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Research Article

Enumeration of k-noncrossing trees and forests

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Abstract: A k-noncrossing tree is a noncrossing tree where each node receives a label in $\{1, 2, \ldots, k\}$ such that the sum of labels along an ascent does not exceed $k+1$, if we consider a path from a fixed vertex called the root. In this paper, we provide a proof for a formula that counts the number of k-noncrossing trees in which the root (labelled by k) has degree d. We also find a formula for the number of forests in which each component is a k-noncrossing tree whose root is labelled by k.

Keywords: k-noncrossing tree, degree, forest

AMS Subject classification: 05A19, 05C05, 05C30

1. Introduction

A *noncrossing tree* is a tree drawn in the plane with its vertices on the boundary of a circle such that the edges are line segments that do not cross inside the circle. In Figure 1, we show a noncrossing tree on 12 vertices.

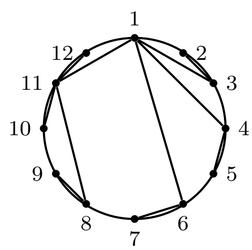

Figure 1. **Noncrossing tree.**

These trees are generalized by combinatorial structures called k-noncrossing trees which were first considered by Pang and Lv [6] in 2010. Formally, a *k-noncrossing* *tree* is a noncrossing tree where each node receives a label in $\{1, 2, \ldots, k\}$ such that the sum of labels along an ascent does not exceed $k+1$, if we consider a path from a fixed vertex called the root. Vertex 1 is normally taken as the root. An *ascent* is an edge (x, y) such that $x < y$. In Figure 1, if we consider the path from the root to vertex 9, then we have the edges $(1, 11)$ and $(8, 9)$ as ascents. Figure 2 gives a 4-noncrossing tree on 12 vertices, where the subscripts are labels of the vertices from the set $\{1, 2, 3, 4\}$.

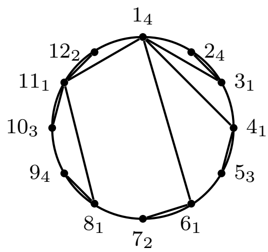

Figure 2. **A 4-noncrossing tree.**

The aforementioned authors showed that these trees with root labelled by k on n vertices are counted by the $(2k+1)$-Catalan number,

$$\frac{1}{2k(n-1)+1}\binom{(2k+1)(n-1)}{n-1}. \tag{1}$$

The same formula counts $(2k+1)$-ary trees, and the authors of [6] constructed a bijection between the two combinatorial structures.
If we set $k = 2$ in (1), we get *2-noncrossing trees*, which were introduced and studied by Yan and Liu [10]. If the root is labelled by 2, then the number of such trees on n vertices is given by

$$\frac{1}{5n-4}\binom{5n-4}{n-1}. \tag{2}$$

Formula (2) also counts the number of 5-ary trees with $n-1$ internal vertices. The number of 2-noncrossing trees on n vertices with root labelled by 1 is given by

$$\frac{2}{5n-3}\binom{5n-3}{n-1}.$$

Several statistics on noncrossing trees have been considered in the literature: An example is the degree of a fixed vertex [4]. In Section 2, we obtain an equivalent formula for k-noncrossing trees. Other statistics include number of end-points or boundary edges [1], maximum degree [1], descents [3], left and right leaves [7] or in- and out-degree sequences [5] among others. Equivalent results for k-noncrossing trees are yet to be obtained. In Section 3, we use generating functions to enumerate forests of k-noncrossing trees.

2. Enumeration of k-noncrossing trees by root degree

Let us review a representation of noncrossing trees introduced by Panholzer and Prodinger in [8]. For any non-root vertex y of a noncrossing tree T, let x be the parent of y. If $x > y$ then the vertex corresponding to y is given label l. Otherwise, it is given label r. Vertex 1 can always be taken as the root. A subtree whose root is labelled by l (resp. r) is said to be a *left* (resp. *right*) *subtree.* In Figure 3, we show the l, r-representation of the noncrossing tree in Figure 1.

