2, \ldots, G_n$ and joining each vertex of G_i to every vertex of G_j whenever i and j are adjacent in G.

This graph operation $G[G_1, G_2, \ldots, G_n]$ is also called the generalized join graph operation in [5] and G-join operation. If $G = K_2$, the K_2-join is the usual join operation, namely $G_1 \nabla G_2$. Herein we follow later name with notation $G[G_1, G_2, \ldots, G_n]$ and call it G-join. In [3], Randić spectrum of G-join of regular graphs was determined.

An integer s is called a proper divisor of n if s divides n, denoted by $s|n$, for $1 < s < n$. Let $s_1, s_2, \ldots, s_t$ be the distinct proper divisors of n. Let Υ_n be the simple graph with vertex set $\{s_1, s_2, \ldots, s_t\}$, in which two distinct vertices are connected by an edge if and only if $n|s_is_j$. Let $r, n_1, n_2, \ldots, n_r$ be positive integers and $p_1, p_2, \ldots, p_r$ be distinct prime numbers so that the prime power factorization is $n = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$. The order of Υ_n is given by

$$|V(\Upsilon_n)| = \prod_{i=1}^{r}(n_i + 1) - 2.$$

As seen in [6], Υ_n is connected. For $1 \leq i \leq t$, consider the following sets

$$A_{s_i} = \{x \in \mathbb{Z}_n : (x, n) = s_i\}.$$

For $i \neq j$, clearly $A_{s_i} \cap A_{s_j} = \phi$. This implies that the sets $A_{s_1}, A_{s_2}, \ldots, A_{s_t}$ are pairwise disjoint and partitions the vertex set of $\Gamma(\mathbb{Z}_n)$ as

$$V(\Gamma(\mathbb{Z}_n)) = A_{s_1} \cup A_{s_2} \cup \cdots \cup A_{s_t}.$$

From the definition of A_{s_i}, a vertex of A_{s_i} is adjacent to the vertex of A_{s_j} in $\Gamma(\mathbb{Z}_n)$ if and only if n divides s_is_j , for $i, j \in \{1, 2, \ldots, t\}$, see [6].
The cardinality of A_{s_i} is given by the following lemma [19].

Lemma 1. *Let s_i be the proper divisor of n. Then $|A_{s_i}| = \phi\left(\frac{n}{s_i}\right)$, for $1 \leq i \leq t$.*

The induced subgraphs $\Gamma(A_{s_i})$ of $\Gamma(\mathbb{Z}_n)$ are either cliques or their complements, as can be seen in the next lemma [6].

Lemma 2. *Let n be a positive integer and s_i be its proper divisor. Then the following hold.*

- **(i)** *For $i \in \{1, 2, \ldots, t\}$, the induced subgraph $\Gamma(A_{s_i})$ of $\Gamma(\mathbb{Z}_n)$ on the vertex set A_{s_i} is either the complete graph $K_{\phi(\frac{n}{s_i})}$ or its complement $\overline{K}_{\phi(\frac{n}{s_i})}$. Indeed, $\Gamma(A_{s_i})$ is $K_{\phi(\frac{n}{s_i})}$ if and only if n divides s_i^2.*
- **(ii)** *For $i, j \in \{1, 2, \ldots, t\}$ with $i \neq j$, a vertex of A_{s_i} is adjacent to either all or none of the vertices in A_{s_j} of $\Gamma(\mathbb{Z}_n)$.*

The next result [6] says that $\Gamma(\mathbb{Z}_n)$ is a generalized join of certain complete graphs and null graphs.

Lemma 3. *Let $\Gamma(A_{s_i})$ be the induced subgraph of $\Gamma(\mathbb{Z}_n)$ on the vertex set A_{s_i}, for $1 \leq i \leq t$. Then $\Gamma(\mathbb{Z}_n) = \Upsilon_n[\Gamma(A_{s_1}), \Gamma(A_{s_2}), \ldots, \Gamma(A_{s_t})]$.*

The following theorem [18] gives the Randić spectrum of G-join of graphs in terms of adjacency spectrum. As an application of this result, we compute the Randić spectrum of zero divisor graphs $\Gamma(\mathbb{Z}_n)$.

