Communications in Combinatorics and Optimization
Vol. 8, No. 2 (2023), pp. 327-348
DOI: 10.22049/CCO.2022.27363.1247

CCO
Commun. Comb. Optim.

Research Article

Unit $\mathbb{Z}_q$-simplex codes of type α and zero divisor $\mathbb{Z}_q$-simplex codes

J. Prabu[†], J. Mahalakshmi[*], S. Santhakumar[‡],

Department of Mathematics, Amrita School of Engineering, Coimbatore, Amrita Vishwa Vidyapeetham, India
[†]j_prabu1@cb.amrita.edu
[*]j_mahalakshmi@cb.amrita.edu
[‡]s_santhakumar@cb.amrita.edu

Received: 7 August 2021; Accepted: 2 March 2022
Published Online: 5 March 2022

Abstract: In this paper, we have punctured unit $\mathbb{Z}_q$-Simplex code and constructed a new code called unit $\mathbb{Z}_q$-Simplex code of type α. In particular, we find the parameters of these codes and have proved that it is an $\left[\phi(q)+2,\ 2,\ \phi(q)+2-\frac{\phi(q)}{\phi(p)}\right]$ $\mathbb{Z}_q$-linear code if $k = 2$ and $\left[\frac{\phi(q)^k-1}{\phi(q)-1}+\phi(q)^{k-2},\ k,\ \frac{\phi(q)^k-1}{\phi(q)-1}+\phi(q)^{k-2}-\left(\frac{\phi(q)}{\phi(p)}\right)\left(\frac{\phi(q)^{k-1}-1}{\phi(q)-1}+\phi(q)^{k-3}\right)\right]$ $\mathbb{Z}_q$-linear code if $k \geq 3$, where p is the smallest prime divisor of q. For q is a prime power and rank $k = 3$, we have given the weight distribution of unit $\mathbb{Z}_q$-Simplex Codes of type α. Also, we have introduced some new code from $\mathbb{Z}_q$-Simplex code called zero divisor $\mathbb{Z}_q$-Simplex code and proved that it is an $\left[\frac{\rho^k-1}{\rho-1},\ k,\ \frac{\rho^k-1}{\rho-1}-\left(\frac{\rho^{(k-1)}-1}{\rho-1}\right)\left(\frac{q}{p}\right)\right]$ $\mathbb{Z}_q$-linear code, where $\rho = q-\phi(q)$ and p is the smallest prime divisor of q. Further, we obtain weight distribution of zero divisor $\mathbb{Z}_q$-Simplex code for rank $k = 3$ and q is a prime power.

Keywords: Unit $\mathbb{Z}_q$-Simplex codes of type α, Unit $\mathbb{Z}_q$-MacDonald code, Zero divisor $\mathbb{Z}_q$-Simplex code and Weight distribution

AMS Subject classification: 11T71, 94B05

1. Introduction

Recently, codes over finite rings have gained significant attention. The binary MacDonald codes were introduced in [15] and q-ary version of MacDonald code over the

* *Corresponding Author*

finite field $\mathbb{F}_q$ was examined in [17]. In 1942, Ronald Fisher[10] detected the binary Simplex code S_k and it is related to the statistical design. Then it is deduced to Galois Field(GF(p^n))[11]. The weight distribution of the code provides a lot of information about the code for practical as well as theoretical purposes. A lot of research has been devoted to the study of weight distribution and weight preserving maps. The most important result in this area is the Duality Theorem of MacWilliams which presents a transformation between the weight distribution of a code and it's dual.
Author Chatouh et al. [3] studied the homogeneous weight and homogeneous Gray map over the ring $\mathbb{F}_2[u_1, u_2, \ldots, u_q]/\langle u_i^2 = 0,\, u_iu_j = u_ju_i\rangle$ for $q > 1$. Wang et al. [19] investigated the Macdonald code over the finite ring $\mathbb{F}_p + v\mathbb{F}_p + v^2\mathbb{F}_p$, where $v^3 = v$ and p is an odd prime. Inspired by the work of simplex codes, Macdonald codes and punctured version of linear codes, which have been analysed [1–9, 12–14, 16, 20]. In this work, we define the unit $\mathbb{Z}_q$-Simplex code of type α and zero divisor $\mathbb{Z}_q$-Simplex code. The parameters and weight distribution of these codes with respect to the hamming distance are examined.
In [4] and [9], initially authors studied the parameters of $\mathbb{Z}_q$-Simplex code for any q and found the weight distribution of $\mathbb{Z}_q$-Simplex code for $k = 2$ and q is a prime power or product of primes. Later in [2], the authors punctured the $\mathbb{Z}_q$-Simplex code and analyzed the parameter of such codes for $k = 2$. Also, they obtained the weight distribution of punctured $\mathbb{Z}_q$-Simplex code when $k = 2$, for any q.
The above work has motivated us to attain the following results. In Section 3, we define the unit $\mathbb{Z}_q$-MacDonald code and brought its parameters for any q. In Section 4, we have constructed unit $\mathbb{Z}_q$-Simplex code of type α and found its parameters for any q. In Section 5, we determine the weight distribution of unit $\mathbb{Z}_q$-Simplex code of type α when q is a prime power and rank $k = 3$. In Sections 6 and 7, we have introduced zero divisor $\mathbb{Z}_q$-Simplex code and obtained its parameters for any q. Further, we found the weight distribution of zero divisor $\mathbb{Z}_q$-Simplex code for q is a prime power and rank $k = 3$.

2. Preliminaries

Let $\mathbb{Z}_q$ denote the ring of integers modulo q. A proper subset of $\mathbb{Z}_q^n$ is called a *code* C of length n over $\mathbb{Z}_q$. Let $x, y \in \mathbb{Z}_q^n$, then the hamming *distance* between x and y is the number of coordinates in which they differ. It is denoted by $d(x, y)$. Clearly, $d(x, y) = wt(x - y)$, the number of non-zero coordinates in $x - y$. The minimum distance d of the code C is defined by

$$d = \min\{d(x, y) \ \mid x, y \in C,\ x \neq y\}.$$

A code C of length n, cardinality M with minimum distance d is denoted by (n, M, d) code. We know that, $\mathbb{Z}_q$ is a ring under addition modulo q and multiplication modulo q. Then $\mathbb{Z}_q^n$ is a free module over $\mathbb{Z}_q$. C is said to be a $\mathbb{Z}_q$-*linear code* if C is a submodule of $\mathbb{Z}_q^n$. A matrix whose rows are basis elements of the $\mathbb{Z}_q$-linear code C

is called a *generator matrix* of C. Every k rank $\mathbb{Z}_q$-linear code with length n and minimum distance d is called $[n,k,d]$ $\mathbb{Z}_q$-linear code.

Let C be a $\mathbb{Z}_q$-linear code. If $x, y \in C$, then $x - y \in C$. Therefore, the minimum distance d of C is

$$\min\{d(x,y) \mid x, y \in C,\ x \neq y\} = \min\{wt(c) \mid c \in C,\ c \neq 0\}.$$

That is, if C is a $\mathbb{Z}_q$-linear code, then the minimum distance of C is the same as the minimum weight of C.
Throughout this paper, we consider $o(x)$ denotes the additive order of x where $x \in \mathbb{Z}_q$ and $N(v)$ denote the number of zero coordinates in the vector v. It is important to mention here the following three theorems which are given in [2].

