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# Pareto-efficient strategies in 3-person games played with staircase-function strategies

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**Abstract:** A tractable method of solving 3-person games in which players' pure strategies are staircase functions is suggested. The solution is meant to be Pareto-efficient. The method considers any 3-person staircase-function game as a succession of 3-person games in which strategies are constants. For a finite staircase-function game, each constant-strategy game is a trimatrix game whose size is likely to be relatively small to solve it in a reasonable time. It is proved that any staircase-function game has a single Pareto-efficient situation if every constant-strategy game has a single Pareto-efficient situation, and vice versa. Besides, it is proved that, whichever the staircase-function game continuity is, any Pareto-efficient situation of staircase function-strategies is a stack of successive Pareto-efficient situations in the constant-strategy games. If a staircase-function game has two or more Pareto-efficient situations, the best efficient situation is one which is the farthest from the triple of the most unprofitable payoffs. In terms of 0-1-standardization, the best efficient situation is the farthest from the triple of zero payoffs.

**Keywords:** game theory, payoff functional, Pareto efficiency, staircase-function strategy, trimatrix game

**AMS Subject classification:** 91A06, 91A10, 91A50, 18F20

## 1. Introduction

In game theory, a 3-person game is used to model a struggle for rationalizing the distribution of limited resources (facilities, access, energy, etc.) among three persons (players) when they cannot be re-personified in two players (or two groups) [1, 2, 4, 19]. Any game is characterized with its qualitative properties and solutions. Qualitative properties of the 3-person game strongly depend on the sets (e. g., whether they are finite or not) of the players' pure strategies [1, 16, 20, 22]. Solution properties are dependent on the sets finiteness and payoffs [11, 15, 20, 21]. The properties including payoff attraction are far simpler for the case of when the sets are countable. The

simplest 3-person game is when the sets are finite. In this case, the game is called trimatrix [14, 20]. The simplest trimatrix game is a dyadic game [15, 20, 21].
Amazingly enough, bimatrix and trimatrix games are intricate models of the struggle process. The struggle process is a selection-and-payoff event or a series of such events, without any differentiation or integration, but the interpretation of the eventual result sometimes appears uncertain enough. First, the optimality or the best decision (solution) has multiple types. This is so because the optimality requires equilibrium, efficiency (profitability), and fairness [5, 12, 20]. These types are often contradictory in a 2-person game, and they may be far more contradictory in a 3-person game [21]. For instance, an equilibrium situation may be efficient for one or two players while it is not profitable for the remaining players. Second, a trimatrix game may have multiple equilibria along with multiple Pareto-efficient situations [7, 15, 19, 20]. This induces the solution uncertainty. The unfairness of the players' payoffs worsens the solution selection. Furthermore, even a trimatrix game may have a continuum of equilibria, wherein the best decision selection is far more difficult.
Infinite (even when a set of pure strategies is countable) and, moreover, continuous 3-person games are far more complicated than trimatrix (i. e., finite) games. Whereas the trimatrix game has at least an equilibrium (generally speaking, in mixed strategies), an infinite game may not have an equilibrium, or it may be indeterminable, or it may be impracticable. The best option, therefore, is to model the struggle process with a finite 3-person game trivially rendered to a trimatrix game [4, 18, 21].
A special attention is paid to the structure of the player's pure strategy. As the pure strategy structure becomes more complex, the practicability of a game solution is further sophisticated. This becomes more ungainly when there is a mixed strategy to be used in a game. This is explained by the simplicity of the pure strategy solution (not requiring repetitions of a game) and the sophistication of the mixed strategy solution (requiring repetitions of a game for a proper implementation of the solution). It is worth to note that a 3-person game may have no Nash equilibrium in pure (stationary) strategies [2]. The most trivial strategy is a decision corresponding to a one-stage event whose duration through time is (usually, negligibly) short. A strategy can be also a multi-stage process like a staircase-function defined on a time interval [6, 7, 9, 12]. In a pure strategy situation, a triple of such staircase-function strategies (from the players in a 3-person game) is mapped into a real value [8, 9, 17]. When each of the players possesses a finite set of such function-strategies, the staircase-function game is finite. It is easily rendered down to a trimatrix game, wherein a pure strategy is a conditional point (just like it is in ordinary finite games), which in reality is a staircase function (seen from a "disclosure" of the conditional point).

## 2. Motivation to the trimatrix game and efficient solution

In real-world practice, the continuity of a process is an ill-posed assumption due to natural constraints imposed by, generally speaking, corpuscular (discontinuous) activity. This is why any process through a definite time interval is a finite set of elementary actions. If the strategy is a time function defined on a closed (time) interval, it is staircase as the elementary action of a player is naturally constrained. Thus, a staircase function during an elementary action must be considered constant. To make a staircase-function game finite, the set of possible values of the player's pure strategy should be finite. In such a staircase-function trimatrix game the player's selection of a pure strategy means using a staircase function on a time interval whereon every pure strategy is defined. The total number of the player's pure strategies in the staircase-function trimatrix game is determined by the number of "stair" subintervals and the number of possible values of the player's pure strategy (staircase function).

The total number of situations in a finite staircase-function game is immensely gigantic. For example, if the number of subintervals is 4, and the number of possible values of the player's pure strategy is just 7, then there are $7^4 = 2401$ possible pure strategies at this player, where every strategy is a 4-subinterval 7-staircased (time) function. The respective trimatrix $2401 \times 2401 \times 2401$ game even in this trivialized case appears to be big enough: there are 13841287201 (over 13.8 billion) situations in pure strategies, which hardly can be handled in a reasonable amount of computational time.

In a more real example, when every strategy, say, is a 10-subinterval 12-staircased function, the respective trimatrix $12^{10} \times 12^{10} \times 12^{10}$ game is intractable gigantic: every player has 61917364224 (over 61.9 billion) strategies, and thus a solution cannot be found in a reasonable amount of computational time by using a reasonably expensive hardware. This means that straightforwardly solving staircase-function trimatrix games is impracticable, whichever practical purposes are.

Another question is what the solution should be. Although the property of solution stability is considered important, the equilibrium in 3-person games often is unprofitable for at least one of the players. For example, in a $2 \times 3 \times 2$ game with payoff matrices (the left and right submatrices correspond to the first and second pure strategies of the third player)

$$\mathbf{F} = \left[ \begin{bmatrix} 1 & 6 & -5 \\ 4 & -11 & -6 \end{bmatrix} \; \begin{bmatrix} 3 & -1 & 4 \\ 10 & 7 & -6 \end{bmatrix} \right] \tag{1}$$

and

$$\mathbf{G} = \left[ \begin{bmatrix} -5 & 0 & -1 \\ 4 & -4 & 3 \end{bmatrix} \; \begin{bmatrix} -4 & -1 & -3 \\ 7 & 11 & 4 \end{bmatrix} \right] \tag{2}$$

and

$$\mathbf{H} = \left[ \begin{bmatrix} -2 & -3 & 3 \\ 7 & -1 & 0 \end{bmatrix} \; \begin{bmatrix} 2 & 6 & 5 \\ 4 & -1 & 1 \end{bmatrix} \right] \tag{3}$$

of the first, second, and third players, respectively, there are two pure strategy equilibria with payoffs

$$\{4,\ 4,\ 7\} \tag{4}$$

and

$$\{7,\ 11,\ -1\}\,. \tag{5}$$

By the way, these equilibria are Pareto-efficient. Besides, there is another Pareto-efficient situation with payoffs

$$\{10,\ 7,\ 4\}\,. \tag{6}$$

Obviously, the situation with payoffs (6) is unprofitable for the third player, whereas it is absolutely profitable for the first player. Situation with payoffs (5) is absolutely unprofitable for the third player, whereas it is absolutely profitable for the second player. Situation with payoffs (4), being worse for the first and second players, is absolutely profitable for the third player. Therefore, the equilibria with payoffs (4) and (5) are unstable solutions. Although payoffs (6) are received in a non-equilibrium situation, the collective payoff here is 21. This is 28.57 % greater than the collective payoff from (4) and 19.05 % greater than the collective payoff from (5). So, based on the greater collective utility (which subsequently can be bargained for some compensations to the second and third players), a stability of the Pareto-efficient situation may be eventually induced. The formal stability of equilibria producing payoffs (4) and (5) is likely to be shattered contrariwise.

This is a quite illustrative example of that Pareto-efficient strategies in 3-person games are first to be checked. Although they are not formally stable, their stability will likely be induced in the way described above. Besides, it is worth noting that sometimes Pareto-efficient strategies happen to be equilibrium strategies.

## 3. Objective and tasks to be fulfilled

Whereas it is absolutely impracticable to straightforwardly solve finite 3-person games played with staircase-function strategies, the objective is to develop a tractable method of solving such games. The solution is meant to be Pareto-efficient. To meet the objective, the following seven tasks are to be fulfilled:

1. To formalize a 3-person game, in which the players' strategies are functions of time. Such function-strategies are presumed to be bounded and Lebesgue-integrable.
2. To formalize a 3-person game, in which the players' strategies are staircase functions of time, whereas the time is discrete. In such a game, the set of the player's pure strategies is a continuum of staircase functions.
3. To consider the property of Pareto efficiency in a staircase-function game.
4. To suggest a method of solving finite 3-person staircase-function games by using the Pareto-efficiency criterion.
5. To give an example of how the suggested method is applied.
6. To discuss practical applicability and scientific significance of the method. It should be emphasized why this method must be important for the game theory and operations research development.
7. To make an appropriate conclusion on it and make an outlook for furthering the study.

## 4. A 3-person game played with function-strategies

In a 3-person game, in which the player's pure strategy is a function, let each of the players use strategies defined almost everywhere on (time) interval $[t_1;\ t_2]$ by $t_2 > t_1$. Denote a strategy of the first player by $x(t)$, a strategy of the second player by $y(t)$, and a strategy of the third player by $z(t)$. Surely, these functions are presumed to be bounded, i. e.

$$a_{\min} \leqslant x(t) \leqslant a_{\max} \ \text{by} \ a_{\min} < a_{\max} \tag{7}$$

and

$$b_{\min} \leqslant y(t) \leqslant b_{\max} \ \text{by} \ b_{\min} < b_{\max} \tag{8}$$

and

$$c_{\min} \leqslant z(t) \leqslant c_{\max} \ \text{by} \ c_{\min} < c_{\max}, \tag{9}$$

and the square of the function-strategy is presumed to be Lebesgue-integrable. Thus, pure strategies of the player belong to a rectangular functional space (of time functions):

$$\boldsymbol{X} = \left\{ x(t),\ t \in [t_1;\ t_2],\ t_1 < t_2 :\ a_{\min} \leqslant x(t) \leqslant a_{\max} \ \text{by} \ a_{\min} < a_{\max} \right\} \subset \mathbb{L}_2[t_1;\ t_2] \tag{10}$$

and

$$\boldsymbol{Y} = \left\{ y(t),\ t \in [t_1;\ t_2],\ t_1 < t_2 :\ b_{\min} \leqslant y(t) \leqslant b_{\max} \ \text{by} \ b_{\min} < b_{\max} \right\} \subset\subset \mathbb{L}_2[t_1;\ t_2] \tag{11}$$

and

$$\boldsymbol{Z} = \left\{ z(t),\ t \in [t_1;\ t_2],\ t_1 < t_2 :\ c_{\min} \leqslant z(t) \leqslant c_{\max} \ \text{by} \ c_{\min} < c_{\max} \right\} \subset\subset \mathbb{L}_2[t_1;\ t_2] \tag{12}$$

are the sets of the players' pure strategies, respectively.