Theorem 1. [3] *Let G be a graph with no isolated vertices and $V(G) = \{1, 2, \ldots, t\}$, and G_i's be r_i-regular graphs on n_i vertices with $r_i \geq 0, n_i \geq 1$, for $i = 1, 2, \ldots, t$ and $G = G[G_1, G_2, \ldots, G_t]$. Let $\boldsymbol{R}_G$ be the Randić matrix of G. Then*

$$Spec_{\boldsymbol{R}_G} = \left(\bigcup_{i=1}^{t} \left(\frac{\rho}{r_i + N_i} \right) : \rho \in Spec(G_i) \smallsetminus \{r_i\} \right) \bigcup Spec(C(G)),$$

where

$$C(G) = (c_{ij})_{t \times t} = \begin{cases} \dfrac{r_i}{r_i + N_i}, & i = j, \\ \sqrt{\dfrac{n_i n_j}{(r_i + N_i)(r_j + N_j)}}, & ij \in E(G), \\ 0 & otherwise, \end{cases} \tag{1}$$

$$N_i = \begin{cases} \displaystyle\sum_{j \in N_G(i)} n_j, & N_G(i) \neq \emptyset, \\ 0, & otherwise. \end{cases}$$

and ρ is adjacency eigenvalue of G_i.

A graph G is called Randić integral graph if all its Randić eigenvalues are integers. The following proposition says when a G-join graph is Randić integral, the proof of which follows trivially from Theorem 1.

Proposition 1. *The G-join graph $G[G_1, G_2, \cdots, G_t]$ is Randić integral if and only if $\dfrac{\rho}{r_i + N_i} \in \mathbb{Z}$, for $i = 1, 2, \ldots, t$ and the matrix $C(G)$ is integral.*

If $G_i \cong \overline{K}_i$, then from Theorem 1, we have $\dfrac{\rho}{r_i + N_i} = 0$. Therefore, in this case $G = G[G_1, G_2, \ldots, G_t]$ is Randić integral if and only if the matrix $C(G)$ is integral. Evidently, $\Gamma(\mathbb{Z}_n)$ is a complete graph if and only if $n = p^2$, for some prime p. Further the adjacency spectrum of K_ω and $\overline{K}_\omega$ on ω vertices are $\{-1^{[\omega-1]}, \omega\}$ and $\{0^{[\omega]}\}$, respectively. By Lemma 2, $\Gamma(A_{s_i})$ is either $K_{\phi(\frac{n}{s_i})}$ or its complement $\overline{K}_{\phi(\frac{n}{s_i})}$, for $1 \leq i \leq t$. So, by Theorem 1, out of $n - \phi(n) - 1$ number of Randić eigenvalues of $\Gamma(\mathbb{Z}_n)$, clearly $n - \phi(n) - 1 - t$ of them are known. The remaining t Randić eigenvalues of $\Gamma(\mathbb{Z}_n)$ will be counted from the zeros of the characteristic polynomial of the matrix $C(G)$ in (1).

We consider an example to find the Randić spectrum with the help of Theorem 1.

Example 1. Randić eigenvalues of zero divisor graph of $\mathbb{Z}_{60}$.

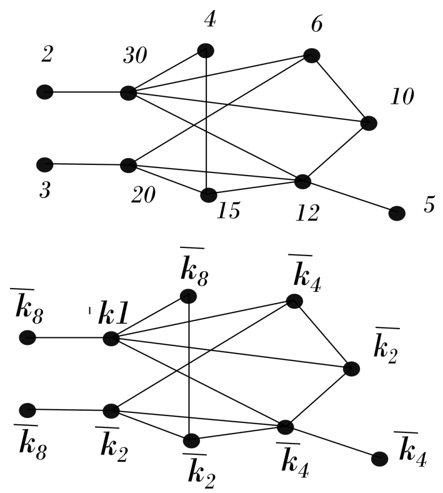

Figure 1. **Zero divisor graph of** $\mathbb{Z}_{60}$

The proper divisors of $n = 60$ are $2, 3, 4, 5, 6, 10, 12, 15, 20,$ and 30. Therefore, Υ_n is the graph as shown in Figure 1. Using Lemma 3, we have