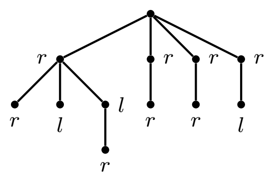

Figure 3. l, r**-representation of a noncrossing tree.**

We begin by proving the following two lemmas.

Lemma 1. *There is a bijection between the set of k-noncrossing trees on n vertices with roots labelled by* 1 *and an ordered k-tuple of k-noncrossing trees with roots labelled by* $1, 2, \ldots, k$ *respectively such that the total number of vertices is $n+k-1$ and for the tree with root labelled by i, the first child of the root (if any) is labelled by $k-i+1$.*

Proof. Our bijection is a modification of the bijection between the set of 2-noncrossing trees on n vertices with roots labelled by 1 and the set of ordered pairs of 2-noncrossing trees with roots labelled by 2 such that the total number of edges is n, obtained by Yan and Liu in [10].
We first give the procedure of obtaining a k-tuple of k-noncrossing trees such that the root of the i-th tree T_i is labelled by i and the first child of T_i is labelled by $k-i+1$, from a k-noncrossing tree T whose root is labelled by 1. Let the root of T be t.
For $1 \leq i \leq k$, let u_i be the first child of t labelled i to the right of $u_1, u_2, \ldots, u_{i-1}$, if such a child exists. We obtain T_i from T as follows:

(a) If there is no u_i, then T_i is the tree that consists of a single root labelled by i.

(b) Otherwise:

 (i) First, relabel u_i to $k-i+1$ and the root t to i.

(ii) Remove all the subtrees rooted at t except the subtree rooted at u_i. Also remove all the right subtrees of u_i.

(iii) For $i \neq k$, root subtrees between u_i and u_j, where $j > i$ such that there is no vertex u_p satisfying $i < p < j$, are attached to vertex u_i in turn as right subtrees, and the right subtrees of vertex u_i are attached to the root t in turn as right subtrees.

(iv) For $i = k$, root subtrees on the right of u_k are attached to u_k in turn as right subtrees, and the right subtrees of u_k are attached to the root t in turn as right subtrees.

We now obtain the reverse procedure: We obtain a k-noncrossing tree T with n vertices such that its root is labelled by 1 from a k-tuple $(T_1, T_2, \ldots, T_k)$ of k-noncrossing trees on $n + k - 1$ vertices such that the roots are labelled by $1, 2, \ldots, k$ respectively and the first child (if there is one) of T_i is labelled by $k - i + 1$ using the following steps:

(a) For $1 \leq i \leq k$, all right subtrees of the first child of T_i are attached to the root of T_i in turn as right subtrees, and the children of the root of T_i other than the first child are attached to the first child in turn as right subtrees.

(b) Relabel the root by 1 and the first child (if there is one) by i.

(c) Now, glue together the roots of T_i for $1 \leq i \leq k$ so that each T_j is on the right of T_m if $j > m$.

The resultant tree is T. □

Figure 4 shows a bijection between 4-noncrossing tree on 15 vertices with root labelled by 1 and four 4-noncrossing trees on 18 vertices with roots labelled by 1, 2, 3 and 4 respectively. The first child of a tree with root labelled by i is labelled by $5 - i$. The subscripts are the labels of the vertices.

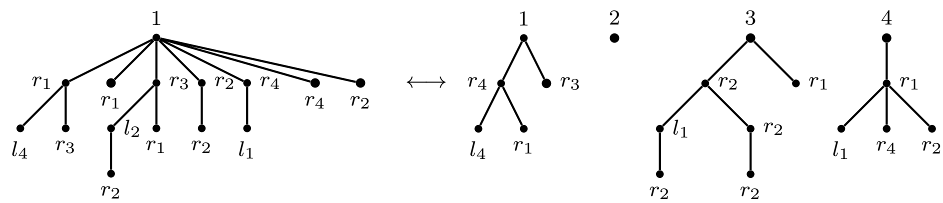

Figure 4. **Example of the bijection in Lemma 1.**

Lemma 2. *There is a bijection between the set of k-noncrossing trees on n vertices with root labelled by i and first child labelled by $k - i + 1$, and the set of k-noncrossing trees on n vertices with root labelled by $i + 1$ and first child labelled by $k - i$.*