Theorem 1. *[2] Let $o(x) = d$ where $x \in \mathbb{Z}_q$. Then every element of order d in $\mathbb{Z}_q$ appears $\frac{\phi(q)}{\phi(d)}$ times in $xU(q)$ where $U(q)$ set of all units of $\mathbb{Z}_q$.*

Theorem 2. *[2] Let d be the smallest divisor of q and $d \neq 1$ and let*

$$G_1 = \left[\begin{array}{c|c|cccc} 1 & 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{\phi(q)} \\ \hline 0 & 1 & 1 & 1 & \cdots & 1 \end{array}\right]$$

where $\{\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{\phi(q)}\}$ is the set of all unit elements in $\mathbb{Z}_q$. Then G_1 generates a $[\phi(q)+2,\ 2,\ 2+\phi(q) - \frac{\phi(q)}{\phi(d)}]$ $\mathbb{Z}_q$-linear code.

Theorem 3. *[2] Let $q \geq 2$ and let*

$$G_2 = \left[\begin{array}{c|c|cccc} 1 & 0 & z_1 & z_2 & \cdots & z_s \\ \hline 0 & 1 & 1 & 1 & \cdots & 1 \end{array}\right]$$

where $\{z_1,\ z_2,\ \cdots,\ z_s\}$ is the set of all zero divisors of $\mathbb{Z}_q$. Then the matrix G_2 generates a $[q-\phi(q)+1,\ 2,\ q-\phi(q)+1-\frac{q}{p}]$ $\mathbb{Z}_q$-linear code where p is the smallest prime divisor of q.

3. Unit $\mathbb{Z}_q$-Macdonald code

In this section, we have introduced some new type of Macdonald code called unit $\mathbb{Z}_q$-Macdonald codes and given its parameters from unit $\mathbb{Z}_q$-Simplex codes.
Let G_2 be a matrix over $\mathbb{Z}_q$ whose columns are one non-zero element from each 1-dimensional submodule of $\mathbb{Z}_q^2$. Then the matrix above is equivalent to the matrix

$$G_2 = \left[\begin{array}{c|c|cccc} 0 & 1 & 1 & 2 & \cdots & q-1 \\ \hline 1 & 0 & 1 & 1 & \cdots & 1 \end{array}\right]$$

By inductively,

$$G_k = \left[\begin{array}{c|c|c|c|c|c} 00\cdots0 & 1 & 11\cdots1 & 22\cdots2 & \cdots & q-1q-1\cdots q-1 \\ \hline & 0 & & & & \\ G_{k-1} & \vdots & G_{k-1} & G_{k-1} & \cdots & G_{k-1} \\ & 0 & & & & \end{array}\right]$$

for $k > 2$.

By [9], the matrix G_k generates $\left[\frac{q^k-1}{q-1}, k, \frac{q}{p}(p-1)\left(\frac{q^{k-1}-1}{q-1}\right)+1\right]$ $\mathbb{Z}_q$-linear code of rank k. The code generated by the matrix G_k is called $\mathbb{Z}_q$-Simplex code.

In [16], authors have defined MacDonald code over $\mathbb{Z}_q$ by deleting the matrix

$$\begin{bmatrix} \mathbf{0} \\ G_u \end{bmatrix}$$

where $2 \le u \le k-1$ and $\mathbf{0}$ is $(k-u) \times \frac{q^u-1}{q-1}$ zero matrix, from the generator matrix G_k of $\mathbb{Z}_q$-Simplex code, that is

$$G_{k,u} = \left(G_k \setminus \begin{pmatrix} \mathbf{0} \\ G_u \end{pmatrix} \right)$$

for $2 \le u \le k-1$ and $\Big(A \setminus B\Big)$ is a matrix obtained from the matrix A by removing the matrix B. By [16], the code generated by the matrix $G_{k,k-1}$ is called $\left[q^{k-1},\ k,\ 1+\left(q-\frac{q}{p}-1\right)\left(\frac{q^{k-1}-1}{q-1}\right)+\frac{q^{k-2}}{q-1}\right]$ $\mathbb{Z}_q$-MacDonald code, where p is the smallest prime divisor of q. It is denoted by $M_{k,k-1}$.

In the matrix G_2, we remove the columns corresponding to the zero divisors and denoted as $G_2(q,u)$, that is,

$$G_2(q,u) = \left[\begin{array}{c|c|cccc} 1 & 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{\phi(q)} \\ \hline 0 & 1 & 1 & 1 & \cdots & 1 \end{array}\right]$$

where $\alpha_1,\ \alpha_2,\ \cdots,\alpha_{\phi(q)}$ are units in $\mathbb{Z}_q$ and $\phi(q)$ is Euler ϕ-function. Then by Theorem 2, the matrix $G_2(q,u)$ generates $\left[\rho+1,\ 2,\ \rho+1-\left(\frac{\rho-1}{p-1}\right)\right]$ $\mathbb{Z}_q$-linear code where $\rho = \phi(q)+1$.

Similarly, we remove the blocks corresponds to the zero divisors from the generator matrix G_k and denoted as $G_k(q,u)$, that is,

$$G_k(q,u) = \left[\begin{array}{c|c|c|c|c|c} 1 & 00\cdots0 & \alpha_1\alpha_1\cdots\alpha_1 & \alpha_2\alpha_2\cdots\alpha_2 & \cdots & \alpha_{\phi(q)}\alpha_{\phi(q)}\cdots\alpha_{\phi(q)} \\ \hline 0 & & & & & \\ \vdots & G_{k-1}(q,u) & G_{k-1}(q,u) & G_{k-1}(q,u) & \cdots & G_{k-1}(q,u) \\ 0 & & & & & \end{array}\right]$$

for $k > 2$.
By [18], this matrix $G_k(q,u)$ generates $\left[\frac{\rho^k-1}{\rho-1},\ k,\ \rho^{k-2}\left(\rho+1-\left(\frac{\rho-1}{p-1}\right)\right)\right]$ $\mathbb{Z}_q$-linear code where $\rho = \phi(q)+1$ and p is a smallest prime divisor of $q \geq 2$, and the code generated by the matrix $G_k(q,u)$ is called unit $\mathbb{Z}_q$-Simplex code.
Consider the generator matrix $G_k(q,u)$ of the unit $\mathbb{Z}_q$-Simplex code of dimension $k \geq 3$. By deleting the matrix

$$\begin{bmatrix} \mathbf{0} \\ G_w(q,u) \end{bmatrix}$$

where $2 \leq w \leq k-1$ and $\mathbf{0}$ is $(k-w) \times \frac{\rho^w-1}{\rho-1}$ zero matrix from $G_k(q,u)$, we have obtained

$$G'_{k,w}(q,u) = \left(G_k(q,u) \setminus \begin{pmatrix} \mathbf{0} \\ G_w(q,u) \end{pmatrix}\right)$$

for $2 \leq w \leq k-1$. A code generated by the matrix $G'_{k,w}(q)$ is called unit $\mathbb{Z}_q$-MacDonald code. It is denoted by $M'_{k,w}(q,u)$.

Theorem 4. *Let $M'_{k,w}(q,u)$ be linear code generated by the matrix $G'_{k,w}(q)$. Then the parameters of the code is, $\left[\frac{\rho^k-\rho^w}{\rho-1},\ k,\ (\rho^{k-2}-\rho^{w-2})\left(\rho+1-\left(\frac{\rho-1}{p-1}\right)\right)\right]$ $\mathbb{Z}_q$-linear code where $\rho = \phi(q)+1$ and p is a smallest prime divisor of $q \geq 2$.*

Proof. The proof is similar to the proof of Theorem 3.5 in [18]. □

4. Unit $\mathbb{Z}_q$-Simplex code of type α

In this section, we have punctured the unit $\mathbb{Z}_q$-Simplex code and introduced some new type of code called an unit $\mathbb{Z}_q$-Simplex code of type α and found its parameters. Here, we removed the matrix

$$\begin{bmatrix} \mathbf{0} \\ G_2(q,u) \end{bmatrix}$$

from the matrix $G_3(q,u)$ denoted as $G'_3(q,u)$.
By inductively, we define

$$G'_k(q,u) = \left[\begin{array}{c|c|c|c|c} 1 & \alpha_1\alpha_1\cdots\alpha_1 & \alpha_2\alpha_2\cdots\alpha_2 & \cdots & \alpha_{\phi(q)}\alpha_{\phi(q)}\cdots\alpha_{\phi(q)} \\ \hline 0 & & & & \\ \vdots & G'_{k-1}(q,u) & G'_{k-1}(q,u) & \cdots & G'_{k-1}(q,u) \\ 0 & & & & \end{array}\right] \tag{1}$$

for $k > 2$.