$$\Gamma(\mathbb{Z}_{60}) = \Upsilon_{30}[\overline{K}_8, \overline{K}_8, \overline{K}_8, \overline{K}_4, \overline{K}_4, \overline{K}_2, \overline{K}_4, \overline{K}_2, \overline{K}_2, K_1].$$

By Theorem 1, the Randić spectrum of $\Gamma(\mathbb{Z}_{60})$ consists of the eigenvalue 0 with multiplicity $7 + 7 + 1 + 7 = 1 + 3 + 1 + 3 + 3 = 33$ and the remaining ten eigenvalues are given by the following matrix

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{13}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3}\sqrt{\frac{2}{3}} & 0 & \frac{2}{\sqrt{39}} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{7}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{2}{3}\sqrt{\frac{2}{5}} & 0 & 0 & \frac{2}{3\sqrt{5}} & \sqrt{\frac{2}{65}} \\
0 & 0 & 0 & 0 & \frac{2}{3}\sqrt{\frac{2}{5}} & 0 & \frac{2}{\sqrt{21}} & 0 & 0 & \sqrt{\frac{2}{117}} \\
0 & 0 & 0 & \frac{2}{\sqrt{7}} & 0 & \frac{2}{\sqrt{21}} & 0 & \frac{2}{7} & \frac{2}{3\sqrt{7}} & \sqrt{\frac{2}{91}} \\
0 & 0 & \frac{2}{3}\sqrt{\frac{2}{3}} & 0 & 0 & 0 & \frac{2}{7} & 0 & \frac{1}{3\sqrt{7}} & 0 \\
0 & \frac{2}{3} & 0 & 0 & \frac{2}{3\sqrt{5}} & 0 & \frac{2}{3\sqrt{7}} & \frac{1}{3\sqrt{7}} & 0 & 0 \\
\frac{2}{\sqrt{13}} & 0 & \frac{2}{\sqrt{39}} & 0 & \sqrt{\frac{2}{65}} & \sqrt{\frac{2}{117}} & \sqrt{\frac{2}{91}} & 0 & 0 & 0
\end{pmatrix}.$$

The approximated eigenvalues of the above matrix are

$$\{1.0271, -0.9096, -0.7669, 0.6764, -0.5403, 0.4838, 0.4252, -0.3865, -0.1726, 0.1634\}.$$

Now, we find the Randić spectrum of $\Gamma(\mathbb{Z}_n)$ for $n \in \{pq, p^2q, p^z\}$, $z \geq 2$ with the help of Theorem 1. Let $n = pq$, where p and q, $p < q$, are primes. By Lemmas 2 and 3, we have

$$\Gamma(\mathbb{Z}_{pq}) = \Upsilon_{pq}[\Gamma(A_p), \Gamma(A_q)] = K_2[\overline{K}_{\phi(p)}, \overline{K}_{\phi(q)}] = \overline{K}_{\phi(p)} \nabla \overline{K}_{\phi(q)} = K_{\phi(p),\phi(q)}. \quad (2)$$

Lemma 4. *The Randić spectrum of* $\Gamma(\mathbb{Z}_n)$ *is* $\{0^{[p+q-2]}, \pm 1\}$.

Proof. Let $n = pq$, where p and q with $p < q$ are distinct primes. The proper divisors of n are p and q, so that Υ_{pq} is K_2. As $r_1 = r_2 = 0$, using Theorem 1 and Equation (2), the Randić spectrum of $\Gamma(\mathbb{Z}_n)$ consists of the eigenvalue 0 with multiplicity $p + q - 4$ and the remaining two eigenvalues are given by the matrix

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

$\square$

Lemma 5. *The Randić spectrum of* $\Gamma(\mathbb{Z}_{p^2q})$ *is*

$$\left\{ 0^{[p^2+pq-2p-3]}, \left(\frac{-1}{pq-2}\right)^{[p-2]}, x_1 = 1, x_2, x_3, x_4 \right\},$$

where x_2, x_3 *and* x_4, $(x_2 \geq x_3 \geq x_4)$, *are the zeros of the characteristic polynomial of the matrix* $C(P_4)$.