The player's payoff in situation

$$\left\{ x(t),\ y(t),\ z(t) \right\} \tag{13}$$

is presumed to be an integral functional [17, 20]. Thus, the first, second, and third players' payoffs in situation (13) are

$$F\big(x(t),\ y(t),\ z(t)\big) = \int\limits_{[t_1;\ t_2]} f\big(x(t),\ y(t),\ z(t),\ t\big) d\mu(t), \tag{14}$$

$$G\big(x(t),\ y(t),\ z(t)\big) = \int\limits_{[t_1;\ t_2]} g\big(x(t),\ y(t),\ z(t),\ t\big) d\mu(t), \tag{15}$$

$$H\big(x(t),\ y(t),\ z(t)\big) = \int\limits_{[t_1;\ t_2]} h\big(x(t),\ y(t),\ z(t),\ t\big) d\mu(t), \tag{16}$$

respectively, where

$$f\left(x\left(t\right),\ y\left(t\right),\ z\left(t\right),\ t\right), \tag{17}$$

$$g\left(x\left(t\right),\ y\left(t\right),\ z\left(t\right),\ t\right), \tag{18}$$

$$h\left(x\left(t\right),\ y\left(t\right),\ z\left(t\right),\ t\right) \tag{19}$$

are functions of $x\left(t\right)$, $y\left(t\right)$, $z\left(t\right)$, explicitly including time $t$. Therefore, the continuous 3-person game

$$\left\langle \left\{\boldsymbol{X},\ \boldsymbol{Y},\ \boldsymbol{Z}\right\},\ \left\{F\left(x\left(t\right),y\left(t\right),z\left(t\right)\right),\ G\left(x\left(t\right),y\left(t\right),z\left(t\right)\right),\ H\left(x\left(t\right),y\left(t\right),z\left(t\right)\right)\right\}\right\rangle \tag{20}$$

is played with function-strategies from respective rectangular functional spaces (10) – (12). Why time $t$ is explicitly included into (17) – (19) will be explained below.

## 5. A 3-person staircase-function game

A staircase-function game is formed naturally. In practical reality, 3-person game (20) with strategies as functions is presumed to be played discretely through time interval $[t_1;\ t_2]$. Then a function-strategy becomes staircase defined by the (physical, economical, biological, social, etc.) laws of a system modeled by the game. The number of subintervals at which the player's pure strategy is constant must be the same for every player.

Denote by $N$ the number of elementary actions could be made by a player, where obviously $N \in \mathbb{N}\backslash\{1\}$. In fact, it is the number of "stair" subintervals at which the player's pure strategy is constant. Then the player's pure strategy is a staircase function which may have at most $N$ different values.

If $\left\{\tau^{(i)}\right\}_{i=1}^{N-1}$ are time points at which the staircase-function strategy changes or may change its value, where

$$t_1 = \tau^{(0)} < \tau^{(1)} < \tau^{(2)} < \ldots < \tau^{(N-1)} < \tau^{(N)} = t_2, \tag{21}$$

then

$$\left\{x\left(\tau^{(i)}\right)\right\}_{i=0}^{N},\ \ \left\{y\left(\tau^{(i)}\right)\right\}_{i=0}^{N},\ \ \left\{z\left(\tau^{(i)}\right)\right\}_{i=0}^{N} \tag{22}$$

are the values of the players' strategies in a play-off of game (20). Time points $\left\{\tau^{(i)}\right\}_{i=0}^{N}$ are not necessarily to be equidistant.

The staircase-function strategies are right-continuous [3]:

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(i)}+\varepsilon\right) = x\left(\tau^{(i)}\right), \tag{23}$$

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} y\left(\tau^{(i)}+\varepsilon\right)=y\left(\tau^{(i)}\right), \tag{24}$$

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} z\left(\tau^{(i)}+\varepsilon\right)=z\left(\tau^{(i)}\right), \tag{25}$$

for $i=\overline{1,\ N-1}$, whereas (if the strategy value changes)

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} x\left(\tau^{(i)}-\varepsilon\right)\neq x\left(\tau^{(i)}\right), \tag{26}$$

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} y\left(\tau^{(i)}-\varepsilon\right)\neq y\left(\tau^{(i)}\right), \tag{27}$$

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} z\left(\tau^{(i)}-\varepsilon\right)\neq z\left(\tau^{(i)}\right), \tag{28}$$

for $i=\overline{1,\ N-1}$. It is easy to see that a strategy value on subinterval $\left[\tau^{(N-1)};\ \tau^{(N)}\right]$ should not change, i. e.

$$x\left(\tau^{(N-1)}\right)=x\left(\tau^{(N)}\right),$$

$$y\left(\tau^{(N-1)}\right)=y\left(\tau^{(N)}\right),$$

$$z\left(\tau^{(N-1)}\right)=z\left(\tau^{(N)}\right).$$

So,

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} x\left(\tau^{(N)}-\varepsilon\right)=x\left(\tau^{(N)}\right), \tag{29}$$

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} y\left(\tau^{(N)}-\varepsilon\right)=y\left(\tau^{(N)}\right), \tag{30}$$

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} z\left(\tau^{(N)}-\varepsilon\right)=z\left(\tau^{(N)}\right). \tag{31}$$

Then constant values (22) by (21) mean that game (20) is a 3-person staircase-function game

$$\left\langle \{\boldsymbol{X}_N,\ \boldsymbol{Y}_N,\ \boldsymbol{Z}_N\}, \left\{F(x(t),\ y(t),\ z(t)),\ G(x(t),\ y(t),\ z(t)),\ H(x(t),\ y(t),\ z(t))\right\}\right\rangle \tag{32}$$

where $\boldsymbol{X}_N\subset\boldsymbol{X}$, $\boldsymbol{Y}_N\subset\boldsymbol{Y}$, $\boldsymbol{Z}_N\subset\boldsymbol{Z}$ are rectangular functional spaces of staircase-function strategies by (21) – (31). The staircase-function game can be thought of as it is a succession of $N$ ordinary 3-person continuous games

$$\left\langle \left\{[a_{\min};\ a_{\max}],\ [b_{\min};\ b_{\max}],\ [c_{\min};\ c_{\max}]\right\},\right. \\ \left.\left\{F(\alpha_i,\ \beta_i,\ \gamma_i),\ G(\alpha_i,\ \beta_i,\ \gamma_i),\ H(\alpha_i,\ \beta_i,\ \gamma_i)\right\}\right\rangle \tag{33}$$

each defined on parallelepiped

$$[a_{\min};\ a_{\max}] \times [b_{\min};\ b_{\max}] \times [c_{\min};\ c_{\max}] \tag{34}$$

by

$$\begin{gathered} \alpha_i = x(t) \in [a_{\min};\ a_{\max}] \ \text{ and} \\ \beta_i = y(t) \in [b_{\min};\ b_{\max}] \ \text{ and} \\ \gamma_i = z(t) \in [c_{\min};\ c_{\max}] \\ \forall\, t \in \left[\tau^{(i-1)};\ \tau^{(i)}\right) \ \text{ for } \ i = \overline{1,\ N-1} \ \text{ and } \ \forall\, t \in \left[\tau^{(N-1)};\ \tau^{(N)}\right], \end{gathered} \tag{35}$$

where the factual players' payoffs in situation

$$\{\alpha_i,\ \beta_i,\ \gamma_i\} \tag{36}$$

are

$$F(\alpha_i,\ \beta_i,\ \gamma_i) = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ \gamma_i,\ t)\, d\mu(t) \ \ \forall\, i = \overline{1,\ N-1} \tag{37}$$

by

$$F(\alpha_N,\ \beta_N,\ \gamma_N) = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ \gamma_N,\ t)\, d\mu(t), \tag{38}$$

$$G(\alpha_i,\ \beta_i,\ \gamma_i) = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ \gamma_i,\ t)\, d\mu(t) \ \ \forall\, i = \overline{1,\ N-1} \tag{39}$$

by

$$G(\alpha_N,\ \beta_N,\ \gamma_N) = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ \gamma_N,\ t)\, d\mu(t), \tag{40}$$

and

$$H(\alpha_i,\ \beta_i,\ \gamma_i) = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} h(\alpha_i,\ \beta_i,\ \gamma_i,\ t)\, d\mu(t) \ \ \forall\, i = \overline{1,\ N-1} \tag{41}$$

by

$$H(\alpha_N,\ \beta_N,\ \gamma_N) = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} h(\alpha_N,\ \beta_N,\ \gamma_N,\ t)\, d\mu(t). \tag{42}$$

The payoff in situation (36) can be thought of as it is the payoff on a “stair” subinterval $i$, which is $\left[\tau^{(i-1)};\ \tau^{(i)}\right)$ for $i=\overline{1,\ N-1}$ and $\left[\tau^{(N-1)};\ \tau^{(N)}\right]$ (when $i=N$). A pure-strategy situation in the staircase-function game (32) is a succession of $N$ situations

$$\left\{\{\alpha_i,\ \beta_i,\ \gamma_i\}\right\}_{i=1}^{N} \tag{43}$$

in games (33), where each situation corresponds to its subinterval. The stack of successive situations (43) is a (staircase) situation in the respective 3-person staircase-function game (32). The succession allows considering players' payoffs in situation (13) of staircase functions in a simpler form.

**Theorem 1.** *In a pure-strategy situation of the staircase-function game (32), represented as a succession of $N$ continuous games (33), functionals (14)–(16) are rewritten as subinterval-wise sums*

$$F\big(x(t),\ y(t),\ z(t)\big)=\sum_{i=1}^{N}F(\alpha_i,\ \beta_i,\ \gamma_i)=$$
$$=\sum_{i=1}^{N-1}\ \int\limits_{\left[\tau^{(i-1)};\,\tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ \gamma_i,\ t)\,d\mu(t)+\int\limits_{\left[\tau^{(N-1)};\,\tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ \gamma_N,\ t)\,d\mu(t), \tag{44}$$

$$G\big(x(t),\ y(t),\ z(t)\big)=\sum_{i=1}^{N}G(\alpha_i,\ \beta_i,\ \gamma_i)=$$
$$=\sum_{i=1}^{N-1}\ \int\limits_{\left[\tau^{(i-1)};\,\tau^{(i)}\right)} g(\alpha_i,\ \beta_i,\ \gamma_i,\ t)\,d\mu(t)+\int\limits_{\left[\tau^{(N-1)};\,\tau^{(N)}\right]} g(\alpha_N,\ \beta_N,\ \gamma_N,\ t)\,d\mu(t), \tag{45}$$

$$H\big(x(t),\ y(t),\ z(t)\big)=\sum_{i=1}^{N}H(\alpha_i,\ \beta_i,\ \gamma_i)=$$
$$=\sum_{i=1}^{N-1}\ \int\limits_{\left[\tau^{(i-1)};\,\tau^{(i)}\right)} h(\alpha_i,\ \beta_i,\ \gamma_i,\ t)\,d\mu(t)+\int\limits_{\left[\tau^{(N-1)};\,\tau^{(N)}\right]} h(\alpha_N,\ \beta_N,\ \gamma_N,\ t)\,d\mu(t). \tag{46}$$