Clearly, the matrix $G'_k(q,u)$ generates $[n_k = \frac{\phi(q)^k-1}{\phi(q)-1} + \phi(q)^{k-2},\ k,\ d]$ $\mathbb{Z}_q$-linear code. The code generated by the matrix $G'_k(q,u)$ is called *unit $\mathbb{Z}_q$-Simplex code of type α*. It is denoted by $S'_k(q,u)$.

This matrix $G'_k(q,u)$ generates the code, $S'_k(q,u) = \{\beta(1\ \boldsymbol{\alpha_1}\ \boldsymbol{\alpha_2} \cdots \boldsymbol{\alpha_{\phi(q)}}) + (0\, c\, c \cdots c) \mid \beta \in \mathbb{Z}_q, c \in S'_{k-1}(q,u)$ and $\boldsymbol{\alpha_i} = \alpha_i\alpha_i \cdots \alpha_i \in \mathbb{Z}_q^{n_{k-1}}\}$.

Lemma 1. *[18] Let $\beta \in \mathbb{Z}_q \backslash \{0\}$. If $o(\beta) = l$ and $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{\phi(q)}$ are units in $\mathbb{Z}_q$, then $\beta(\alpha_1,\ \alpha_2,\ \ldots, \alpha_{\phi(q)}) =$ permutation of $(\beta_1,\ \beta_2,\ \ldots, \beta_{\phi(l)},\ \beta_1,\ \beta_2,\ \ldots, \beta_{\phi(l)},$ $\ldots, \beta_1,\ \beta_2,\ \ldots, \beta_{\phi(l)})$ and the repetition of the vector $\left(\beta_1,\ \beta_2,\ \ldots, \beta_{\phi(l)}\right)$ is $\frac{\phi(q)}{\phi(l)}$ where $\{\beta_1,\ \beta_2,\ \ldots, \beta_{\phi(l)}\}$ is the set of all elements of order l.*

Note that, if $o(\beta) = l$ is minimum in the above lemma, then the repetition of the vector $(\beta_1,\ \beta_2,\ \ldots, \beta_{\phi(l)})$ will be maximum.

Lemma 2. *Let C be an $[n,k,d]$ $\mathbb{Z}_q$-linear code generated by the matrix $G'_k(q,u)$ and $c_1 \in C$, $\beta \in \mathbb{Z}_q$ such that $o(\beta) = l$. Define $\overline{c_1} = \beta(1\, \boldsymbol{\alpha_1}\, \boldsymbol{\alpha_2} \cdots \boldsymbol{\alpha_{\phi(q)}}) + (0\, c_1\, c_1 \cdots c_1)$. Then $wt(\overline{c_1}) = n\phi(q) + 1 - \frac{\phi(q)}{\phi(l)} N_l(c_1)$ where $N_l(c_1)$ denotes the number of l order elements in the codeword c_1.*

Proof. Clearly, the length of the codeword $\overline{c_1}$ is $n\phi(q)+1$. It is given that $o(\beta) = l$. By Lemma 1,
$\beta(\alpha_1\alpha_2 \ldots \alpha_{\phi(q)}) =$ permutation of $\left(\beta_1\beta_2 \ldots \beta_{\phi(l)}\, \beta_1\beta_2 \ldots \beta_{\phi(l)} \ldots \beta_1\beta_2 \ldots \beta_{\phi(l)}\right)$.
Consider

$$\begin{aligned} wt(\overline{c_1}) &= n\phi(q) + 1 - \left\{ N(\boldsymbol{\alpha_1} + c_1) + N(\boldsymbol{\alpha_2} + c_1) + \ldots + N(\boldsymbol{\alpha_{\phi(q)}} + c_1) \right\} \\ &= n\phi(q) + 1 - \left(\frac{\phi(q)}{\phi(l)} \{N(\boldsymbol{\beta_1} + c_1) + N(\boldsymbol{\beta_2} + c_1) + \ldots + N(\boldsymbol{\beta_{\phi(l)}} + c_1)\} \right) \\ wt(\overline{c_1}) &= n\phi(q) + 1 - \frac{\phi(q)}{\phi(l)} N_l(c_1) \end{aligned}$$

That is, the weight of the codeword $\overline{c_1}$ is $n\phi(q) + 1 - \frac{\phi(q)}{\phi(l)} N_l(c_1)$. $\square$

Let $B_p = \{x \in \mathbb{Z}_q \mid o(x) = p\}$ where p is the smallest prime divisor of q. Define $C'_2(q,u) = \{\alpha(1\, 0\, \alpha_1\, \alpha_2\ \cdots \alpha_{\phi(q)}) + \beta(0\, 1\, 1\ \cdots\ 1) \mid \alpha, \beta \in B_p\}$. Also, by inductively we define $C'_k(q,u) = \{\beta(1\, \boldsymbol{\alpha_1}\, \boldsymbol{\alpha_2} \cdots \boldsymbol{\alpha_{\phi(q)}}) + (0\, c\, c \cdots c) \mid \beta \in B_p, c \in C'_{k-1}(q,u),\ \boldsymbol{\alpha_i} = (\alpha_i\alpha_i \cdots \alpha_i) \in \mathbb{Z}_q^{n_{k-1}}\}$.

Lemma 3. *Let $c \in C'_k(q,u)$ then, $wt(c) \le \frac{\phi(q)^k-1}{\phi(q)-1} + \phi(q)^{k-2}$ and max $\{wt(c) \mid c \in C'_k(q,u)\} = \frac{\phi(q)^k-1}{\phi(q)-1} + \phi(q)^{k-2}$.*

Proof. By induction on k, we can prove this lemma. $\square$

Let $S'_{k-1}(q,u)$ be an $[n,k-1,d]$ code and $S'_k(q,u) = \{\beta(1\,\boldsymbol{\alpha_1}\,\boldsymbol{\alpha_2}\cdots\boldsymbol{\alpha_{\phi(q)}}) + (0\,c\,c\cdots c) \mid \beta\in\mathbb{Z}_q, c\in S'_{k-1}(q,u)$ and $\boldsymbol{\alpha_i} = (\alpha_i\alpha_i\cdots\alpha_i)\in\mathbb{Z}_q^n\}$. Then the minimum distance of $S'_k(q,u)$ is

$$d(S'_k(q,u)) = \min\{wt(\beta(1\,\boldsymbol{\alpha_1}\,\boldsymbol{\alpha_2}\cdots\boldsymbol{\alpha_{\phi(q)}}) + (0\,c\,c\cdots c)\,) \mid c\in S'_{k-1}(q,u), \beta\in\mathbb{Z}_q \text{ and } \boldsymbol{\alpha_i}\in\mathbb{Z}_q^n\}.$$

If $\beta=0$, then $\min\{wt(0\,c\,c\cdots c)\mid c\in C,\ c\neq 0\} = d\phi(q)$. If $c=0$, then $\min\{wt(\beta(1\,\boldsymbol{\alpha_1}\,\boldsymbol{\alpha_2}\cdots\boldsymbol{\alpha_{\phi(q)}}))\mid \beta\in\mathbb{Z}_q,\ \beta\neq 0\} = n\phi(q)+1$.