Proof. Let $n = p^2q$, where p and q are distinct primes. Since proper divisors of n are p, q, pq, p^2, so Υ_{p^2q} is the path $P_4 : q \sim p^2 \sim pq \sim p$. By Lemma 3, we have

$$\Gamma(\mathbb{Z}_{p^2q}) = \Upsilon_{p^2q}[\Gamma(A_q), \Gamma(A_{P^2}), \Gamma(A_{pq}), \Gamma(A_p)] = P_4[\overline{K}_{\phi(p^2)}, \overline{K}_{\phi(q)}, K_{\phi(p)}, \overline{K}_{\phi(pq)}].$$

Now, by Theorem 1, $\frac{N_1}{r_1+N_1} = \frac{N_2}{r_2+N_2} = \frac{N_4}{r_4+N_4} = 1$ and $\frac{N_3}{r_3+N_3} = \frac{pq-p}{pq-2}$. The Randić spectrum of $\Gamma(\mathbb{Z}_{p^2q})$ consists of the eigenvalue 0 with multiplicity $p^2 + pq - p - 3$, the eigenvalue $\dfrac{-1}{pq-2}$ with multiplicity $p-2$ and the remaining four eigenvalues are given by the matrix

$$C(P_4) = \begin{pmatrix} 0 & \sqrt{\frac{p^2-p}{p^2-1}} & 0 & 0 \\ \sqrt{\frac{p^2-p}{p^2-1}} & 0 & \sqrt{\frac{q-1}{(p-1)(pq-2)}} & 0 \\ 0 & \sqrt{\frac{q-1}{(p-1)(pq-2)}} & \frac{pq-p}{pq-2} & \sqrt{\frac{pq-p-q+1}{pq-2}} \\ 0 & 0 & \sqrt{\frac{pq-p-q+1}{pq-2}} & 0 \end{pmatrix}.$$

□

Now, we obtain the Randić eigenvalues of $\Gamma(\mathbb{Z}_n)$, when n is a prime power.

Theorem 2. *Let $n = p^{2m}$, where $p > 2$ is prime and $m \geq 2$ is a positive integer. Then the Randić spectrum of $\Gamma(\mathbb{Z}_n)$ consists of the eigenvalue 0 with multiplicity $p^{2m-1} - p^m - m + 1$, the eigenvalue $\frac{-1}{p^i - 2}$ with multiplicity $\phi(p^{2m-i}) - 1$, where $i = m, m+1, \ldots, 2m-2, 2m-1$. The other Randić eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the eigenvalues of matrix (3).*

Proof. Let $n = p^{2m}$, for some positive integer $m \geq 2$. Then the proper divisors of n are

$$\{p, p^2, \ldots, p^{m-1}, p^m, p^{m+1}, \ldots, p^{2m-2}, p^{2m-1}\}.$$

We note that the vertex p of $\Upsilon_{p^{2m}}$ is adjacent to the vertex p^{2m-1} and the vertex p^2 is adjacent to both p^{2m-1} and p^{2m-2}. Thus, in general, vertices p^i and p^j of $\Upsilon_{p^{2m}}$ are adjacent if and only if $i + j \geq 2m$, with $1 \leq i \leq 2m-1$ and $i \neq j$. Now, $N_1 = \phi(p) = p - 1$, $N_2 = \phi(p) + \phi(p^2) = p - 1 + p^2 - p = p^2 - 1$. So, by following the similar steps and using the fact that $\sum\limits_{i=1}^{b} p^i = p^b - 1$, we have