Proof. Consider a k-noncrossing tree T on n vertices whose root is labelled by i and the first child (if there is one) is labelled by $k-i+1$. Let $v_1, v_2, \ldots, v_m$ be the children of the root that are labelled by $k-i+1$, from left to right. The first child of the root is thus v_1. We obtain a k-noncrossing tree T' on n vertices with root labelled by $i+1$ and first child of label $k-i$ by the following steps:

(a) Give the new label $i+1$ to the root, new label $k-i$ to v_1 and detach all subtrees from the root except those to the left of v_2. If there is no v_2, all the root subtrees are kept.

(b) Change the labels of $v_2, v_3, \ldots, v_m$ to $i+1$ and make them all children of v_1, attached to the right of the previously existing children.

(c) For $2 \leq j < m$, the old root subtrees that lie between v_j and v_{j+1} become the new right subtrees of v_j and the old right subtrees of v_j become new subtrees attached to v_1 between v_j and v_{j+1}.

(d) The old root subtrees that lie on the right of v_m become the new right subtrees of v_m and the old right subtrees of v_m become new subtrees attached to v_1 as the rightmost subtrees of v_1.

The resultant tree is T'.
We now obtain the reverse procedure: Consider a k-noncrossing tree T' on n vertices whose root is labelled by $i+1$ and the first child (if there is one) is labelled by $k-i$. Let v_1 be the first child of the root labelled by $k-i$. Moreover, let $v_2, v_3, \ldots, v_m$ be the children of v_1 labelled by $i+1$, from left to right. We obtain a corresponding k-noncrossing tree T on n vertices with root labelled by i and first child of label $k-i+1$ by the following steps:

(a) Give the new label i to the root, new label $k-i+1$ to v_1.

(b) Change the labels of $v_2, v_3, \ldots, v_m$ to $k-i+1$ and make them all children of the root, attached to the right of the previously existing children.

(c) For $2 \leq j < m$, the old right subtrees of v_j become new root subtrees that lie between v_j and v_{j+1} and the old subtrees of v_1 that lie between v_j and v_{j+1} become new right subtrees of v_j.

(d) The old right subtrees of v_m become new root subtrees that lie on the right of v_m and the old subtrees of v_1 that lie on the right of v_m become new right subtrees of v_m.

□

An example of the bijection is illustrated in Figure 5.
Let $\mathcal{T}_i$ be the set of k-noncrossing trees on n vertices whose roots are labelled by i and first children of the root are labelled by $k-i+1$. By induction on i in Lemma 2, we have

$$|\mathcal{T}_1| = |\mathcal{T}_2| = \cdots = |\mathcal{T}_k|$$

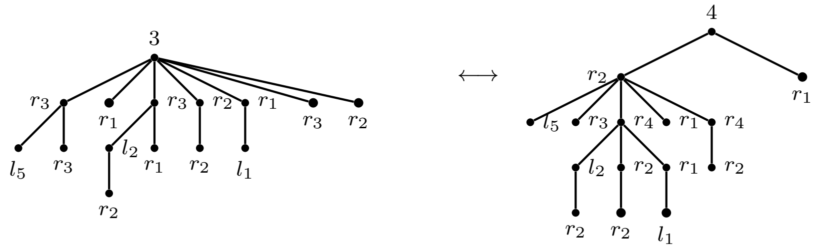

Figure 5. **Example of the bijection in Lemma 2 with $n = 15, k = 5$ and $i = 3$.**

and hence by Equation (1),

$$|\mathcal{T}_i| = \frac{1}{2k(n-1)+1}\binom{(2k+1)(n-1)}{n-1}$$

for all $i \in \{1, 2, \ldots, k\}$.
We now prove the following theorem:

Theorem 1. *There is a bijection between the set of k-noncrossing trees on n vertices with roots labelled by 1 and an ordered k-tuple of k-noncrossing trees with roots labelled by k such that the total number of vertices is $n + k - 1$.*

Proof. Let $\mathcal{S}_T$ be the set of k-noncrossing trees on n vertices such that the roots are labelled by 1. Also, let $\mathcal{S}_O$ be the set of $k-$tuples of k-noncrossing trees with $n+k-1$ vertices, such that the roots of these trees are labelled by k. Applying Lemma 1, and then the observation from Lemma 2 that $|\mathcal{T}_1| = |\mathcal{T}_2| = \cdots = |\mathcal{T}_k|$, we obtain a bijection $\phi : \mathcal{S}_T \longrightarrow \mathcal{S}_O$. □

We require the following definition in the next theorem: A *complete t-ary tree* is an ordered tree such that each internal vertex has t children.

Theorem 2. *There is a bijection between the set of k-noncrossing trees of order n such that roots are labelled by k and the set of complete $(2k + 1)$-ary trees with $n - 1$ internal vertices.*

Proof. We mimic the proof of Yan and Liu in [10] where they proved the theorem for the case $k = 2$. We construct a bijection φ from the set of k-noncrossing trees of order n such that roots are labelled by k to the set of complete $(2k+1)$-ary trees with $n - 1$ internal vertices. We start by noting that all the children of the root labelled by k in the k-noncrossing tree are labelled by 1 and are also r-labelled (using the representation of Panholzer and Prodinger [8]). By bijection ϕ, obtained in Theorem 1 above, we construct an inductive map φ. Let T be a k-noncrossing tree with n vertices whose root v is labelled by k. If $n = 1$, then we define $\varphi(T)$ to be a single vertex. Now, suppose that $u_1, u_2, \ldots, u_m$ are the left to right children of v. For $i = 1, 2, \ldots, m$, let L_i (resp. R_i) be the labelled ordered tree with a root labelled by 1 and containing the vertex u_i and all the l-labelled (resp. r-labelled) subtrees of u_i. For $i = 1, 2, \ldots, m$, let $\bar{L}_i$ be the k-noncrossing tree whose root is labelled by 1 obtained from L_i by changing all the l-labelled children of the root to r-labelled. Let $\mathcal{S}_O$ be the set of k-tuples of k-noncrossing trees with roots labelled by k. Suppose that $\phi(\bar{L}_i) = (\bar{L}_{i_1}, \ldots, \bar{L}_{i_k}) \in \mathcal{S}_O$ and $\phi(R_i) = (R_{i_1}, \ldots, R_{i_k}) \in \mathcal{S}_O$ for $i = 1, 2, \ldots, m$. We construct a complete $(2k+1)$-ary tree recursively in which

(a) there are m internal vertices $u'_1, u'_2, \ldots, u'_m$ on the longest rightmost path from the root u'_1.

(b) for $i = 1, 2, \ldots, m$, we have

$$\varphi(\bar{L}_{i_1}), \varphi(\bar{L}_{i_2}), \ldots, \varphi(\bar{L}_{i_k}), \varphi(R_{i_1}), \varphi(R_{i_2}), \ldots, \varphi(R_{i_k})$$

as the left to right subtrees of u'_i.

The resultant tree $\varphi(T)$ is a complete $(2k+1)$-ary tree with $n-1$ internal vertices. The procedure is reversible. □

From [9], we know that the number of lattice paths (consisting of $(1,0)$ and $(0,1)$ steps), from (i,j) to (n, mn) that are below, and do not cross, the line $y = mx$, is given by

$$\frac{mi - j + 1}{(m+1)n - i - j + 1}\binom{(m+1)n - i - j + 1}{n - i}.$$

So, Equation (1) gives the number of lattice paths from $(0,0)$ to $(n-1, 2kn-2k)$ that lie below the line $y = 2kx$, and do not cross it. These paths are said to be *good paths*.