By Lemma 3, let $c'\in S'_{k-1}(q,u)$ such that $wt(c') = max\{wt(c)\mid c\in C'_{k-1}(q,u)\}$. Then $N(c') = n - wt(c')$ and $wt(c') = N_p(c')$. If $c\neq 0$, $\beta\neq 0$ and $o(\beta)=l$, then by Lemma 2,

$$\begin{aligned} wt\left(\beta\left(1\,\boldsymbol{\alpha_1}\,\boldsymbol{\alpha_2}\cdots\boldsymbol{\alpha_{\phi(q)}}\right) + (0\,c\,c\cdots c)\right) &= n\phi(q)+1-\frac{\phi(q)}{\phi(l)}N_l(c)\\ &\geq n\phi(q)+1-\frac{\phi(q)}{\phi(p)}N_p(c')\\ &\geq n\phi(q)+1-\frac{\phi(q)}{\phi(p)}wt(c') \end{aligned}$$

Therefore, $wt\left(\beta\left(1\,\boldsymbol{\alpha_1}\,\boldsymbol{\alpha_2}\cdots\boldsymbol{\alpha_{\phi(q)}}\right) + (0\,c\,c\cdots c)\right)\geq n\phi(q)+1-\frac{\phi(q)}{\phi(p)}wt(c')$ where p is the smallest prime divisor of q. Thus, we have

Theorem 5. *Let $S'_k(q,u)$ be a $\mathbb{Z}_q$-linear code generated by the matrix $G'_k(q,u)$. Then the minimum distance of $S'_k(q,u)$ is $min\{d\phi(q),\ n\phi(q)+1,\ n\phi(q)+1-\frac{\phi(q)}{\phi(p)}wt(c')\}$ where $wt(c') = max\{wt(c)\mid c\in C'_{k-1}(q,u)\}$ and p is the smallest prime divisor of q.*

Theorem 6. *Let $S'_k(q,u)$ be linear code generated by the matrix $G'_k(q,u)$. Then the parameters of the code is,*

$$\begin{cases}\left(\phi(q)+2,\quad 2,\quad \phi(q)+2-\frac{\phi(q)}{\phi(p)}\right) & if\ k=2\\ \left(\frac{\phi(q)^k-1}{\phi(q)-1}+\phi(q)^{k-2},\quad k,\quad \frac{\phi(q)^k-1}{\phi(q)-1}+\phi(q)^{k-2}-\left(\frac{\phi(q)}{\phi(p)}\right)\left(\frac{\phi(q)^{k-1}-1}{\phi(q)-1}+\phi(q)^{k-3}\right)\right) & if\ k\geq 3\end{cases}$$

where p is the smallest prime divisor of q.

Proof. We prove this by induction on k. For $k=2$,

$$S'_2(q,u) = \{\beta(10\alpha_1\alpha_2\cdots\alpha_{\phi(q)}) + \gamma(0111\cdots 1)\mid \beta,\gamma\in\mathbb{Z}_q\}$$

is a code of length $\phi(q)+2$. Let $x=(10\alpha_1\alpha_2\cdots\alpha_{\phi(q)})$ and $y=(0111\cdots 1)$. Then the minimum distance of $S_2'(q,u)$ is

$$d(S_2'(q,u)) \;=\; min\{wt(\beta x), wt(\gamma y), wt(\beta x+\gamma y) \mid \beta,\gamma\in\mathbb{Z}_q\backslash\{0\}\}.$$

Clearly, $min\{wt(\beta x)\mid \beta\neq 0\}=\phi(q)+1$ and $min\{wt(\gamma y)\mid \gamma\neq 0\}=\phi(q)+1$. Consider

$$\begin{aligned} min\{wt(\beta x+\gamma y)\ \mid \beta,\,\gamma\in\mathbb{Z}_q\backslash\{0\}\} &= min\left\{wt(\beta x+\gamma y)\mid \beta,\ \gamma\in B_p\right\}\\ &= \phi(q)+2-\frac{\phi(q)}{\phi(p)}\\ min\{wt(\beta x+\gamma y)\ \mid \beta,\,\gamma\in\mathbb{Z}_q\backslash\{0\}\} &= \phi(q)+2-\frac{\phi(q)}{\phi(p)}. \end{aligned}$$

Therefore, $d(S_2'(q,u))=min\left\{\phi(q),\ \phi(q)+2-\frac{\phi(q)}{\phi(p)}\right\}=\phi(q)+2-\frac{\phi(q)}{\phi(p)}$.

Assume that the theorem is true for all $i<k$. That is, the parameter of the code $S_i'(q,u)$ is $\left(\frac{\phi(q)^i-1}{\phi(q)-1}+\phi(q)^{i-2},\quad i,\quad \frac{\phi(q)^i-1}{\phi(q)-1}+\phi(q)^{i-2}-\left(\frac{\phi(q)}{\phi(p)}\right)\left(\frac{\phi(q)^{i-1}-1}{\phi(q)-1}+\phi(q)^{i-3}\right)\right)$ $\mathbb{Z}_q$-linear code.

Now, we are going to prove the theorem for k. By Theorem 5,

$$d(S_k'(q,u))=min\left\{d\phi(q),\ n\phi(q)+1,\ n\phi(q)+1-\frac{\phi(q)}{\phi(p)}wt(c')\right\}$$

where $n=\frac{\phi(q)^k-2}{\phi(q)-1}+\phi(q)^{k-3}$ and
$d=\frac{\phi(q)^{k-1}-1}{\phi(q)-1}+\phi(q)^{k-3}-\left(\frac{\phi(q)}{\phi(p)}\right)\left(\frac{\phi(q)^{k-2}-1}{\phi(q)-1}+\phi(q)^{k-4}\right)$ are length and minimum distance of the code $M_{k-1}(q,u)$, respectively.

By Lemma 3, $wt(c')=\frac{\phi(q)^k-1}{\phi(q)-1}+\phi(q)^{k-2}$. Therefore, by induction hypothesis,

$$\begin{aligned} d(S_k'(q,u)) &= min\left\{\phi(q)\left(\frac{\phi(q)^{k-1}-1}{\phi(q)-1}+\phi(q)^{k-3}-\left(\frac{\phi(q)}{\phi(p)}\right)\left(\frac{\phi(q)^{k-2}-1}{\phi(q)-1}+\phi(q)^{k-4}\right)\right),\right.\\ &\qquad \left.\frac{\phi(q)^k-1}{\phi(q)-1}+\phi(q)^{k-2}-\left(\frac{\phi(q)}{\phi(p)}\right)\left(\frac{\phi(q)^{k-1}-1}{\phi(q)-1}+\phi(q)^{k-3}\right)\right\} \end{aligned}$$

and hence, $d(S_k'(q,u))=\frac{\phi(q)^k-1}{\phi(q)-1}+\phi(q)^{k-2}-\left(\frac{\phi(q)}{\phi(p)}\right)\left(\frac{\phi(q)^{k-1}-1}{\phi(q)-1}+\phi(q)^{k-3}\right)$. □

The following example illustrates Theorem 6.