$$N_i = p^i - 1, \text{ for } i = 1, 2, 3, \ldots, m-2, m-1.$$

Similarly, for $i = m, m+1, \ldots, 2m-2, 2m-1$, we have

$$N_i = \sum_{j=1}^{i} \phi(p^j) - \phi(p^{2m-i}) = p^i - 1 - \phi(p^{2m-i}).$$

Also, for $i = 1, 2, \ldots, m-2, m-1$, we see that $(p^i)^2$ is not a multiple of p^{2m}, so that $G_i = \overline{K}_{\phi(p^{2m-i})}$. For $i = m, m+1, \ldots, 2m-2, 2m-1$, we observe that p^{2m} divides $(p^i)^2$ and hence $G_i = K_{\phi(p^{2m-i})}$. This implies that $r_i = 0$, for $i = 1, 2, \ldots, m-2, m-1$ and $r_i = \phi(p^{2m-i}) - 1$, for $i = m, m+1, \ldots, 2m-2, 2m-1$. Further, $2r_i + N_i = p^i - 1$, for $i = 1, 2 \ldots, m-1$ and $2r_i + N_i = p^i + \phi(p^{2m-i}) - 3$, for $i = m, \ldots, 2m-2, 2m-1$. Therefore, by Theorem 1, we see that 0 is the Randić eigenvalue of $\Gamma(\mathbb{Z}_n)$ with multiplicity $\phi(p^{2m-1}) - 1 + \phi(p^{2m-2}) - 1 + \cdots + \phi(p^{m+2}) - 1 + \phi(p^{m+1}) - 1 = p^{2m-1} - p^m - m + 1$. Also, for $i = m, m+1, \ldots, 2m-2, 2m-1$, we have $r_i + N_i = \phi(p^{2m-i}) - 1 + p^i - 1 - \phi(p^{2m-i}) = p^i - 2$. So, by Theorem 1, $\frac{-1}{p^i - 2}$ is the Randić eigenvalue of $\Gamma(\mathbb{Z}_n)$ with multiplicity $\phi(p^{2m-i}) - 1$. The other Randić eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the eigenvalues of following matrix (3).

$$\begin{pmatrix} A_{m-1} & & & B_{(m-1)\times(m-1)} & & \\ & c_m & a_{m,m+1} & \cdots & a_{m,2m-2} & a_{m,2m-1} \\ & a_{m+1,m} & c_{m+1} & \cdots & a_{m+1,2m-2} & a_{m+1,2m-1} \\ B^T & \vdots & \vdots & \ddots & \vdots & \vdots \\ & a_{2m-2,m} & a_{2m-2,m+1} & \cdots & c_{2m-2} & a_{2m-2,2m-1} \\ & a_{2m-1,m} & a_{2m-1,m+1} & \cdots & a_{2m-1,2m-2} & c_{2m-1} \end{pmatrix}, \tag{3}$$

where $A_{m-1} = diag(0, 0, \ldots, 0, 0)$,

$$B = \begin{pmatrix} 0 & 0 & \ldots & 0 & a_{1,2m-1} \\ 0 & 0 & \ldots & a_{2,2m-2} & a_{2,2m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & a_{m-2,2m-2} & a_{m-2,2m-1} \\ 0 & a_{m-1,m+1} & \ldots & a_{m-1,2m-2} & a_{m,2m-1} \end{pmatrix}$$

and $a_{i,j} = a_{j,i} = \sqrt{\dfrac{n_i n_j}{(r_i + N_i)(r_j + N_j)}}$, for $1 \leq i, j \leq 2m-1$; $c_i = \dfrac{r_i}{r_i + N_i}$, for $i = m, m+1, \ldots, 2m-1$. □

By using the similar procedure as in Theorem 2, the case $\Gamma(\mathbb{Z}_{p^{2m+1}})$ can be proved.