Let us describe the bijection between complete $(2k+1)$-ary trees with $n-1$ internal nodes and good paths from $(0,0)$ to $(n-1, 2kn-2k)$ where each step is of the form $(1,0)$ or $(0,1)$. Traversing a complete $(2k+1)$-ary tree with $n-1$ internal vertices in preorder (i.e., visit vertex, rightmost-child, second rightmost-child, third rightmost-child etc in this order) and drawing a $(1,0)$ step for each internal vertex and a $(0,1)$ step for each leaf (except the last one), we obtain a good path ending at $(n-1, 2kn-2k)$. Thus given a k-noncrossing tree, one obtains a good path via the associated complete $(2k+1)$-ary tree. This procedure is reversible and therefore proves Equation (1).

It is observed that if the degree of vertex 1 in the k-noncrossing tree is d then there is a horizontal run of length d at the beginning of the good path, followed by a vertical step. This is because if the degree of the root in the k-noncrossing tree is d, then its corresponding complete $(2k+1)$-ary tree will have longest rightmost path of length d starting at the root (See the proof of Theorem 2). Thus the number of k-noncrossing trees of order n such that the root is of degree d and labelled by k is equal to the number of good paths starting at $(d,1)$ and ending at $(n-1, 2kn-2k)$. This provides a bijective proof of the following theorem.

Theorem 3. *The number of k-noncrossing trees on n vertices such that the root is labelled by k and is of degree d is given by*

$$\frac{2kd}{(2k+1)(n-1)-d}\binom{(2k+1)(n-1)-d}{n-d-1}. \tag{3}$$

Now, considering good paths from $(d,0)$ to $(n-1, 2kn-2k)$ we have:

Corollary 1. *There are*

$$\frac{2kd+1}{(2k+1)(n-1)-d+1}\binom{(2k+1)(n-1)-d+1}{n-d-1}.$$

k-noncrossing trees on n vertices with root labelled by k and of degree greater than or equal to d.

Setting $k=1$ in Equation (3), we recover the equivalent result for noncrossing trees obtained in [4]. Also setting $d=1$, we obtain the following corollary.

Corollary 2. *The number of k-noncrossing trees on n vertices with root labelled by k such that the root has degree 1 is given by*

$$\frac{2k}{(2k+1)(n-1)-1}\binom{(2k+1)(n-1)-1}{n-2},$$

and the proportion of such trees among all k-noncrossing trees with root labelled by k on n vertices is given by

$$\frac{4nk^2-4k^2+2k}{4nk^2+4nk+n-4k^2-6k-2}.$$

The asymptotic proportion of k-noncrossing trees with roots labelled by k and of degree 1, among all k-noncrossing trees with root labelled by k is given by $\frac{4k^2}{4k^2+4k+1}$.

3. Forests of k-noncrossing trees

In this section, we enumerate k-noncrossing forests with roots labelled by k. We consider forests of k-noncrossing trees with the following two properties:

- each component is rooted at a vertex whose label is smallest.
- the components are k-noncrossing trees with the root labelled by k, and the components do not intersect each other.

We shall use generating functions to prove the following result:

Theorem 4. *The number of forests on n vertices and r components such that each component is a k-noncrossing tree whose root is labelled by k is given by*

$$\frac{1}{(2k+1)n-2kr}\binom{n}{r-1}\binom{(2k+1)n-2kr}{n-r}.$$

Proof. Let T be the generating function for the components, i.e., k-noncrossing trees with roots labelled by k. We decompose forests according to components that contain vertex 1. If the component has m vertices then there are m spaces to be filled with further forests $F_1, F_2, \ldots, F_m$ (possibly empty).