*Proof.* Situation (36) is tied to half-subinterval $\left[\tau^{(i-1)};\ \tau^{(i)}\right)$ by $i=\overline{1,\ N-1}$ and to subinterval $\left[\tau^{(N-1)};\ \tau^{(N)}\right]$ by $i=N$. Each of functions (17)–(19) in this situation is some function of time $t$. Denote a function corresponding to (17) by $\psi_i(t)$. For situation (36) function

$$\psi_i(t)=0\ \ \forall\,t\notin\left[\tau^{(i-1)};\ \tau^{(i)}\right), \tag{47}$$

and for situation

$$\{\alpha_N,\ \beta_N,\ \gamma_N\} \tag{48}$$

function

$$\psi_N(t) = 0\ \ \forall\, t \notin \left[\tau^{(N-1)};\ \tau^{(N)}\right]. \tag{49}$$

Therefore,

$$f\big(x(t),\ y(t),\ z(t),\ t\big) = \sum_{i=1}^{N} \psi_i(t) \tag{50}$$

in a pure-strategy situation (13) of the staircase-function game (32), by using (47) and (49). Consequently,

$$F\big(x(t),\ y(t),\ z(t)\big) = \int\limits_{[t_1;\ t_2]} f\big(x(t),\ y(t),\ z(t),\ t\big) d\mu(t) =$$
$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} \psi_i(t)\, d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} \psi_N(t)\, d\mu(t) =$$
$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f(\alpha_i,\ \beta_i,\ \gamma_i,\ t)\, d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f(\alpha_N,\ \beta_N,\ \gamma_N,\ t)\, d\mu(t) =$$
$$= \sum_{i=1}^{N} F(\alpha_i,\ \beta_i,\ \gamma_i) \tag{51}$$

in a pure-strategy situation (13) of the staircase-function game (32). Obviously, subinterval-wise sums (45) and (46) are proved similarly to (47) – (51). □

Along with (37) – (42), Theorem 1 helps in understanding the following. If time $t$ is not explicitly included into the function under the integrals in (14) – (16), then the payoff value would depend only on the subinterval length (in the same situation on different subintervals). If the subinterval length does not change, there are $N$ identical (ordinary) 3-person continuous games (33). The triviality of the equal-length-subinterval case is explained by a standstill of the players' strategies. Time variable $t$ therefore is explicitly included into (37) – (42) to make the system change (and make the players modify their actions) as time goes by.

When the staircase-function game (32) is studied, Theorem 1 allows considering each game (33) separately. Although Theorem 1 does not provide a method of solving the 3-person staircase-function game, it provides a fundamental decomposition of the game. By this decomposition each subinterval game (33) can be solved separately, whereupon the subinterval games solutions are stacked (stitched) together.

## 6. When a Pareto-efficient stack is single

The occurrence when every subinterval 3-person game has a single Pareto-efficient situation is rare. The likelihood of such an occurrence even for finite staircase-function games is roughly less than 1 %. Nevertheless, there is an interesting assertion addressed to this case.

**Theorem 2.** *If each of $N$ games (33) by (21) – (31) and (35), (37) – (42) has a single Pareto-efficient situation, then the respective 3-person staircase-function game (32) has a single Pareto-efficient situation, which is the stack of successive Pareto-efficient situations in games (33).*

*Proof.* Let

$$\{\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\} \tag{52}$$

be the single efficient situation in the game on "stair" subinterval $i$. This implies that the triple of simultaneous strict inequalities

$$F(\alpha_i,\ \beta_i,\ \gamma_i) > F(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*), \tag{53}$$

$$G(\alpha_i,\ \beta_i,\ \gamma_i) > G(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*), \tag{54}$$

$$H(\alpha_i,\ \beta_i,\ \gamma_i) > H(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*) \tag{55}$$

is impossible for any $\alpha_i \in [a_{\min};\ a_{\max}]$, $\beta_i \in [b_{\min};\ b_{\max}]$, $\gamma_i \in [c_{\min};\ c_{\max}]$. The triple of simultaneous nonstrict inequalities

$$F(\alpha_i,\ \beta_i,\ \gamma_i) \geqslant F(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*), \tag{56}$$

$$G(\alpha_i,\ \beta_i,\ \gamma_i) \geqslant G(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*), \tag{57}$$

$$H(\alpha_i,\ \beta_i,\ \gamma_i) \geqslant H(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*) \tag{58}$$

is impossible due to efficient situation (52) is single. Therefore, the six triples of simultaneous inequalities (56), (54), (55), inequalities (53), (57), (55), inequalities (53), (54), (58), inequalities (56), (57), (55), inequalities (56), (54), (58), inequalities (53), (57), (58) are impossible as well. Without losing generality, suppose that $\exists\ \alpha_i^{(0)} \in [a_{\min};\ a_{\max}]$, $\exists\ \beta_i^{(0)} \in [b_{\min};\ b_{\max}]$, $\exists\ \gamma_i^{(0)} \in [c_{\min};\ c_{\max}]$ such that inequality

$$F\left(\alpha_i^{(0)},\ \beta_i^{(0)},\ \gamma_i^{(0)}\right) > F(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*) \tag{59}$$

holds. Then one of the four pairs of simultaneous inequalities

$$G\left(\alpha_i^{(0)},\ \beta_i^{(0)},\ \gamma_i^{(0)}\right) \leqslant G(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*), \tag{60}$$

$$H\left(\alpha_i^{(0)},\ \beta_i^{(0)},\ \gamma_i^{(0)}\right) < H(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*), \tag{61}$$

and

$$G\left(\alpha_i^{(0)},\ \beta_i^{(0)},\ \gamma_i^{(0)}\right) < G\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right), \tag{62}$$

$$H\left(\alpha_i^{(0)},\ \beta_i^{(0)},\ \gamma_i^{(0)}\right) \leqslant H\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right), \tag{63}$$

and

$$G\left(\alpha_i^{(0)},\ \beta_i^{(0)},\ \gamma_i^{(0)}\right) > G\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right), \tag{64}$$

$$H\left(\alpha_i^{(0)},\ \beta_i^{(0)},\ \gamma_i^{(0)}\right) < H\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right), \tag{65}$$

and

$$G\left(\alpha_i^{(0)},\ \beta_i^{(0)},\ \gamma_i^{(0)}\right) < G\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right), \tag{66}$$

$$H\left(\alpha_i^{(0)},\ \beta_i^{(0)},\ \gamma_i^{(0)}\right) > H\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right) \tag{67}$$

must hold due to the triple of simultaneous inequalities (53), (57), (58) is impossible. This means that situation

$$\left\{\alpha_i^{(0)},\ \beta_i^{(0)},\ \gamma_i^{(0)}\right\} \tag{68}$$

is efficient, which is impossible due to (52) is the single efficient situation. Therefore, inequality (59) is impossible, and impossibility of the remaining cases of inequalities (64) and (67) is proved symmetrically. Without losing generality, suppose that $\exists\ \alpha_i^{(0)} \in [a_{\min};\ a_{\max}]$, $\exists\ \beta_i^{(0)} \in [b_{\min};\ b_{\max}]$, $\exists\ \gamma_i^{(0)} \in [c_{\min};\ c_{\max}]$ such that inequalities (59) and (64) simultaneously hold. Then inequality (65) must hold. This means that situation (68) is efficient, which is impossible due to (52) is the single efficient situation. Therefore, simultaneous inequalities (59) and (64) are impossible, and impossibility of the remaining cases of inequalities, including (56), (57), (58), is proved symmetrically. So, any of inequalities (53) – (58) is impossible for every $i = \overline{1,\ N}$. As they are impossible then each of the inequalities

$$\sum_{i=1}^{N} F\left(\alpha_i,\ \beta_i,\ \gamma_i\right) > \sum_{i=1}^{N} F\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right), \tag{69}$$

$$\sum_{i=1}^{N} G\left(\alpha_i,\ \beta_i,\ \gamma_i\right) > \sum_{i=1}^{N} G\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right), \tag{70}$$

$$\sum_{i=1}^{N} H\left(\alpha_i,\ \beta_i,\ \gamma_i\right) > \sum_{i=1}^{N} H\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right), \tag{71}$$

$$\sum_{i=1}^{N} F\left(\alpha_i,\ \beta_i,\ \gamma_i\right) \geqslant \sum_{i=1}^{N} F\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right), \tag{72}$$

$$\sum_{i=1}^{N} G\left(\alpha_i,\ \beta_i,\ \gamma_i\right) \geqslant \sum_{i=1}^{N} G\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right), \tag{73}$$

$$\sum_{i=1}^{N} H\left(\alpha_i,\ \beta_i,\ \gamma_i\right) \geqslant \sum_{i=1}^{N} H\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right) \tag{74}$$

for any $\alpha_i \in [a_{\min};\ a_{\max}]$ and $\beta_i \in [b_{\min};\ b_{\max}]$ and $\gamma_i \in [c_{\min};\ c_{\max}]$ is impossible as well, and any combination of simultaneous inequalities (69) – (74) is impossible. By the efficiency definition, owing to Theorem 1, this implies that stack

$$\left\{\left\{\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right\}\right\}_{i=1}^{N} \tag{75}$$

is a Pareto-efficient situation in the respective 3-person staircase-function game (32). Suppose that there is another stack which is also Pareto-efficient. Consider the case when $N = 2$. First, let stack

$$\left\{\left\{\alpha_1^{(0)},\ \beta_1^*,\ \gamma_1^*\right\},\ \left\{\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right\}\right\} \tag{76}$$

be a Pareto-efficient situation by $\alpha_1^{(0)} \neq \alpha_1^*$. This implies that a triple of simultaneous inequalities

$$F\left(\alpha_1,\ \beta_1,\ \gamma_1\right) + F\left(\alpha_2,\ \beta_2,\ \gamma_2\right) > F\left(\alpha_1^{(0)},\ \beta_1^*,\ \gamma_1^*\right) + F\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \tag{77}$$

$$G\left(\alpha_1,\ \beta_1,\ \gamma_1\right) + G\left(\alpha_2,\ \beta_2,\ \gamma_2\right) > G\left(\alpha_1^{(0)},\ \beta_1^*,\ \gamma_1^*\right) + G\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \tag{78}$$

$$H\left(\alpha_1,\ \beta_1,\ \gamma_1\right) + H\left(\alpha_2,\ \beta_2,\ \gamma_2\right) > H\left(\alpha_1^{(0)},\ \beta_1^*,\ \gamma_1^*\right) + H\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right) \tag{79}$$

is impossible for any

$$\begin{aligned} &\alpha_1 \in [a_{\min};\ a_{\max}],\ \ \beta_1 \in [b_{\min};\ b_{\max}],\ \ \gamma_1 \in [c_{\min};\ c_{\max}],\\ &\alpha_2 \in [a_{\min};\ a_{\max}],\ \ \beta_2 \in [b_{\min};\ b_{\max}],\ \ \gamma_2 \in [c_{\min};\ c_{\max}]. \end{aligned} \tag{80}$$