Theorem 3. *Let $n = p^{2m+1}$, where $p > 2$ is a prime and $m \geq 2$ is a positive integer. Then the Randić spectrum of $\Gamma(\mathbb{Z}_n)$ consists of the eigenvalue 0 with multiplicity $p^{2m-1} - p^m - m$, the eigenvalue $\dfrac{-1}{p^i - 2}$ with multiplicity $\phi(p^{2m+1-i}) - 1$, where $i = m+1, m+2 \ldots, 2m-1, 2m$. The other Randić eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the eigenvalues of matrix (4).*

$$\begin{pmatrix} A_m & & & B_{(m)\times(m)} & & \\ & c_{m+1} & a_{m+1,m+2} & \cdots & a_{m+1,2m-1} & a_{m+1,2m} \\ & a_{m+2,m+1} & c_{m+2} & \cdots & a_{m+2,2m-1} & a_{m+2,2m} \\ B^T & \vdots & \vdots & \ddots & \vdots & \vdots \\ & a_{2m-1,m+1} & a_{2m-1,m+2} & \cdots & c_{2m-1} & a_{2m-1,2m} \\ & a_{2m,m+1} & a_{2m,m+2} & \cdots & a_{2m,2m-1} & c_{2m} \end{pmatrix}, \tag{4}$$

where $A_m = diag(0, 0, \ldots, 0, 0)$,

$$B = \begin{pmatrix} 0 & 0 & \ldots & 0 & a_{1,2m} \\ 0 & 0 & \ldots & a_{2,2m-1} & a_{2,2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{m-1,m+2} & \ldots & a_{m-1,2m-1} & a_{m-1,2m} \\ a_{m,m+1} & a_{m,m+2} & \ldots & a_{m,2m-1} & a_{m,2m} \end{pmatrix},$$

$a_{i,j} = a_{j,i} = \sqrt{\dfrac{n_i n_j}{(r_i + N_i)(r_j + N_j)}}$, for $1 \leq i, j \leq 2m$; $c_i = \dfrac{r_i}{r_i + N_i}$, for $i = m+1, m+2, \ldots, 2m$ and $r_i \begin{cases} 0 & \text{for } i = 1, 2, \ldots, m-1, m \\ \phi(p^{2m+1-i}) - 1 & \text{for } i = m+1, m+2, \ldots, 2m-1, 2m. \end{cases}$

For $n = p^2$, we have the following consequence of Theorem 2.

Corollary 1. *For any prime p and $n = p^2$, the Randić spectrum of $\Gamma(\mathbb{Z}_n)$ is*

$$\left\{ -1, \left(\frac{1}{p-2} \right)^{[p-2]} \right\}.$$

Proof. Since $\Gamma(\mathbb{Z}_{p^2}) = \Gamma(A_p)$ is the complete graph K_{p-1}, the result follows. $\square$

For $n = p^3$ in Theorem 3, we have following result.

Corollary 2. *Let $n = p^3$. Then the Randić spectrum of $\Gamma(\mathbb{Z}_n)$ is*

$$\left\{0^{[p^2-p-1]}, 1, \left(\frac{-1}{p^2-2}\right)^{[p-2]}, \frac{p^2-p}{p^2-2}\right\}.$$

Proof. Since the proper divisors of n are p and p^2, so Υ_n is $K_2 : p \sim p^2$. By Lemma 3,

$$\Gamma(\mathbb{Z}_{p^3}) = \Upsilon_{p^3}[\Gamma(A_p), \Gamma(A_{p^2})] = K_2[\overline{K}_{\phi(p^2)}, \overline{K}_{\phi(p)}] = \overline{K}_{p(p-1)} \nabla K_{p-1}.$$

This implies that $\Gamma(\mathbb{Z}_{p^3})$ is a complete split graph of order p^2-1, having independent set of cardinality $p(p-1)$ and clique of order $p-1$. By Theorem 1, Randić spectrum of $\Gamma(\mathbb{Z}_n)$ consists of 0 with multiplicity p^2-p-1, the eigenvalue $\frac{-1}{p^2-2}$ with multiplicity $p-2$ and other two eigenvalues are given by

$$C(K_2) = \begin{pmatrix} 0 & \sqrt{\frac{p^2-p}{p^2-2}} \\ \sqrt{\frac{p^2-p}{p^2-2}} & \frac{p-2}{p^2-2} \end{pmatrix}.$$

Now, it is easy to see that the eigenvalues of $C(K_2)$ are $\left\{1, \frac{p-p^2}{p^2-2}\right\}$. $\square$

If $m = 2$ in Theorem 2, we obtain the following result.