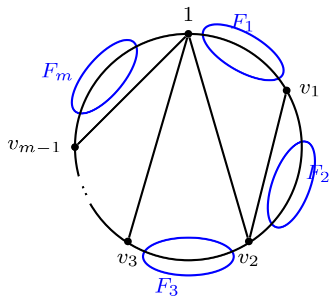

Figure 6. **Decomposition of forests according to node number 1.**

So if $T(z) = \sum_{n\geq 1} t_n z^n$, then the generating function $F(z,w)$ for forests where w marks the number of components satisfies

$$F(z,w) = 1 + w\sum_{m\geq 1} t_m z^m F(z,w)^m = 1 + wT(zF(z,w)).$$

Let $T(z) = z(1+u(z))$. From the functional equation for the generating function for these trees, i.e., $T^{2k+1} - z^{2k-1}T + z^{2k} = 0$ (see [6]), we have $u(z) = z(1+u(z))^{2k+1}$. Set $F(z,w) = 1 + wy$. Then, $1 + wy = 1 + wT(z(1+wy))$ or $y = T(z(1+wy))$. Define q as $q(t) = \frac{t}{T^{-1}(t)}$. Since $z(1+wy) = T^{-1}(y)$, then $z(1+wy) = \frac{y}{q(y)}$. If follows that $y = z(1+wy)q(y)$. By the Lagrange inversion formula, the number of forests on n vertices and r components such that each component is a k-noncrossing tree whose root is labelled by k is

$$\begin{aligned}[z^n w^r]F(z,w) &= [z^n w^{r-1}]y\\ &= \frac{1}{n}[t^{n-1}w^{r-1}]((1+wt)q(t))^n\\ &= \frac{1}{n}\binom{n}{r-1}[t^{n-r}]q(t)^n\end{aligned}$$

It remains to obtain $[t^{n-r}]q(t)^n$. By definition, $t = T\left(\frac{t}{q(t)}\right)$.
Writing $T(z) = z(1+u(z))$, we have

$$t = \frac{t}{q(t)}\left(1 + u\left(\frac{t}{q(t)}\right)\right)$$

or

$$q(t) = 1 + u\left(\frac{t}{q(t)}\right)$$

where $u(z)$ satisfies $u(z) = z(1+u(z))^{2k+1}$. So,

$$q(t) - 1 = u\left(\frac{t}{q(t)}\right) = \frac{t}{q(t)}\left(1 + u\left(\frac{t}{q(t)}\right)\right)^{2k+1} = \frac{t}{q(t)} \cdot q(t)^{2k+1}.$$

Therefore, $q(t) = 1 + tq(t)^{2k}$. If we set $tq(t^{2k}) = p(t)$, then

$$p(t) = tq(t^{2k}) = t\left(1 + t^{2k}q(t^{2k})^{2k}\right) = t(1 + p(t)^{2k}).$$

By Lagrange inversion,

$$\begin{aligned}
[t^{n-r}]q(t)^n &= [t^{2kn-2kr}]q(t^{2k})^n \\
&= [t^{2kn-2kr}]\left(\frac{p(t)}{t}\right)^n \\
&= [t^{(2k+1)n-2kr}]p(t)^n \\
&= \frac{n}{(2k+1)n-2kr}[s^{2kn-2kr}](1+s^{2k})^{(2k+1)n-2kr} \\
&= \frac{n}{(2k+1)n-2kr}\binom{(2k+1)n-2kr}{n-r}
\end{aligned}$$

Finally, the number of k-noncrossing forests with n vertices and r components is

$$\begin{aligned}
[z^n w^r]F(z,w) &= \frac{1}{n}\binom{n}{r-1}[t^{n-r}]q(t)^n \\
&= \frac{1}{n}\binom{n}{r-1}\frac{n}{(2k+1)n-2kr}\binom{(2k+1)n-2kr}{n-r} \\
&= \frac{1}{(2k+1)n-2kr}\binom{n}{r-1}\binom{(2k+1)n-2kr}{n-r}.
\end{aligned}$$

□

Setting $k = 2$ in (3), we obtain the formula for the number of forests of 2-noncrossing trees with n vertices and r components such that the root of each tree is labelled by 2. Also, setting $k = 1$ in the same equation, we get the number of forests of noncrossing trees of order n with r components initially obtained by Flajolet and Noy [2].

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