Table 1. **Parameters of $\mathbb{Z}_q$-Simplex Codes of type α.**

Generator matrix $G'_k(25,u)$ and its parameters	
$G'_2(25,u)$	[22, 2, 17]
$G'_3(25,u)$	[441, 3, 336]
$G'_4(25,u)$	[8821, 4, 6800]
$G'_5(25,u)$	[176421, 5, 132316]
$G'_6(25,u)$	[3528421, 6, 2646316]
$G'_7(25,u)$	[67351614, 7, 50521514]

Example 1. Let $q = 25$. Then $\{1,\ 2,\ 4,\ 6,\ 7,\ 8,\ 9,\ 11,\ 12,\ 13,\ 14,\ 16,\ 17,\ 18,\ 19,\ 21,\ 22,\ 23,\ 24\}$ are units in $\mathbb{Z}_{25}$ and the generator matrix of $S'_2(q,u)$ is,

$$G'_2(q,u) = \begin{bmatrix} 1 & 0 & 1 & 2 & 3 & 4 & 6 & 7 & 8 & 9 & 11 & 12 & 13 & 14 & 16 & 17 & 18 & 19 & 21 & 22 & 23 & 24 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The Matrix

$$G'_k(q,u) = \left[\begin{array}{c|c|c|c|c} 1 & \alpha_1\alpha_1\cdots\alpha_1 & \alpha_2\alpha_2\cdots\alpha_2 & \cdots & \alpha_{\phi(q)}\alpha_{\phi(q)}\cdots\alpha_{\phi(q)} \\ \hline 0 & & & & \\ \vdots & G'_{k-1,u} & G'_{k-1,u} & \cdots & G'_{k-1,u} \\ 0 & & & & \end{array}\right]$$

for $k > 2$.
By Theorem 6, the parameters of the code generated by the matrix $G'_2(25,u)$, $G'_3(25,u)$, $G'_4(25,u)$, $G'_5(25,u)$, $G'_6(25,u)$ and $G'_7(25,u)$, are listed in Table 1.

5. Weight distribution of unit $\mathbb{Z}_q$-Simplex code of type α of dimension 3 for $q = p^m$ where p is a prime integer and $m \geq 2$

In this section, we obtain the weight distribution of unit $\mathbb{Z}_q$-Simplex code of type α generated by matrix $G'_3(q,u)$ for $q = p^m$ where p is a prime integer and $m \geq 2$.

Let $B_{p^i} = \{x \in \mathbb{Z}_q \mid o(x) = p^i\}$ for $0 \leq i \leq m$. Then $\{B_{p^i} \mid 0 \leq i \leq m\}$ forms a partition for $\mathbb{Z}_q$. Clearly, $B_p \cup \{0\}$ is a group with respect to addition and B_{p^i} is closed under additive inverse $\forall\ i$.

Throughout this paper, the pair $\left(b, B_{p^i}\right)$ denotes a word c with elements of B_{p^i} appear b number of places. We define the set

$$D = \{\beta(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{\phi(q)}) + (a,a,a,\ldots,a) \mid a,\beta \in \mathbb{Z}_q\backslash\{0\}\}.$$

Then, we have the following obvious results

Lemma 4. *[18] For each $i \in \{2,3,\ldots,m\}$ and $j \in \{1,2,\ldots,i-1\}$, let $\beta \in B_{p^i}$ and $a \in B_{p^j}$, then D has the words of the form $\left(\phi(q),\ B_{p^i}\right)$.*

Lemma 5. *[18] For each $i \in \{1,2,3,\ldots,m-1\}$ and $j \in \{i+1,i+2,\ldots,m\}$, let $\beta \in B_{p^i}$, then D has the words of the form $\left(\phi(q),\ B_{p^j}\right)$.*

Lemma 6. *[18] For each $i \in \{1,2,3,\ldots,m\}$, let β and $a \in B_{p^i}$, then D has the words of the form $\left(\frac{\phi(q)}{\phi(p^i)}(p^i-2p^{i-1}),\ B_{p^i}\right)$, $\left(\frac{\phi(q)}{\phi(p^i)},\ B_1\right)$ and $\left(\frac{\phi(q)}{\phi(p^i)}(p^j-p^{j-1}),\ B_{p^j}\right)$ for $1 \leq j \leq i-1$.*

Now, we are going to discuss the coordinates of the code $S_2'(q,u)$. If $k=2$, then the matrix $G_2'(q,u)$ generates the code

$$S_2'(q,u) = \{\beta_1(1\,0\,\alpha_1\,\alpha_2\cdots\alpha_{\phi(q)}) + \beta_2(0\,1\,1\,1\cdots 1) \mid \beta_1,\beta_2 \in \mathbb{Z}_q\}.$$

Case i. Clearly, if $\beta_1,\ \beta_2 \in B_1$, then $S_2'(q,u)$ has the codeword of the form $\left(\phi(q)+2,\ B_1\right)$, that is, $wt(c)=0$.

Case ii. It is easy to see that, if $\beta_1 \in B_1$ and $\beta_2 \in B_{p^i}$ or $\beta_1 \in B_{p^i}$ and $\beta_2 \in B_1$ for $1 < i \leq m$, then $S_2'(q,u)$ has the codewords of the form $\left(\frac{\phi(q)}{\phi(p^i)}\left(p^i-p^{i-1}\right),\ B_{p^i}\right)$. That is, $wt(c)=\phi(q)+1$.

Case iii. Choose the scalars β_1 and $\beta_2 \in B_{p^i}$ for $1 < i \leq m$. Then by Lemma 6, $S_2'(q,u)$ has the codewords of the form $\left(\frac{\phi(q)}{\phi(p^i)}(p^i-2p^{i-1})+2,\ B_{p^i}\right)$, $\left(\frac{\phi(q)}{\phi(p^i)},\ B_1\right)$ and $\left(\frac{\phi(q)}{\phi(p^i)}(p^j-p^{j-1}),\ B_{p^j}\right)$ for $1 \leq j \leq i-1$. That is, $wt(c)=\phi(q)+2-\frac{\phi(q)}{\phi(p^i)}$.

Case iv. Choose the scalars $\beta_1 \in B_{p^i}$ and $\beta_2 \in B_{p^j}$ for $i \neq j, 1 < i,j \leq m$. If $i > j$, then by Lemma 4, $S_2'(q,u)$ has the codewords of the form $\left(\frac{\phi(q)}{\phi(p^i)}(p^i-p^{i-1})+1,\ B_{p^i}\right)$ and $\left(1, B_{p^j}\right)$. If $i < j$, then by Lemma 5, $M_2(q,u)$ has the codewords of the form $\left(\frac{\phi(q)}{\phi(p^i)}(p^i-p^{i-1})+1,\ B_{p^j}\right)$ and $\left(1, B_{p^i}\right)$. That is, $wt(c)=\phi(q)+2$.

By the above discussion, the number of distinct weight codewords in $S_2'(q,u)$ is $m+2$.

In the same way, we are going to discuss the coordinates of the code $S_3'(q,u)$. If $k=3$, then the matrix $G_3'(q,u)$ generates the code

$$M_3(q,u) = \{\beta_3\left(1\ \boldsymbol{\alpha_1}\ \boldsymbol{\alpha_2}\cdots\boldsymbol{\alpha_{\phi(q)}}\right) + 0\ c\ c\cdots c \mid \beta_3 \in \mathbb{Z}_q, c \in S_2'(q,u),$$
$$\text{and } \boldsymbol{\alpha_i} = \alpha_i\alpha_i\cdots\alpha_i, \in \mathbb{Z}_q^{\phi(q)+2}\}.$$

Case i. If we fix the scalar β_3 in B_1 then this case is similar to the previous case $k=2$, that is, if $c_2 \in S'_2(q,u)$ has the weight w, then $c_3 \in S'_3(q,u)$ has the weight $\phi(q)w$.