Plugging $\alpha_2 = \alpha_2^*$ and $\beta_2 = \beta_2^*$ and $\gamma_2 = \gamma_2^*$ in the left sides of inequalities (77) – (79) gives a triple of inequalities

$$F\left(\alpha_1,\ \beta_1,\ \gamma_1\right) > F\left(\alpha_1^{(0)},\ \beta_1^*,\ \gamma_1^*\right), \tag{81}$$

$$G\left(\alpha_1,\ \beta_1,\ \gamma_1\right) > G\left(\alpha_1^{(0)},\ \beta_1^*,\ \gamma_1^*\right), \tag{82}$$

$$H\left(\alpha_1,\ \beta_1,\ \gamma_1\right) > H\left(\alpha_1^{(0)},\ \beta_1^*,\ \gamma_1^*\right). \tag{83}$$

If the triple of simultaneous inequalities (81) – (83) is impossible then situation

$$\left\{\alpha_1^{(0)},\ \beta_1^*,\ \gamma_1^*\right\} \tag{84}$$

must be efficient. Therefore, the supposition about Pareto-efficiency of situation (76) is contradictory.
Second, let stack

$$\left\{\left\{\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^*\right\},\, \{\alpha_2^*,\, \beta_2^*,\, \gamma_2^*\}\right\} \tag{85}$$

be a Pareto-efficient situation by $\alpha_1^{(0)} \neq \alpha_1^*$ and $\beta_1^{(0)} \neq \beta_1^*$. This implies that a triple of simultaneous inequalities

$$F(\alpha_1,\, \beta_1,\, \gamma_1) + F(\alpha_2,\, \beta_2,\, \gamma_2) > F\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^*\right) + F(\alpha_2^*,\, \beta_2^*,\, \gamma_2^*), \tag{86}$$

$$G(\alpha_1,\, \beta_1,\, \gamma_1) + G(\alpha_2,\, \beta_2,\, \gamma_2) > G\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^*\right) + G(\alpha_2^*,\, \beta_2^*,\, \gamma_2^*), \tag{87}$$

$$H(\alpha_1,\, \beta_1,\, \gamma_1) + H(\alpha_2,\, \beta_2,\, \gamma_2) > H\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^*\right) + H(\alpha_2^*,\, \beta_2^*,\, \gamma_2^*) \tag{88}$$

is impossible for any (80). Plugging $\alpha_2 = \alpha_2^*$ and $\beta_2 = \beta_2^*$ and $\gamma_2 = \gamma_2^*$ in the left sides of inequalities (86) – (88) gives a triple of inequalities

$$F(\alpha_1,\, \beta_1,\, \gamma_1) > F\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^*\right), \tag{89}$$

$$G(\alpha_1,\, \beta_1,\, \gamma_1) > G\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^*\right), \tag{90}$$

$$H(\alpha_1,\, \beta_1,\, \gamma_1) > H\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^*\right). \tag{91}$$

If the triple of simultaneous inequalities (89) – (91) is impossible then situation

$$\left\{\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^*\right\} \tag{92}$$

must be efficient. Therefore, the supposition about Pareto-efficiency of situation (85) is contradictory.
The Pareto-efficiency impossibility of other versions of 2-subinterval stacks is proved symmetrically. The Pareto-efficiency impossibility of $N$-subinterval stacks by $N \geqslant 3$ is proved similarly by ascending induction. □

So, if each of the subinterval 3-person games has a single Pareto-efficient solution, Theorem 2 allows finding the Pareto-efficient solution of the respective 3-person staircase-function game in a very simple way, just by stacking the subinterval solutions. It is easy to see that the assertion of Theorem 2 is reversible.

**Theorem 3.** *If a 3-person staircase-function game (32) has a single Pareto-efficient situation, then each of the respective $N$ games (33) by (21) – (31) and (35), (37) – (42) has a single Pareto-efficient situation.*

*Proof.* Let stack (75) be a single Pareto-efficient situation in a 3-person staircase-function game (32). This implies that the triple of simultaneous inequalities (72) – (74) is impossible for any

$$\alpha_i \in \{[a_{\min};\ a_{\max}]\} \setminus \{\alpha_i^*\} \text{ and } \beta_i \in \{[b_{\min};\ b_{\max}]\} \setminus \{\beta_i^*\} \text{ and } \gamma_i \in \{[c_{\min};\ c_{\max}]\} \setminus \{\gamma_i^*\}.$$

Plugging $\alpha_k = \alpha_k^*$ and $\beta_k = \beta_k^*$ and $\beta_k = \beta_k^*\ \forall\, k = \overline{2,\ N}$ in the left sides of inequalities (72) – (74) gives a triple of simultaneous inequalities

$$F(\alpha_1,\ \beta_1,\ \gamma_1) > F(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*), \tag{93}$$

$$G(\alpha_1,\ \beta_1,\ \gamma_1) > G(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*), \tag{94}$$

$$H(\alpha_1,\ \beta_1,\ \gamma_1) > H(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*), \tag{95}$$

which is impossible as well. Hence, situation

$$\{\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\} \tag{96}$$

is efficient. The efficiency of the remaining subinterval situations is proved in the same way.

Suppose that, along with efficient situation (96), situation

$$\left\{\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right\} \tag{97}$$

is efficient also. Thus, a triple of simultaneous inequalities

$$F(\alpha_1,\ \beta_1,\ \gamma_1) > F\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right), \tag{98}$$

$$G(\alpha_1,\ \beta_1,\ \gamma_1) > G\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right), \tag{99}$$

$$H(\alpha_1,\ \beta_1,\ \gamma_1) > H\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) \tag{100}$$

is impossible for any

$$\alpha_1 \in \{[a_{\min};\ a_{\max}]\} \setminus \left\{\alpha_1^{(0)},\ \alpha_1^*\right\} \text{ and } \beta_1 \in \{[b_{\min};\ b_{\max}]\} \setminus \left\{\beta_1^{(0)},\ \beta_1^*\right\} \text{ and}$$
$$\gamma_1 \in \{[c_{\min};\ c_{\max}]\} \setminus \left\{\gamma_1^{(0)},\ \gamma_1^*\right\}.$$

Stack

$$\left\{\left\{\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right\},\ \{\{\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\}\}_{i=2}^{N}\right\} \tag{101}$$

must not be efficient. This implies that a triple of simultaneous inequalities

$$F\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right)+\sum_{i=2}^{N} F(\alpha_i,\ \beta_i,\ \gamma_i) < F(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*)+\sum_{i=2}^{N} F(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*), \tag{102}$$

$$G\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) + \sum_{i=2}^{N} G\left(\alpha_i,\ \beta_i,\ \gamma_i\right) \leqslant G\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right) + \sum_{i=2}^{N} G\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right), \quad (103)$$

$$H\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) + \sum_{i=2}^{N} H\left(\alpha_i,\ \beta_i,\ \gamma_i\right) \leqslant H\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right) + \sum_{i=2}^{N} H\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right) \quad (104)$$

holds, or a triple of simultaneous inequalities

$$F\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) + \sum_{i=2}^{N} F\left(\alpha_i,\ \beta_i,\ \gamma_i\right) \leqslant F\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right) + \sum_{i=2}^{N} F\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right), \quad (105)$$

$$G\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) + \sum_{i=2}^{N} G\left(\alpha_i,\ \beta_i,\ \gamma_i\right) < G\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right) + \sum_{i=2}^{N} G\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right), \quad (106)$$

(104) holds, or a triple of simultaneous inequalities (105), (103),

$$H\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) + \sum_{i=2}^{N} H\left(\alpha_i,\ \beta_i,\ \gamma_i\right) < H\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right) + \sum_{i=2}^{N} H\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right) \quad (107)$$

holds, or a triple of simultaneous inequalities (102), (106), (104) holds, or a triple of simultaneous inequalities (102), (103), (107) holds, or a triple of simultaneous inequalities (105), (106), (107) holds. Plugging $\alpha_k = \alpha_k^*$ and $\beta_k = \beta_k^*$ and $\gamma_k = \gamma_k^*$ $\forall\, k = \overline{2,\ N}$ in the left sides of inequalities (102)–(104), inequalities (105), (106), (104), inequalities (105), (103), (107), inequalities (102), (106), (104), inequalities (102), (103), (107), inequalities (105), (106), (107) gives a triple of simultaneous inequalities

$$\begin{aligned} &F\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) < F\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right), \\ &G\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) \leqslant G\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right), \\ &H\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) \leqslant H\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right), \end{aligned} \quad (108)$$

or a triple of simultaneous inequalities

$$\begin{aligned} &F\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) \leqslant F\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right), \\ &G\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) < G\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right), \\ &H\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) \leqslant H\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right), \end{aligned} \quad (109)$$

or a triple of simultaneous inequalities

$$F\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) \leqslant F\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right),$$
$$G\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) \leqslant G\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right),$$
$$H\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) < H\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right), \tag{110}$$

or a triple of simultaneous inequalities

$$F\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) < F\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right),$$
$$G\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) < G\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right),$$
$$H\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) \leqslant H\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right), \tag{111}$$

or a triple of simultaneous inequalities

$$F\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) < F\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right),$$
$$G\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) \leqslant G\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right),$$
$$H\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) < H\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right), \tag{112}$$

or a triple of simultaneous inequalities

$$F\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) \leqslant F\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right),$$
$$G\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) < G\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right),$$
$$H\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) < H\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right). \tag{113}$$

The possibility of inequality triples (108) – (113) means that situation (97) is not efficient. Such a contradiction is similarly proved for any other subinterval situation. Without losing generality, suppose that, along with efficient situations (96) and

$$\left\{\alpha_k^*,\ \beta_k^*,\ \gamma_k^*\right\} \ \text{by}\ \ k \in \left\{\overline{2,\ N}\right\},$$

situations (97) and

$$\left\{\alpha_k^{(0)},\ \beta_k^{(0)},\ \gamma_k^{(0)}\right\}$$

are efficient also. Then, anyway, stack (101) must not be efficient, which is the contradiction due to (102) – (113). Such a contradiction is similarly proved for any other combinations of subinterval situations. □

So, Theorem 3 asserts that when a Pareto-efficient stack is single, it does directly mean that every subinterval 3-person game must have a single Pareto-efficient situation. The question about multiple Pareto-efficient stacks is cleared right below.