Corollary 3. *The Randić spectrum of $\Gamma(\mathbb{Z}_n)$, where $n = p^4$ is*

$$\left\{0^{[p^3-p^2-1]}, 1, \left(\frac{-1}{p^3-2}\right)^{[p-2]}, \left(\frac{-1}{p^2-2}\right)^{[p^2-p-1]}, \frac{2-2p^2+2p^3-p^4 \pm \sqrt{D}}{2(p^5-2p^3-2p^2+4)}\right\},$$

where $D = 4p^{10} - 8p^9 - 7p^8 + 8p^7 + 24p^6 - 40p^4 + 8p^3 + 8p^2 + 4$.

Proof. As proper divisors of n are p, p^2 and p^3, so Υ_n is $P_3 : p \sim p^3 \sim p^2$. By Lemmas 1, 2 and 3, we have $\Gamma(A_p) = \overline{K}_{\phi(p^3)} = \overline{K}_{p^2(p-1)}, \Gamma(A_{p^2}) = K_{\phi(p^2)} = K_{p(p-1)}$, and $\Gamma(A_{p^3}) = K_{\phi(p)} = K_{p-1}$. Therefore,

$$\begin{aligned}\Gamma(\mathbb{Z}_{p^4}) &= \Upsilon_{p^3}[\Gamma(A_p), \Gamma(A_{p^3}), \Gamma(A_{p^2})] \\ &= P_3[\overline{K}_{p^2(p-1)}, K_{p-1}, K_{p(p-1)}] \\ &= K_{p-1} \nabla (\overline{K}_{p^2(p-1)} \cup K_{p(p-1)}).\end{aligned}$$

Thus, by Theorem 1, the Randić spectrum of $\Gamma(\mathbb{Z}_n)$ consists of eigenvalue 0 with multiplicity $p^3 - p^2 - 1$, the eigenvalue $\frac{-1}{p^3-2}$ with multiplicity $p-2$, the eigenvalue $\frac{-1}{p^2-2}$ with multiplicity $p^2 - p - 1$ and the remaining three eigenvalues are given by

$$C(P_3) = \begin{pmatrix} 0 & \sqrt{\frac{p^3-p^2}{p^3-2}} & 0 \\ \sqrt{\frac{p^3-p^2}{p^3-2}} & \frac{p-2}{p^3-2} & \sqrt{\frac{(p-1)(p^2-p)}{(p^2-2)(p^3-2)}} \\ 0 & \sqrt{\frac{(p-1)(p^2-p)}{(p^2-2)(p^3-2)}} & \frac{p^2-p-1}{p^2-2} \end{pmatrix}.$$

Hence the result. □

If $n = p^{2m}, m \geq 2$ or $n = p^{2m+1}, m \geq 1$, then by Corollaries 1, 2 and 3, $\Gamma(\mathbb{Z}_n)$ is not Randić integral. Also, by Example 1, we see that $\Gamma(\mathbb{Z}_n)$ cannot be Randić integral as $\Gamma(\mathbb{Z}_n)$ may not be complete bipartite, when n is a product of more than three primes. However, by Lemma 4, if $n = pq$, where $p < q$ are primes, then $\Gamma(\mathbb{Z}_n)$ is the complete bipartite graph and is Randić integral.

Proposition 2. *The zero divisor graph $\Gamma(\mathbb{Z}_n)$ is Randić integral if and only if n is a product of two distinct primes.*

Acknowledgements. The authors are grateful to the anonymous referees for their valuable suggestions. The research of S. Pirzada is supported by SERB-DST, New Delhi under the research project number CRG/2020/000109.

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