Case ii. If we fix two scalars β_l in B_1 for $l=1,2$ and β_3 in B_{p^i} for $1<i\leq m$. Then $wt(c)=\frac{\phi(q)^3-1}{\phi(q)-1}+\phi(q)$.

Case iii. If we fix one of the scalar β_l in B_1 for $l=1,2$ and others in B_{p^i} for $1<i\leq m$. Then by Lemma 6, $S'_3(q,u)$ has the codewords of the form $\left(\frac{\phi(q)^2}{\phi(p^i)}(p^i-2p^{i-1})+2,\ B_{p^i}\right)$, $\left(\frac{\phi(q)^2}{\phi(p^i)},\ B_1\right)$ and $\left(\frac{\phi(q)^2}{\phi(p^i)}(p^j-p^{j-1}),\ B_{p^j}\right)$ for $1\leq j\leq i-1$. That is, $wt(c)=\frac{\phi(q)^3-1}{\phi(q)-1}+\phi(q)-\frac{\phi(q)^2}{\phi(p^i)}$.

Case iv. Choose all the scalars β_l in B_{p^i} for $1\leq l\leq 3,\ 1\leq i\leq m$. Then by Lemma 4 and Lemma 6, the codewords in $S'_3(q,u)$ has of the form

$$\left(\left(\frac{\phi(q)}{\phi(p^i)}\right)^2\left(\left(p^i-2p^{i-1}\right)^2+\sum_{l=1}^{i-1}\left(p^l-p^{l-1}\right)+1\right)+2\frac{\phi(q)}{\phi(p^i)}\left(p^i-2p^{i-1}\right)+3,\ B_{p^i}\right),$$

$$\left(2\frac{\phi(q)}{\phi(p^i)}+\left(\frac{\phi(q)}{\phi(p^i)}\right)^2(p^i-2p^{i-1}),\ B_1\right) \text{ and}$$

$$\left(2\frac{\phi(q)}{\phi(p^i)}(p^j-p^{j-1})+\left(\frac{\phi(q)}{\phi(p^i)}\right)^2(p^i-2p^{i-1})(p^j-p^{j-1}),\ B_{p^j}\right) \text{ for } 1\leq j\leq i-1.$$

That is, $wt(c)=\frac{\phi(q)^3-1}{\phi(q)-1}+\phi(q)-2\frac{\phi(q)}{\phi(p^i)}-\left(\frac{\phi(q)}{\phi(p^i)}\right)^2(p^i-2p^{i-1})$.

Case v. Choose the scalar β_3 in B_{p^j} and other scalars β_1,β_2 from B_{p^i} for $i>j,\ 1<i,j\leq m$. Then by Lemma 5 and Lemma 6, the codewords in $S'_3(q,u)$ has of the form

$$\left(\frac{\phi(q)}{\phi(p^j)}\left(\frac{\phi(q)}{\phi(p^i)}(p^i-2p^{i-1})+2\right),\quad B_{p^i}\right),\left(\frac{\phi(q)}{\phi(p^j)}\frac{\phi(q)}{\phi(p^i)}(p^j-p^{j-1}),\quad B_1\right),$$

$$\left(\frac{\phi(q)}{\phi(p^j)}\frac{\phi(q)}{\phi(p^i)}(p^e-p^{e-1}),\quad B_{p^e}\right) \text{ for } j+1\leq e\leq i-1 \text{ and}$$

$$\left(\frac{\phi(q)}{\phi(p^j)}\frac{\phi(q)}{\phi(p^i)}(p^j-2p^{j-1})+\frac{\phi(q)}{\phi(p^j)}\frac{\phi(q)}{\phi(p^i)}\left(\sum_{l=1}^{j-1}\left(p^l-p^{l-1}\right)\right)+1,\quad B_{p^j}\right)$$

$$\left(\frac{\phi(q)}{\phi(p^j)}\frac{\phi(q)}{\phi(p^i)}(p^e-p^{e-1}),\ B_{p^e}\right) \text{ for } 1\leq e\leq j-1.$$

That is, $wt(c)=\frac{\phi(q)^3-1}{\phi(q)-1}+\phi(q)-\frac{\phi(q)}{\phi(p^j)}\frac{\phi(q)}{\phi(p^i)}(p^j-p^{j-1})$.

Case vi. Choose the scalars β_1,β_2 in B_{p^j} and β_3 from B_{p^i} for $1\leq l\leq 3,\ 1<i<j\leq m$. Then the codewords in $S'_3(q,u)$ has of the form $\left(\frac{\phi(q)^3-1}{\phi(q)-1}+\phi(q),B_{p^i}\right)$. That is, $wt(c)=\frac{\phi(q)^3-1}{\phi(q)-1}+\phi(q)$.

Case vii. Choose the scalar β_1 in B_{p^i}, β_2 in B_{p^j} and β_l in B_{p^r} for $1\leq i,j,r\leq$

m and $i \neq j \neq r$. Then $wt(c) = \frac{\phi(q)^3-1}{\phi(q)-1} + \phi(q)$.
From the above, we conclude that, the number of distinct weight codewords in $S_3'(q,u)$ is

$$\begin{cases} 8 & \text{for } m=2 \text{ and } p=2 \\ 10 & \text{for } m=2 \text{ and } p \geq 3 \\ 11 & \text{for } m=3 \text{ and } p=2 \\ 4m+3 & \text{for } m \geq 3,\ q \geq 3 \text{ and } m \geq 4,\ p=2. \end{cases}$$

Thus, we have

Theorem 7. *Let $q = p^m$ where p is a prime integer and $m \geq 2$. Then the weight distribution of unit $\mathbb{Z}_q$-Simplex Codes of type α generated by the matrix $G_3'(q,u)$ is*

$$\begin{aligned}
A_0 &= 1, \\
A_{\phi(q)^2+\phi(q)} &= 2(q-1) + 3\phi(q)^2, \\
A_{\frac{\phi(q)^3-1}{\phi(q)-1}+\phi(q)} &= (q-1) + 2\phi(p^i)\phi(p^j) + \phi(p^i)\phi(p^j)^2, \ \textit{for } i \neq j \\
&\quad \textit{and } 1 \leq i,j \leq m, \\
A_{\phi(q)\left(\phi(q)+2-\frac{\phi(q)}{\phi(p^i)}\right)} &= \phi(p^i)^2 \ \textit{for } 1 \leq i \leq m-1, \\
A_{\frac{\phi(q)^3-1}{\phi(q)-1}+\phi(q)-\frac{\phi(q)^2}{\phi(p^i)}} &= 2\left(\phi(p^i)^2\right) 1 \leq i \leq m, \\
A_{\frac{\phi(q)^3-1}{\phi(q)-1}+\phi(q)-1} &= \phi(p^i)\phi(p^j) \ \textit{for } i \neq j \ \textit{and } 1 \leq i,j \leq m, \\
A_{\frac{\rho^3-1}{\rho-1}-2\frac{\phi(q)}{\phi(p^i)}-\left(\frac{\phi(q)}{\phi(p^i)}\right)^2(p^i-2p^{i-1})} &= (\phi(p^i)^3) \ \textit{for } 1 \leq i \leq m, \\
A_{\frac{\rho^3-1}{\rho-1}-\frac{\phi(q)}{\phi(p^j)}\frac{\phi(q)}{\phi(p^i)}(p^j-p^{j-1})} &= \phi(p^i)^2\phi(p^j) \ \textit{for } i > j \ \textit{and } 1 \leq i,j \leq m, \\
A_{\frac{\phi(q)^3-1}{\phi(q)-1}+\phi(q)} &= \phi(p^i)\phi(p^j)\phi(p^r) \ \textit{for } i \neq j \neq r \ \textit{and } 1 \leq i,j,r \leq m.
\end{aligned}$$

Here, we support Theorem 7 by the following example.