## 7. What a Pareto-efficient stack consists of

The player in a finite 3-person staircase-function game (32) may have multiple Pareto-efficient strategies. For example, a game with 2-subinterval 2-staircased function-strategies at the first player, 2-subinterval 3-staircased function-strategies at the second player, and 2-subinterval 2-staircased function-strategies at the third player represented with respective matrices

$$\mathbf{F}_1 = \left[ \begin{bmatrix} 1 & 6 & -5 \\ 4 & -11 & -6 \end{bmatrix} \begin{bmatrix} 3 & -1 & 4 \\ 10 & 7 & -6 \end{bmatrix} \right],$$
$$\mathbf{F}_2 = \left[ \begin{bmatrix} 1 & 6 & -5 \\ 4 & -11 & -6 \end{bmatrix} \begin{bmatrix} 3 & -1 & 4 \\ 10 & 7 & -6 \end{bmatrix} \right] \tag{114}$$

and

$$\mathbf{G}_1 = \left[ \begin{bmatrix} -5 & 0 & -1 \\ 4 & -4 & 3 \end{bmatrix} \begin{bmatrix} -4 & -1 & -3 \\ 7 & 11 & 4 \end{bmatrix} \right],$$
$$\mathbf{G}_2 = \left[ \begin{bmatrix} -5 & 0 & -1 \\ 4 & -4 & 3 \end{bmatrix} \begin{bmatrix} -4 & -1 & -3 \\ 7 & 11 & 4 \end{bmatrix} \right] \tag{115}$$

and

$$\mathbf{H}_1 = \left[ \begin{bmatrix} -2 & -3 & 3 \\ 7 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 6 & 5 \\ 4 & -1 & 1 \end{bmatrix} \right],$$
$$\mathbf{H}_2 = \left[ \begin{bmatrix} -2 & -3 & 3 \\ 7 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 6 & 5 \\ 4 & -1 & 1 \end{bmatrix} \right] \tag{116}$$

has 5 Pareto-efficient situations. They are the stack of efficient situations with payoffs $\{1, 3, 2\}$ and $\{5, 3, 3\}$, the stack of efficient situations with payoffs $\{1, 3, 2\}$ and $\{0, 5, -1\}$, the stack of efficient situations with payoffs $\{1, 4, 1\}$ and $\{5, 3, 3\}$, the stack of efficient situations with payoffs $\{1, 4, 1\}$ and $\{0, 5, -1\}$, and the stack of efficient situations with payoffs $\{5, 2, 4\}$ and $\{5, 3, 3\}$. By the way, the stack of efficient situations with payoffs $\{5, 2, 4\}$ and $\{0, 5, -1\}$ is not an efficient situation. Indeed, whereas the efficient (stacked) situations produce payoffs $\{6, 6, 5\}$, $\{1, 8, 1\}$, $\{6, 7, 4\}$, $\{1, 9, 0\}$, $\{10, 5, 7\}$, respectively (in the order of listing them), the non-efficient stack produce payoffs $\{5, 7, 3\}$. Obviously, a continuous 3-person staircase-function game may have multiple Pareto-efficient situations as well.

**Theorem 4.** *Any Pareto-efficient situation in a 3-person staircase-function game (32) is a stack of successive Pareto-efficient situations in games (33) by (21) – (31) and (35), (37) – (42).*

*Proof.* Let stack (75) be a Pareto-efficient situation in the respective 3-person staircase-function game (32), where (52) is a Pareto-efficient situation in the game on "stair" subinterval $i$. Suppose that situation (97) is not efficient in game (33) on the first "stair" subinterval, but stack (101) is an efficient situation in staircase-function game (32). Then at least one inequality of inequalities

$$F\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) + \sum_{i=2}^{N} F\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) \geqslant F\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) + \sum_{i=2}^{N} F\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right), \quad (117)$$

$$G\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) + \sum_{i=2}^{N} G\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) \geqslant G\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) + \sum_{i=2}^{N} G\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right), \quad (118)$$

$$H\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) + \sum_{i=2}^{N} H\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) \geqslant H\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) + \sum_{i=2}^{N} H\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) \quad (119)$$

must hold. Without losing generality, suppose that inequality (117) holds. Then inequality

$$F\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) \geqslant F\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) \quad (120)$$

holds. The pair of simultaneous inequalities

$$G\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) \geqslant G\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right), \quad (121)$$

$$H\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) \geqslant H\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) \quad (122)$$

here is impossible due to situation (96) is efficient and situation (97) is not efficient. So, there must be a pair of simultaneous inequalities

$$G\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) < G\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) \quad (123)$$

and (122), or a pair of simultaneous inequalities (121) and

$$H\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) < H\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right), \quad (124)$$

or a pair of simultaneous inequalities (123) and (124). As situation (97) is not efficient, the triple of simultaneous inequalities (120), (123), (122) is possible only by

$$F\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) = F\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right), \;\; H\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) = H\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right), \quad (125)$$

which is followed by

$$F\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) + \sum_{i=2}^{N} F\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) = F\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) + \sum_{i=2}^{N} F\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right), \quad (126)$$

$$G\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) + \sum_{i=2}^{N} G\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) < G\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) + \sum_{i=2}^{N} G\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right), \quad (127)$$

$$H\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) + \sum_{i=2}^{N} H\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) = H\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) + \sum_{i=2}^{N} H\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right). \quad (128)$$

The triple of simultaneous relationships (126)–(128) means that stack (101) is not an efficient situation. As situation (97) is not efficient, the triple of simultaneous inequalities (120), (121), (124) is possible only by

$$F\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) = F\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right), \;\; G\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) = G\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right), \quad (129)$$

which is followed by (126),

$$G\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) + \sum_{i=2}^{N} G\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) = G\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) + \sum_{i=2}^{N} G\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right), \quad (130)$$

$$H\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) + \sum_{i=2}^{N} H\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) < H\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) + \sum_{i=2}^{N} H\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right). \quad (131)$$

The triple of simultaneous relationships (126), (130), (131) means that stack (101) is not an efficient situation. As situation (97) is not efficient, the triple of simultaneous inequalities (120), (123), (124) is possible only by

$$F\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) = F\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right), \quad (132)$$

which is followed by (126), (127), (131). The triple of simultaneous relationships (126), (127), (131) means that stack (101) is not an efficient situation. Such contradictions implying that stack (101) cannot be efficient are similarly proved for any other subinterval situation and inequalities (118), (119).
Suppose now that situations (97) and

$$\left\{\alpha_2^{(0)},\, \beta_2^{(0)},\, \gamma_2^{(0)}\right\} \quad (133)$$

are not efficient in the first two subinterval games, but stack

$$\left\{\left\{\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right\},\, \left\{\alpha_2^{(0)},\, \beta_2^{(0)},\, \gamma_2^{(0)}\right\},\, \left\{\left\{\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right\}\right\}_{i=3}^{N}\right\} \quad (134)$$

is an efficient situation in staircase-function game (32). Then at least one inequality of inequalities

$$F\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) + F\left(\alpha_2^{(0)},\, \beta_2^{(0)},\, \gamma_2^{(0)}\right) + \sum_{i=3}^{N} F\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) \geqslant$$
$$\geqslant F\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) + F\left(\alpha_2^*,\, \beta_2^*,\, \gamma_2^*\right) + \sum_{i=3}^{N} F\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right), \quad (135)$$

$$G\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) + G\left(\alpha_2^{(0)},\, \beta_2^{(0)},\, \gamma_2^{(0)}\right) + \sum_{i=3}^{N} G\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) \geqslant$$

$$\geqslant G\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) + G\left(\alpha_2^*,\, \beta_2^*,\, \gamma_2^*\right) + \sum_{i=3}^{N} G\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right), \tag{136}$$

$$H\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) + H\left(\alpha_2^{(0)},\, \beta_2^{(0)},\, \gamma_2^{(0)}\right) + \sum_{i=3}^{N} H\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) \geqslant$$

$$\geqslant H\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) + H\left(\alpha_2^*,\, \beta_2^*,\, \gamma_2^*\right) + \sum_{i=3}^{N} H\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) \tag{137}$$

must hold. Without losing generality, suppose that inequality (135) holds. Then inequality

$$F\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) + F\left(\alpha_2^{(0)},\, \beta_2^{(0)},\, \gamma_2^{(0)}\right) \geqslant F\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) + F\left(\alpha_2^*,\, \beta_2^*,\, \gamma_2^*\right) \tag{138}$$

holds. Consider a pair of simultaneous inequalities

$$G\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) + G\left(\alpha_2^{(0)},\, \beta_2^{(0)},\, \gamma_2^{(0)}\right) \geqslant G\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) + G\left(\alpha_2^*,\, \beta_2^*,\, \gamma_2^*\right), \tag{139}$$

$$H\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) + H\left(\alpha_2^{(0)},\, \beta_2^{(0)},\, \gamma_2^{(0)}\right) \geqslant H\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) + H\left(\alpha_2^*,\, \beta_2^*,\, \gamma_2^*\right), \tag{140}$$

a pair of simultaneous inequalities

$$G\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) + G\left(\alpha_2^{(0)},\, \beta_2^{(0)},\, \gamma_2^{(0)}\right) < G\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) + G\left(\alpha_2^*,\, \beta_2^*,\, \gamma_2^*\right) \tag{141}$$

and (140), a pair of simultaneous inequalities (139) and

$$H\left(\alpha_1^{(0)},\, \beta_1^{(0)},\, \gamma_1^{(0)}\right) + H\left(\alpha_2^{(0)},\, \beta_2^{(0)},\, \gamma_2^{(0)}\right) < H\left(\alpha_1^*,\, \beta_1^*,\, \gamma_1^*\right) + H\left(\alpha_2^*,\, \beta_2^*,\, \gamma_2^*\right), \tag{142}$$

a pair of simultaneous inequalities (141) and (142). One of these four pairs must follow inequality (138). As either of situations (97) and (133) is not efficient, then one of the six triples of simultaneous inequalities (108)–(113) is true, and one of the six triples of simultaneous inequalities

$$\begin{aligned} F\left(\alpha_2^{(0)},\, \beta_2^{(0)},\, \gamma_2^{(0)}\right) &< F\left(\alpha_2^*,\, \beta_2^*,\, \gamma_2^*\right), \\ G\left(\alpha_2^{(0)},\, \beta_2^{(0)},\, \gamma_2^{(0)}\right) &\leqslant G\left(\alpha_2^*,\, \beta_2^*,\, \gamma_2^*\right), \\ H\left(\alpha_2^{(0)},\, \beta_2^{(0)},\, \gamma_2^{(0)}\right) &\leqslant H\left(\alpha_2^*,\, \beta_2^*,\, \gamma_2^*\right), \end{aligned} \tag{143}$$

$$\begin{aligned} F\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) &\leqslant F\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \\ G\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) &< G\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \\ H\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) &\leqslant H\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \end{aligned} \tag{144}$$

$$\begin{aligned} F\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) &\leqslant F\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \\ G\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) &\leqslant G\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \\ H\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) &< H\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \end{aligned} \tag{145}$$

$$\begin{aligned} F\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) &< F\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \\ G\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) &< G\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \\ H\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) &\leqslant H\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \end{aligned} \tag{146}$$

$$\begin{aligned} F\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) &< F\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \\ G\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) &\leqslant G\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \\ H\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) &< H\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \end{aligned} \tag{147}$$

$$\begin{aligned} F\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) &\leqslant F\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \\ G\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) &< G\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \\ H\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) &< H\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right) \end{aligned} \tag{148}$$

is true as well. After summing up inequalities (108) – (113) and (143) – (148) sidewise, where 36 inequality triples are obtained, there may be possible only inequality

$$F\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) + F\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) \leqslant F\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right) + F\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right) \tag{149}$$

along with a pair of simultaneous inequalities (141) and (142), or with a pair of simultaneous inequalities (141) and

$$H\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right) + H\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right) \leqslant H\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right) + H\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right), \tag{150}$$

or with a pair of simultaneous inequalities

$$G\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right)+G\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right)\leqslant G\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right)+G\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right) \quad (151)$$

and (142). The triple of simultaneous inequalities (149), (141), (142) implies that

$$F\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right)+F\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right)+\sum_{i=3}^{N}F\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right)\leqslant$$
$$\leqslant F\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right)+F\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right)+\sum_{i=3}^{N}F\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right), \quad (152)$$

$$G\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right)+G\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right)+\sum_{i=3}^{N}G\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right)<$$
$$< G\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right)+G\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right)+\sum_{i=3}^{N}G\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right), \quad (153)$$

$$H\left(\alpha_1^{(0)},\ \beta_1^{(0)},\ \gamma_1^{(0)}\right)+H\left(\alpha_2^{(0)},\ \beta_2^{(0)},\ \gamma_2^{(0)}\right)+\sum_{i=3}^{N}H\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right)<$$
$$< H\left(\alpha_1^*,\ \beta_1^*,\ \gamma_1^*\right)+H\left(\alpha_2^*,\ \beta_2^*,\ \gamma_2^*\right)+\sum_{i=3}^{N}H\left(\alpha_i^*,\ \beta_i^*,\ \gamma_i^*\right), \quad (154)$$

i. e. stack (134) is not an efficient situation in staircase-function game (32). Both the triples of simultaneous inequalities (149), (141), (150), and (149), (151), (142) imply the same. Such contradictions implying that stack (134) cannot be efficient are similarly proved for any other two subinterval situations and inequalities (136), (137). Furthermore, by using the considered ascending induction, such contradictions implying that a stack including non-efficient subinterval situations cannot be efficient are similarly proved for any number of situations. □

So, Theorem 4 directly answers the question of what a Pareto-efficient stack consists of. Every efficient situation in a 3-person staircase-function game (32) is built out of Pareto-efficient situations in "stair" subinterval games. Theorem 4 does not mean that any stack of successive efficient situations will be efficient. However, Theorem 4 does mean that if every subinterval (continuous) game has a finite number of Pareto-efficient situations, then all the Pareto-efficient situations in the respective 3-person staircase-function game (32) can be determined by just running over all possible stacks (whose number is finite) and selecting such stacks (75) for which none of triples of the following simultaneous inequalities is possible: (69), (73), (74), and (72), (70), (74), and (72), (73), (71), and (69), (70), (74), and (69), (73), (71), and (72), (70), (71).

## 8. Solving a finite 3-person staircase-function game

In a finite 3-person staircase-function game, players (forcedly or deliberately) act within a finite subset of possible values of their pure strategies. That is, these values are

$$a_{\min} = a^{(0)} < a^{(1)} < a^{(2)} < \ldots < a^{(M-1)} < a^{(M)} = a_{\max} \tag{155}$$

and

$$b_{\min} = b^{(0)} < b^{(1)} < b^{(2)} < \ldots < b^{(Q-1)} < b^{(Q)} = b_{\max} \tag{156}$$

and

$$c_{\min} = c^{(0)} < c^{(1)} < c^{(2)} < \ldots < c^{(S-1)} < c^{(S)} = c_{\max} \tag{157}$$

for the first, second, and third players, respectively, where $M \in \mathbb{N}$, $Q \in \mathbb{N}$, $S \in \mathbb{N}$ (i. e., the player's function-strategy must have at least two different values). Then the pure strategy sets of the players in finite 3-person staircase-function game (32) are

$$\boldsymbol{X}_N = \left\{ x(t) :\ x(t) \in \left\{ a^{(m-1)} \right\}_{m=1}^{M+1} \ \forall\, t \in \left[ \tau^{(i-1)};\ \tau^{(i)} \right) \text{ for } i = \overline{1,\ N-1} \text{ and } \forall\, t \in \left[ \tau^{(N-1)};\ \tau^{(N)} \right] \right\} \subset \boldsymbol{X} \tag{158}$$

and

$$\boldsymbol{Y}_N = \left\{ y(t) :\ y(t) \in \left\{ b^{(q-1)} \right\}_{q=1}^{Q+1} \ \forall\, t \in \left[ \tau^{(i-1)};\ \tau^{(i)} \right) \text{ for } i = \overline{1,\ N-1} \text{ and } \forall\, t \in \left[ \tau^{(N-1)};\ \tau^{(N)} \right] \right\} \subset \boldsymbol{Y} \tag{159}$$

and

$$\boldsymbol{Z}_N = \left\{ z(t) :\ z(t) \in \left\{ c^{(s-1)} \right\}_{s=1}^{S+1} \ \forall\, t \in \left[ \tau^{(i-1)};\ \tau^{(i)} \right) \text{ for } i = \overline{1,\ N-1} \text{ and } \forall\, t \in \left[ \tau^{(N-1)};\ \tau^{(N)} \right] \right\} \subset \boldsymbol{Z}. \tag{160}$$

Subsequently, the succession of $N$ continuous games (33) by (21) – (31) and (35), (37) – (42) becomes a succession of $N$ trimatrix games

$$\left\langle \left\{ \left\{ a^{(m-1)} \right\}_{m=1}^{M+1},\ \left\{ b^{(q-1)} \right\}_{q=1}^{Q+1},\ \left\{ c^{(s-1)} \right\}_{s=1}^{S+1} \right\},\ \{ \mathbf{F}_i,\ \mathbf{G}_i,\ \mathbf{H}_i \} \right\rangle \tag{161}$$

with first player's payoff matrices

$$\mathbf{F}_i = \left[ \varphi_{imqs} \right]_{(M+1)\times(Q+1)\times(S+1)}$$

whose elements are

$$\varphi_{imqs} = F\left(a^{(m-1)},\ b^{(q-1)},\ c^{(s-1)}\right) = \\ = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} f\left(a^{(m-1)},\ b^{(q-1)},\ c^{(s-1)},\ t\right) d\mu\,(t) \ \text{ for } \ i = \overline{1,\ N-1} \qquad (162)$$

and

$$\varphi_{Nmqs} = F\left(a^{(m-1)},\ b^{(q-1)},\ c^{(s-1)}\right) = \\ = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} f\left(a^{(m-1)},\ b^{(q-1)},\ c^{(s-1)},\ t\right) d\mu\,(t), \qquad (163)$$

with second player's payoff matrices

$$\mathbf{G}_i = \left[\rho_{imqs}\right]_{(M+1)\times(Q+1)\times(S+1)}$$

whose elements are

$$\rho_{imqs} = G\left(a^{(m-1)},\ b^{(q-1)},\ c^{(s-1)}\right) = \\ = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} g\left(a^{(m-1)},\ b^{(q-1)},\ c^{(s-1)},\ t\right) d\mu\,(t) \ \text{ for } \ i = \overline{1,\ N-1} \qquad (164)$$

and

$$\rho_{Nmqs} = G\left(a^{(m-1)},\ b^{(q-1)},\ c^{(s-1)}\right) = \\ = \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} g\left(a^{(m-1)},\ b^{(q-1)},\ c^{(s-1)},\ t\right) d\mu\,(t), \qquad (165)$$

and with third player's payoff matrices

$$\mathbf{H}_i = \left[\theta_{imqs}\right]_{(M+1)\times(Q+1)\times(S+1)}$$

whose elements are

$$\theta_{imqs} = H\left(a^{(m-1)},\ b^{(q-1)},\ c^{(s-1)}\right) = \\ = \int\limits_{\left[\tau^{(i-1)};\ \tau^{(i)}\right)} h\left(a^{(m-1)},\ b^{(q-1)},\ c^{(s-1)},\ t\right) d\mu\,(t) \ \text{ for } \ i = \overline{1,\ N-1} \qquad (166)$$

and

$$\theta_{Nmqs} = H\left(a^{(m-1)},\ b^{(q-1)},\ c^{(s-1)}\right) =$$
$$= \int\limits_{\left[\tau^{(N-1)};\ \tau^{(N)}\right]} h\left(a^{(m-1)},\ b^{(q-1)},\ c^{(s-1)},\ t\right) d\mu\,(t), \tag{167}$$

for $m = \overline{1,\ M+1}$ and $q = \overline{1,\ Q+1}$ and $s = \overline{1,\ S+1}$.
Let

$$\left\{\alpha^*_{ij_i},\ \beta^*_{ij_i},\ \gamma^*_{ij_i}\right\} \tag{168}$$

be an efficient situation in trimatrix game (161), where $j_i \in \left\{\overline{1,\ J_i}\right\}$ and $J_i \in \mathbb{N}$. So, trimatrix game (161) has $J_i$ efficient situations. It is unknown whether "participation" of situation (168) in a stack makes the stack efficient or not. Let

$$\left\{\left\{\alpha^*_{ij_i},\ \beta^*_{ij_i},\ \gamma^*_{ij_i}\right\}\right\}_{i=1}^{N} \tag{169}$$

be a stack in a 3-person staircase-function game, which is the succession of $N$ trimatrix games (161), where

$$\alpha^*_{ij_i} \in \left\{a^{(m-1)}\right\}_{m=1}^{M+1},\quad \beta^*_{ij_i} \in \left\{b^{(q-1)}\right\}_{q=1}^{Q+1},\quad \gamma^*_{ij_i} \in \left\{c^{(s-1)}\right\}_{s=1}^{S+1}.$$

Thus, stack (169) produces payoffs

$$\left\{u^*_l,\ v^*_l,\ w^*_l\right\} =$$
$$= \left\{\sum_{i=1}^{N} F\left(\alpha^*_{ij_i},\ \beta^*_{ij_i},\ \gamma^*_{ij_i}\right),\ \sum_{i=1}^{N} G\left(\alpha^*_{ij_i},\ \beta^*_{ij_i},\ \gamma^*_{ij_i}\right),\ \sum_{i=1}^{N} H\left(\alpha^*_{ij_i},\ \beta^*_{ij_i},\ \gamma^*_{ij_i}\right)\right\}$$
$$\text{by}\ \ l = \overline{1,\ \prod_{i=1}^{N} J_i}. \tag{170}$$

Let $L$ be the number of efficient stacks, where $L \in \left\{\overline{1,\ \prod_{i=1}^{N} J_i}\right\}$.
It is worth to remember that the case of when $L = 1$ is only possible if $J_i = 1$ $\forall\, i = \overline{1,\ N}$ (see Theorem 3). Without losing generality, presume that namely the first $L$ payoffs in (170) are produced by the efficient stacks (for instance, this can be done after sorting all the possible stacks just by separating the efficient from the non-efficient stacks). The best efficient stack can be found by a method suggested in [14]. Payoffs (170) are 0-1-standardized and an $l_*$-th stack is found at which the respective efficient payoffs

$$\left\{u^*_{l_*},\ v^*_{l_*},\ w^*_{l_*}\right\}$$

are the farthest from the zero payoffs $\{0,\ 0,\ 0\}$ (the most unprofitable payoffs):

$$l_* \in \arg \max_{l=\overline{1,\ L}} \sqrt{\left(\frac{u_l^* - \min\limits_{k=\overline{1,\ L}} u_k^*}{\max\limits_{k=\overline{1,\ L}} u_k^* - \min\limits_{k=\overline{1,\ L}} u_k^*}\right)^2 + \left(\frac{v_l^* - \min\limits_{k=\overline{1,\ L}} v_k^*}{\max\limits_{k=\overline{1,\ L}} v_k^* - \min\limits_{k=\overline{1,\ L}} v_k^*}\right)^2 + \left(\frac{w_l^* - \min\limits_{k=\overline{1,\ L}} w_k^*}{\max\limits_{k=\overline{1,\ L}} w_k^* - \min\limits_{k=\overline{1,\ L}} w_k^*}\right)^2}. \quad (171)$$