Example 2. Let

$$G_2'(4,u) = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$G_2'(9,u) = \begin{bmatrix} 1 & 0 & 1 & 2 & 4 & 5 & 7 & 8 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$G_2'(25,u) = \begin{bmatrix} 1 & 0 & 1 & 2 & 3 & 4 & 6 & 7 & 8 & 9 & 11 & 12 & 13 & 14 & 16 & 17 & 18 & 19 & 21 & 22 & 23 & 24 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

where $\{1,\ 3\}$, $\{1,\ 2,\ 4,\ 5,\ 7,\ 8\}$ and $\{1,\ 2,\ 3,\ 4,\ 6,\ 7,\ 8,\ 9,\ 11,\ 12,\ 13,\ 14,\ 16,$ $17,\ 18,\ 19,\ 21,\ 22,\ 23,\ 24\}$ are units of $\mathbb{Z}_4$, $\mathbb{Z}_9$ and $\mathbb{Z}_{25}$, respectively. By Theorem 7, using matlab, the weight distribution of the code generated by the matrices $G_3'(4,u)$, $G_3'(9,u)$ and $G_3'(25,u)$ are listed in Table 2.

Table 2. **Weight distribution of unit $\mathbb{Z}_q$-Simplex code of type α of dimension 3.**

weight distribution of $G'_3(q,u)$	
$G'_3(4,u)$	$A_0=1,\ A_3=2,\ A_4=1,\ A_5=1,\ A_6=26,$ $A_7=16,\ A_8=4$ and $A_9=13$
$G'_3(9,u)$	$A_0=1,\ A_{28}=8,\ A_{30}=4,\ A_{34}=8,\ A_{42}=268,$ $A_{43}=72,\ A_{44}=216,\ A_{46}=48,$ $A_{48}=24$ and $A_{49}=80$
$G'_3(25,u)$	$A_0=1,\ A_{336}=32,\ A_{340}=16,\ A_{356}=64,\ A_{420}=4448,$ $A_{421}=1600,\ A_{424}=8000,\ A_{436}=640$ $A_{440}=160$ and $A_{441}=664$

6. Zero divisor $\mathbb{Z}_q$-Simplex code

In this section, we have introduced some new type of $\mathbb{Z}_q$-Simplex code called a zero divisor $\mathbb{Z}_q$-Simplex code and find its parameters.
Let

$$G_2(q,z)=\left[\begin{array}{c|c|cccc} 1 & 0 & z_1 & z_2 & \cdots & z_s \\ \hline 0 & 1 & 1 & 1 & \cdots & 1 \end{array}\right] \tag{2}$$

where $z_1, z_2, \ \cdots, z_s$ are zerodivisors in $\mathbb{Z}_q$ and $s=q-\phi(q)-1$ where $\phi(q)$ is a Euler ϕ-function. Then by Theorem 3, the matrix $G_2(q,z)$ generates $\left[\rho+1,\ 2,\ \rho+1-\left(\frac{q}{p}\right)\right]$ $\mathbb{Z}_q$-linear code where $\rho=q-\phi(q)$.
Now, we define inductively

$$G_k(q,z)=\left[\begin{array}{c|c|c|c|c|c} 1 & 00\cdots0 & z_1z_1\cdots z_1 & z_2z_2\cdots z_2 & \cdots & z_sz_s\cdots z_s \\ \hline 0 & & & & & \\ \vdots & G_{k-1}(q,z) & G_{k-1}(q,z) & G_{k-1}(q,z) & \cdots & G_{k-1}(q,z) \\ 0 & & & & & \end{array}\right] \tag{3}$$

for $k>2$.
Clearly, this matrix $G_k(q,z)$ generates $\left[\frac{\rho^k-1}{\rho-1},\ k,\ d\right]$ $\mathbb{Z}_q$-linear code where $\rho=q-\phi(q)$. Then the code generated by the matrix $G_k(q,z)$ is called *zero divisor $\mathbb{Z}_q$-Simplex code.* It is denoted by $S_k(q,z)$. This matrix $G_k(q,z)$ generates the code

$$S_k(q,z)=\{\beta\,(1\,\mathbf{0}\,\boldsymbol{z_1}\,\boldsymbol{z_2}\ldots\boldsymbol{z_s})+(0\,c\,c\ldots c)\mid\beta\in\mathbb{Z}_q, c\in S_{k-1}(q,z),$$
$$\boldsymbol{z_i}=z_iz_i\ldots z_i,\mathbf{0}=00\ldots0\in\mathbb{Z}_q^{n_{k-1}}\}.$$

Lemma 7. *Let $\beta\in\mathbb{Z}_q\backslash\{0\}$. If $o(\beta)=l$ and $z_1,\ z_2,\ \ldots,\ z_s$ are zerodivisors in $\mathbb{Z}_q$ then $\beta(0,z_1,\ z_2,\ \ldots,z_s)$ has elements of order l with repetitions $\frac{q}{l}-\frac{\phi(q)}{\phi(l)}$ and elements of order d with repetitions $\frac{q}{d}$ for $1\le d<l$ and d divides l.*

Lemma 8. *Let C be an $[n,k,d]$ $\mathbb{Z}_q$-linear code generated by the matrix $G_k(q,z)$ and $c_1 \in C$, $\beta \in \mathbb{Z}_q$ such that $o(\beta) = l$. Define $\overline{c_1} = \beta(1\,\mathbf{0}\,\boldsymbol{z_1}\,\boldsymbol{z_2}\cdots\boldsymbol{z_s}) + (0\,c_1\,c_1\,c_1\cdots c_1)$ then $wt(\overline{c_1}) = \rho n + 1 - \left(N(c_1) + \left(\frac{q}{l} - \frac{\phi(q)}{\phi(l)}\right) N_l(c_1) + \sum_{d|l}\left(\frac{q}{d}\right) N_d(c_1)\right)$ where $N_l(c_1)$ and $N_d(c_1)$ denotes the number elements of order l and number elements of order d in the codeword c_1, respectively.*

Proof. Clearly, the length of the codeword $\overline{c_1}$ is $\rho n + 1$. It is given that $o(\beta) = l$. By Lemma 7, $\beta(z_1, z_2, \ldots, z_s)$ has l order elements with repetitions $\frac{q}{l} - \frac{\phi(q)}{\phi(l)}$ and d order elements with repetitions $\frac{q}{d}$ for $1 \le d < l$ and d divides l.

$$\begin{aligned}
wt(\overline{c_1}) &= \rho n + 1 - \left\{N(\mathbf{0} + c_1) + N(\boldsymbol{z_1} + c_1) + N(\boldsymbol{z_2} + c_1) + \ldots + N(\boldsymbol{z_s} + c_1)\right\} \\
&= \rho n + 1 - \Bigg(N(c_1) + \left(\frac{q}{l} - \frac{\phi(q)}{\phi(l)}\right) N_l(c_1) + \\
&\qquad \sum_{d|l} \frac{q}{d}\Big(N(\boldsymbol{y_1} + c_1) + N(\boldsymbol{y_2} + c_1) + \ldots + N(\boldsymbol{y_{(\frac{q}{d})}} + c_1)\Big)\Bigg) \\
wt(\overline{c_1}) &= \rho n + 1 - \left(N(c_1) + \left(\frac{q}{l} - \frac{\phi(q)}{\phi(l)}\right) N_l(c_1) + \sum_{d|l}\left(\frac{q}{d}\right) N_d(c_1)\right)
\end{aligned}$$