Thus, (171) provides the $l_*$-th stack to be the best.
Consider an example case in which $t \in [0.5\pi;\ 0.9\pi]$,

$$\left\{\tau^{(i)}\right\}_{i=0}^{4} = \{0.5\pi,\ 0.6\pi,\ 0.7\pi,\ 0.8\pi,\ 0.9\pi\}, \quad (172)$$

the set of pure strategies of the first player is

$$\boldsymbol{X}_4 = \left\{x(t):\ x(t) \in \{0.7 + 0.1m\}_{m=1}^{5}\ \forall t \in \left[\tau^{(i-1)};\ \tau^{(i)}\right) \text{ for } i = \overline{1,\ 3} \text{ and } \forall t \in \left[\tau^{(3)};\ \tau^{(4)}\right]\right\}$$
$$\subset \boldsymbol{X} = \left\{x(t),\ t \in [0.5\pi;\ 0.9\pi]:\ 0.8 \leqslant x(t) \leqslant 1.2\right\} \subset \mathbb{L}_2[0.5\pi;\ 0.9\pi], \quad (173)$$

the set of pure strategies of the second player is

$$\boldsymbol{Y}_4 = \left\{y(t):\ y(t) \in \{1 + 0.5q\}_{q=1}^{4}\ \forall t \in \left[\tau^{(i-1)};\ \tau^{(i)}\right) \text{ for } i = \overline{1,\ 3} \text{ and } \forall t \in \left[\tau^{(3)};\ \tau^{(4)}\right]\right\}$$
$$\subset \boldsymbol{Y} = \left\{y(t),\ t \in [0.5\pi;\ 0.9\pi]:\ 1.5 \leqslant y(t) \leqslant 3\right\} \subset \mathbb{L}_2[0.5\pi;\ 0.9\pi], \quad (174)$$

and the set of pure strategies of the third player is

$$\boldsymbol{Z}_4 = \left\{z(t):\ z(t) \in \{3.3 + 0.2s\}_{s=1}^{7}\ \forall t \in \left[\tau^{(i-1)};\ \tau^{(i)}\right) \text{ for } i = \overline{1,\ 3} \text{ and } \forall t \in \left[\tau^{(3)};\ \tau^{(4)}\right]\right\}$$
$$\subset \boldsymbol{Z} = \left\{z(t),\ t \in [0.5\pi;\ 0.9\pi]:\ 3.5 \leqslant z(t) \leqslant 4.7\right\} \subset \mathbb{L}_2[0.5\pi;\ 0.9\pi], \quad (175)$$

where values of the pure strategies can change only at time points

$$\left\{\tau^{(i)}\right\}_{i=1}^{3} = \{0.6\pi,\ 0.7\pi,\ 0.8\pi\}. \quad (176)$$

The first player's payoff functional is

$$F\left(x(t),\ y(t),\ z(t)\right) = \int\limits_{[0.5\pi;\ 0.9\pi]} \sin\left(0.85xyt - \frac{2\pi}{9}\right) d\mu(t), \quad (177)$$

the second player's payoff functional is

$$G\left(x\left(t\right),\ y\left(t\right),\ z\left(t\right)\right)=\int\limits_{[0.5\pi;\ 0.9\pi]}\sin\left(1.4t\left(y+zt\right)-\frac{3\pi}{7}\right)d\mu\left(t\right),\tag{178}$$

and the third player's payoff functional is

$$H\left(x\left(t\right),\ y\left(t\right),\ z\left(t\right)\right)=\int\limits_{[0.5\pi;\ 0.9\pi]}\sin\left(\frac{0.5zt}{x}-\frac{\pi}{4}\right)d\mu\left(t\right).\tag{179}$$

Consequently, this game can be thought of as it is defined on parallelepipedal lattice

$$\{0.7+0.1m\}_{m=1}^{5}\times\{1+0.5q\}_{q=1}^{4}\times\{3.3+0.2s\}_{s=1}^{7}\subset$$
$$\subset[0.8;\ 1.2]\times[1.5;\ 3]\times[3.5;\ 4.7],\tag{180}$$

that is this game is a succession of four finite $5\times4\times7$ (trimatrix) games (161):

$$\left\langle\left\{\{0.7+0.1m\}_{m=1}^{5},\ \{1+0.5q\}_{q=1}^{4},\ \{3.3+0.2s\}_{s=1}^{7}\right\},\ \{\mathbf{F}_i,\ \mathbf{G}_i,\ \mathbf{H}_i\}\right\rangle\tag{181}$$

with first player's payoff matrices $\left\{\mathbf{F}_i=\left[\varphi_{imqs}\right]_{5\times4\times7}\right\}_{i=1}^{4}$ whose elements are

$$\varphi_{imqs}=$$
$$=\int\limits_{\left[0.5\pi+0.1\cdot(i-1)\pi;\ 0.5\pi+0.1i\pi\right)}\sin\left(0.85\cdot(0.7+0.1m)\left(1+0.5q\right)t-\frac{2\pi}{9}\right)d\mu\left(t\right)$$
$$\text{for}\ \ i=\overline{1,\ 3}\tag{182}$$

and

$$\varphi_{4mqs}=$$
$$=\int\limits_{[0.8\pi;\ 0.9\pi]}\sin\left(0.85\cdot(0.7+0.1m)\left(1+0.5q\right)t-\frac{2\pi}{9}\right)d\mu\left(t\right),\tag{183}$$

with second player's payoff matrices $\left\{\mathbf{G}_i=\left[\rho_{imqs}\right]_{5\times4\times7}\right\}_{i=1}^{4}$ whose elements are

$$\rho_{imqs}=$$
$$=\int\limits_{\left[0.5\pi+0.1\cdot(i-1)\pi;\ 0.5\pi+0.1i\pi\right)}\sin\left(1.4t\cdot(1+0.5q+3.3t+0.2st)-\frac{3\pi}{7}\right)d\mu\left(t\right)$$
$$\text{for}\ \ i=\overline{1,\ 3}\tag{184}$$

and

$$\rho_{4mqs} = \\ = \int\limits_{[0.8\pi;\ 0.9\pi]} \sin\left(1.4t \cdot (1 + 0.5q + 3.3t + 0.2st) - \frac{3\pi}{7}\right) d\mu\,(t), \quad (185)$$

and with third player's payoff matrices $\left\{\mathbf{H}_i = \left[\theta_{imqs}\right]_{5\times4\times7}\right\}_{i=1}^{4}$ whose elements are

$$\theta_{imqs} = \\ = \int\limits_{\left[0.5\pi+0.1\cdot(i-1)\pi;\ 0.5\pi+0.1i\pi\right)} \sin\left(\frac{0.5t\,(3.3+0.2s)}{0.7+0.1m} - \frac{\pi}{4}\right) d\mu\,(t) \\ \text{for } \ i = \overline{1,\ 3} \quad (186)$$

and

$$\theta_{4mqs} = \\ = \int\limits_{[0.8\pi;\ 0.9\pi]} \sin\left(\frac{0.5t\,(3.3+0.2s)}{0.7+0.1m} - \frac{\pi}{4}\right) d\mu\,(t). \quad (187)$$

It is worth noting that this finite 3-person game is rendered to a trimatrix $625 \times 256 \times 2401$ game. Such a trimatrix game cannot be solved in a reasonable amount of computational time because there are 384160000 pure-strategy situations (searching for efficient situations through this number of situations would take too long).
The four trimatrix $5 \times 4 \times 7$ games (181) with (182) – (187) have 30, 21, 14, 8 Pareto-efficient situations, respectively. Therefore, there are $\prod\limits_{i=1}^{4} J_i = 30 \cdot 21 \cdot 14 \cdot 8 = 70560$ stacks of such situations. The respective 3-person staircase-function game by (172) – (179) has 736 Pareto-efficient stacks (which are a subset of those 70560 ones due to Theorem 4) presented in Figure 1, wherein a three-dimensional scatter cloud of 70560 stack payoffs

$$\left\{u_l^*,\ v_l^*,\ w_l^*\right\} = \\ = \left\{\sum_{i=1}^{4} F\left(\alpha_{ij_i}^*,\ \beta_{ij_i}^*,\ \gamma_{ij_i}^*\right),\ \sum_{i=1}^{4} G\left(\alpha_{ij_i}^*,\ \beta_{ij_i}^*,\ \gamma_{ij_i}^*\right),\ \sum_{i=1}^{4} H\left(\alpha_{ij_i}^*,\ \beta_{ij_i}^*,\ \gamma_{ij_i}^*\right)\right\} \\ \text{by } \ l = \overline{1,\ 70560}$$

can be seen as well. The single best efficient payoffs point calculated by (171) as

$$l_* \in \arg \max_{l=\overline{1,\ 736}} \\ \sqrt{\left(\frac{u_l^* - \min\limits_{k=\overline{1,\ 736}} u_k^*}{\max\limits_{k=\overline{1,\ 736}} u_k^* - \min\limits_{k=\overline{1,\ 736}} u_k^*}\right)^2 + \left(\frac{v_l^* - \min\limits_{k=\overline{1,\ 736}} v_k^*}{\max\limits_{k=\overline{1,\ 736}} v_k^* - \min\limits_{k=\overline{1,\ 736}} v_k^*}\right)^2 + \left(\frac{w_l^* - \min\limits_{k=\overline{1,\ 736}} w_k^*}{\max\limits_{k=\overline{1,\ 736}} w_k^* - \min\limits_{k=\overline{1,\ 736}} w_k^*}\right)^2} \quad (188)$$

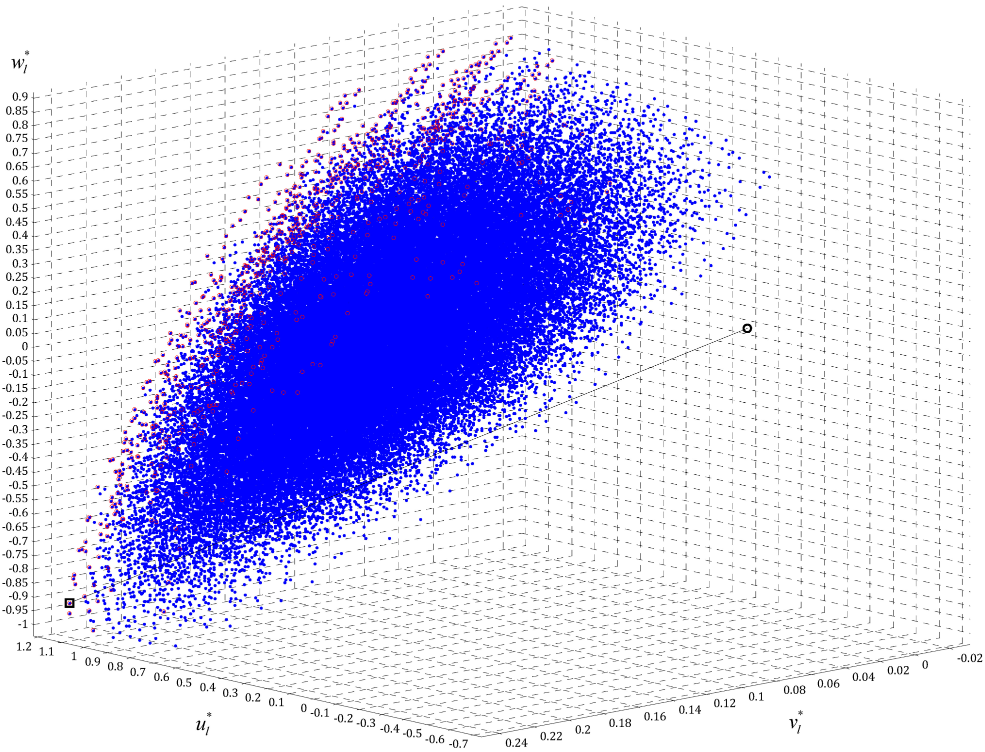


**Figure 1.** **A three-dimensional scatter cloud of 70560 stack payoffs in the finite 3-person staircase-function game and 736 payoffs (circles) by the efficient stacks (the best efficient payoffs point is squared).**

corresponds to the best Pareto-efficient situation, whose first, second, and third players' strategies $x^*(t)$, $y^*(t)$, and $z^*(t)$, respectively, are shown in Figure 2. The best efficient payoffs are

$$\left\{u^*_{l_*},\ v^*_{l_*},\ w^*_{l_*}\right\} = \left\{u^*_{17},\ v^*_{17},\ w^*_{17}\right\} = \left\{1.2129,\ 0.2298,\ -0.9461\right\}. \tag{189}$$

Note that those 736 Pareto-efficient situations are not sorted in any order, so index $l_* = 17$ does not relate to anything.

One may notice also that the third player's best efficient payoff is negative, whereas the payoff ranges from $-1.0033$ to 0.9194 and its negativity seems to be weird. The second player's efficient payoff ranges from $-0.0308$ to 0.2443, whose best efficient payoff is only 5.949 % less than the maximum. Moreover, the first player's efficient payoff ranges from $-0.7309$ to 1.214, whose best efficient payoff is only 0.09 % less than the maximum. So, why is the third player's best efficient payoff so close to its minimum rather than maximum? The explanation lies in the relative collective utility, which after 0-1-standardization by (188) appears to be the largest just for payoffs (189).

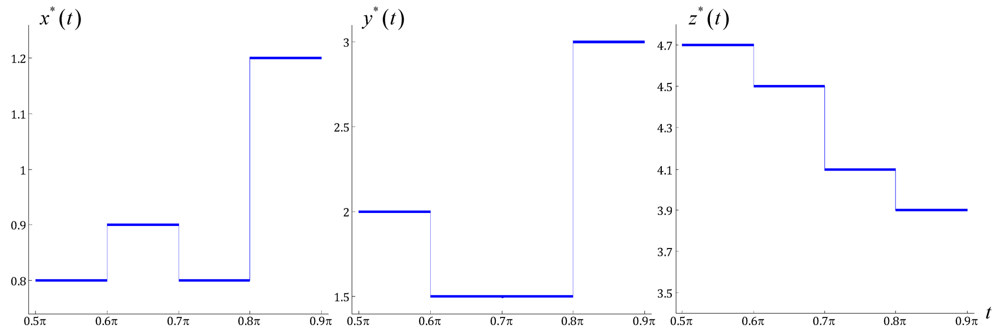


**Figure 2.** **The best efficient staircase-function strategies producing payoffs (189).**

In the considered example, the first player has 625 staircase-function strategies (one of which is shown in Figure 2). The second and third players have 256 and 2401 staircase-function strategies (see Figure 2), respectively. Even in this not-a-big example, the total number of stacks of subinterval Pareto-efficient situations ("made" of staircase-function strategies triples) is indeed that large (it is 70560 which is less than 0.019 % of all possible pure-strategy situations). The number of "real" Pareto-efficient situations (it is 736) is slightly greater than 1.04 % of the total number of stacks of subinterval Pareto-efficient situations. Therefore, the example shows that seeking for the efficiency in a finite 3-person staircase-function game is an extremely hard computational task. This task, however, is dramatically simplified by considering the respective succession of trimatrix games. Another, concomitant, task (existing when the conditions of Theorem 2 do not hold) is the selection of the best Pareto-efficient situation among $L$ ("real") Pareto-efficient situations.

## 9. Discussion

The core of the method of solving 3-person games played with staircase-function strategies consists in finding all Pareto-efficient situations in every subinterval 3-person game. The computation time depends on the number of subintervals, i. e., on the "length" of the staircase-function game. Nevertheless, too lengthy staircase-function games are rare in practice.

If a subinterval game has a single efficient situation, this helps much in solving the staircase-function game. An efficient situation in the staircase-function game will definitely have the single efficient situation on the given subinterval. This directly follows from Theorem 4. The size of a finite subinterval game is defined by the sets of possible values of players' pure strategies. The size influences the computation time also. In particular cases, solving a continuous subinterval game may cause considerable delay or be just intractable itself. Then the continuous subinterval game must be approximated with a finite (i. e., trimatrix) game using the known techniques [16, 18]. Usually, an ordinary 3-person game has multiple Pareto-efficient situations. A 3-

person game having a single Pareto-efficient situation is a great rarity (unless it is a $2 \times 2 \times 2$ game). However, this does not diminish the value of Theorem 2 whose proof directly follows from Theorem 4. The reversibility of Theorem 2 gives a definite practical impact. If it is known that a staircase-function game has a single Pareto-efficient situation then, according to Theorem 3, its search is organized by the principle of the early stop – once an efficient situation in a subinterval 3-person game is found, the next subinterval game is solved.

It is clear that staircase-function trimatrix games are solved easier. Moreover, there is no universal method to finding all Pareto-efficient situations in an infinite or continuous 3-person game. The finite approximation may become a necessary intermediate in solving a staircase-function game.

## 10. Conclusion

A finite 3-person game whose players use staircase function-strategies is rendered to a trimatrix game owing to the finiteness of the pure strategy sets. However, a finite 3-person staircase-function game can hardly be solved as the trimatrix game because there commonly is a gigantic and intractable number of pure-strategy situations. The best way is to consider any 3-person staircase-function game as a succession of 3-person games in which the players' strategies are constants.

For a continuous (infinite) staircase-function game, where the player has a continuum (infinite, countable or uncountable set) of staircase function-strategies, each constant-strategy game is a classical continuous (infinite) 3-person game. For a finite staircase-function game, each constant-strategy game is a trimatrix game whose size is likely to be relatively small to solve it in a reasonable time.

Whichever the staircase-function game continuity is, any Pareto-efficient situation of staircase function-strategies is a stack of successive Pareto-efficient situations in the constant-strategy games (Theorem 4). If every constant-strategy game has a single Pareto-efficient situation, then the staircase-function game has a single Pareto-efficient situation (Theorem 2). The inverse assertion is correct as well (Theorem 3). If a staircase-function game has two or more Pareto-efficient situations, the best efficient situation is one which is the farthest from the triple of the most unprofitable payoffs. In terms of 0-1-standardization, the best efficient situation is the farthest from the triple of zero payoffs.

The suggested method is a significant contribution to the mathematical 3-person game theory and practice. It significantly simplifies a 3-person game played with staircase-function strategies by just "deeinstellungizing" them [10, 16, 17, 22] (in fact, both the game and such strategies). Thus, too complicated solutions resulting from game continuities, pure strategy structure complexity, and equilibrium, profitability, fairness uncertainty are avoided. The method is practically applicable owing to its tractability and simplicity, although the efficient situations search may be optimized for particular game classes. As pure strategies are only considered, the method fits nonrepeatable games as well [8, 9, 13].

The case when staircase function-strategies constitute an efficient Nash equilibrium should be considered separately. Then it is quite interesting whether the core of the presented assertions and conclusions changes. For their extension to a general consideration, the case of four players and more will be studied also.

## References


[1] A.S. Belenky, *A 3-person game on polyhedral sets*, Comput. Math. Appl. **28** (1994), no. 5, 53–56.

[2] E. Boros, V. Gurvich, M. Milanič, V. Oudalov, and J. Vičič, *A three-person deterministic graphical game without nash equilibria*, Discrete Appl. Math. **243** (2018), 21–38.

[3] R.E. Edwards, *Functional Analysis: Theory and Applications*, Holt, Rinehart and Winston, 1965.

[4] G. Fandel and J. Trockel, *Avoiding non-optimal management decisions by applying a three-person inspection game*, European Journal of Operational Research **226** (2013), no. 1, 85–93.

[5] J.C. Harsanyi and R. Selten, *A General Theory of Equilibrium Selection in Games*, The MIT Press, 1988.

[6] D. Hirshleifer, D. Jiang, and Y.M. DiGiovanni, *Mood beta and seasonalities in stock returns*, Journal of Financial Economics **137** (2020), no. 1, 272–295.

[7] J. Ke, X. Li, H. Yang, and Y. Yin, *Pareto-efficient solutions and regulations of congested ride-sourcing markets with heterogeneous demand and supply*, Transportation Research Part E: Logistics and Transportation Review **154** (2021), Article ID: 102483.

[8] A. F Kleimenov and M.A. Schneider, *Cooperative dynamics in a repeated three-person game with finite number of strategies*, IFAC Proceedings Volumes **38** (2005), no. 1, 171–175.

[9] Y. Lin and W. Zhang, *Pareto efficiency in the infinite horizon mean-field type cooperative stochastic differential game*, Journal of the Franklin Institute **358** (2021), no. 10, 5532–5551.

[10] F. Loesche and T. Ionescu, *Mindset and einstellung effect, in: Encyclopedia of Creativity*, pp. 174–178, Academic Press, 2020.

[11] H. Moulin, *Théorie des jeux pour l'économie et la politique*, Hermann, 1981.

[12] N. Nisan, T. Roughgarden, É. Tardos, and V.V. Vazirani, *Algorithmic game theory*, Cambridge University Press, 2007.

[13] V.V. Romanuke, *Convergence and estimation of the process of computer implementation of the optimality principle in matrix games with apparent play horizon*, Journal of Automation and Information Sciences **45** (2013), no. 10, 49–56.

[14] ———, *Maximization of collective utility and minimization of payoff parity losses for ordering and scaling efficient nash equilibria in trimatrix games with asym-*

*metric payoffs*, Herald of Khmelnytskyi national university. Technical sciences (2018), no. 3, 279–281.

[15] ———, *Ecological-economic balance in fining environmental pollution subjects by a dyadic 3-person game model*, Applied Ecology and Environmental Research **17** (2019), no. 2, 1451–1474.

[16] ———, *Adaptive finite approximation of continuous noncooperative games*, Journal of Automation and Information Sciences **52** (2020), no. 10, 31–41.

[17] ———, *Finite approximation of continuous noncooperative two-person games on a product of linear strategy functional spaces*, Journal of Mathematics and Applications **43** (2020), 123–138.

[18] V.V. Romanuke and V.V. Kamburg, *Approximation of isomorphic infinite two-person non-cooperative games by variously sampling the players' payoff functions and reshaping payoff matrices into bimatrix game*, Applied Computer Systems **20** (2016), no. 1, 5–14.

[19] L. Sääksvuori and A. Ramalingam, *Bargaining under surveillance: Evidence from a three-person ultimatum game*, Journal of Economic Psychology **51** (2015), 66–78.

[20] N.N. Vorob'yov, *Game theory fundamentals. noncooperative games*, 1984.

[21] ———, *Game theory for economists-cyberneticists*, 1985.

[22] J. Yang, Y.-S. Chen, Y. Sun, H.-X. Yang, and Y. Liu, *Group formation in the spatial public goods game with continuous strategies*, Physica A: Statistical Mechanics and its Applications **505** (2018), 737–743.