That is, the weight of the codeword $\overline{c_1}$ is

$$\rho n + 1 - \Big(N(c_1) + \left(\frac{q}{l} - \frac{\phi(q)}{\phi(l)}\right) N_l(c_1) + \sum_{d|l}\left(\frac{q}{d}\right) N_d(c_1)\Big).$$

□

Let $B_p = \{x \in \mathbb{Z}_q \mid o(x) = p\}$ where p is the smallest prime divisor of q. Let $S_{k-1}(q,z)$ be an $[n, k-1, d]$ code and $S_k(q,z) = \{\beta(1\,\mathbf{0}\,\boldsymbol{z_1}\,\boldsymbol{z_2}\cdots\boldsymbol{z_{\phi(q)}}) + (0\,c\,c\,c\cdots c) \mid \beta \in \mathbb{Z}_q, c \in S_{k-1}(q,z),\ \boldsymbol{z_i} = (z_i z_i \cdots z_i), \mathbf{0} = (00\cdots 0) \in \mathbb{Z}_q^n\}$. Then the minimum distance of $S_k(q,z)$ is

$$\begin{aligned}
d(S_k(q,z)) = \min\{&wt(\beta(1\,\mathbf{0}\,\boldsymbol{z_1}\,\boldsymbol{z_2}\ldots\boldsymbol{z_s}) + (0\,c\,c\,c\ldots c)\,) \mid \\
&c \in S_{k-1}(q,z), \beta \in \mathbb{Z}_q, \boldsymbol{z_i} \in \mathbb{Z}_q^n\}.
\end{aligned}$$

If $\beta = 0$, then $\min\{wt(0\,c\,c\,c\cdots c) \mid c \in C,\ c \neq 0\} = \rho d$. If $c = 0$, then $\min\{wt(\beta(1\,\mathbf{0}\,\boldsymbol{z_1}\,\boldsymbol{z_2}\cdots\boldsymbol{z_s})) \mid \beta \in \mathbb{Z}_q,\ \beta \neq 0\} = \{wt(\beta(1\,\mathbf{0}\,\boldsymbol{z_1}\,\boldsymbol{z_2}\cdots\boldsymbol{z_s})) \mid \beta \in B_p\} = n\rho + 1 - n\left(\frac{q}{p}\right)$.

Assume $c' \in S_{k-1}(q,z)$ such that $wt(c') = d$. If $c \neq 0$, $\beta \neq 0$ and $o(\beta) = l$, then by Lemma 8,

$$\begin{aligned} wt\left(\beta\left(1\,\mathbf{0}\,\boldsymbol{z_1}\,\boldsymbol{z_2}\ldots\boldsymbol{z_s}\right)+\left(0\,c\,c\,c\ldots c\right)\right) &= \rho n+1-\left(N(c)+\left(\frac{q}{l}-\frac{\phi(q)}{\phi(l)}\right)N_l(c)+\sum_{d|l}\left(\frac{q}{d}\right)N_d(c)\right)\\ &\geq \rho n+1-\left(N(c')+\frac{q}{p}N_p(c')\right)\\ &\geq \rho n+1-\left(n-wt(c')+\frac{q}{p}wt(c')\right). \end{aligned}$$

Therefore, $wt\left(\beta\left(1\,\mathbf{0}\,\boldsymbol{z_1}\,\boldsymbol{z_2}\ldots\boldsymbol{z_s}\right)+\left(0\,c\,c\,c\ldots c\right)\right) \geq \rho n+1-\left(n-wt(c')+\frac{q}{p}wt(c')\right)$ where p is the smallest prime divisor of q. Thus, we have

Theorem 8. *Let $S_k(q,z)$ be a $\mathbb{Z}_q$-linear code generated by the matrix $G_k(q,z)$. Then the minimum distance of $S_k(q,z)$ is $min\{\rho d,\ n\rho+1-n\left(\frac{q}{p}\right),\ \rho n+1-\left(n-wt(c')+\frac{q}{p}wt(c')\right)\}$ where $wt(c') = min\{wt(c)\mid c \in S_{k-1}(q,z)\}$, $\rho = q-\phi(q)$ and p is the smallest prime divisor of q.*

Theorem 9. *Let $S_k(q,z)$ be a $\mathbb{Z}_q$-linear code generated by the matrix $G_k(q,z)$. Then the parameters of the code is $\left(\frac{\rho^k-1}{\rho-1},\ k,\ \frac{\rho^k-1}{\rho-1}-\left(\frac{\rho^{(k-1)}-1}{\rho-1}\right)\left(\frac{q}{p}\right)\right)$ where $\rho = q-\phi(q)$ and p is the smallest prime divisor of q.*

Proof. We prove this by induction on k. For $k = 2$,

$$S_2(q,z) = \{\beta(10z_1z_2\ldots z_s)+\gamma(0111\ldots 1)\mid \beta,\gamma\in\mathbb{Z}_q\}$$

is a code of length $\rho+1$. Let $x = (10z_1z_2\ldots z_{\phi(q)})$ and $y = (0111\ldots 1)$. Then the minimum distance of $S_2(q,z)$ is

$$d(S_2(q,u)) = min\{wt(\beta x), wt(\gamma y), wt(\beta x+\gamma y)\mid \beta,\gamma\in\mathbb{Z}_q\backslash\{0\}\}.$$

Clearly, $min\{wt(\beta x)\mid \beta\neq 0\} = \rho$ and $min\{wt(\gamma y)\mid\gamma\neq 0\} = \rho+1-\frac{q}{p}$.

Case i. If $\beta\neq 0$ and $\gamma\neq 0$ then assume that $B_p = \{x\in\mathbb{Z}_q\mid o(x)=p\}$ where p is the smallest prime divisor of q and $p^2|q$. Consider

$$\begin{aligned} min\{wt(\beta x+\gamma y)\mid\beta,\gamma\in\mathbb{Z}_q\backslash\{0\}\} &= min\left\{wt(\beta x+\gamma y)\mid\beta\in B_{p_1^2}\text{ and }\gamma\in B_{p_1}\right\}\\ &= \rho+1-\left(\frac{q}{p_1^2}\right)\\ min\{wt(\beta x+\gamma y)\mid\beta,\gamma\in\mathbb{Z}_q\backslash\{0\}\} &= \rho+1-\left(\frac{q}{p_1^2}\right). \end{aligned}$$

Case ii. If $p^2 \mid q$. Then

$$\begin{aligned} min\{wt(\beta x+\gamma y)\ \mid \beta,\, \gamma\in\mathbb{Z}_q\backslash\{0\}\} &= min\left\{wt(\beta x+\gamma y) \mid \beta\in B_p \text{ and } \gamma\in B_p\right\} \\ &= \rho+1-\left(\frac{q}{p}-\frac{\phi(q)}{\phi(p)}\right) \\ min\{wt(\beta x+\gamma y)\ \mid \beta,\, \gamma\in\mathbb{Z}_q\backslash\{0\}\} &= \rho+1-\frac{q}{p}+\frac{\phi(q)}{\phi(p)}. \end{aligned}$$

Therefore,

$$\begin{aligned} d(S_2(q,z)) &= min\left\{\rho,\ \rho+1-\left(\frac{q}{p}\right),\rho+1-\left(\frac{q}{p_1^2}\right),\ \rho+1-\frac{q}{p}+\frac{\phi(q)}{\phi(p)}\right\} \\ &= \rho+1-\left(\frac{q}{p}\right), \end{aligned}$$

where p is the smallest prime divisor of q.
Assume that the theorem is true for all i less than k. That is, the parameter of the code $S_i(q,z)$ is $\left[\frac{\rho^i-1}{\rho-1},\, i,\, \frac{\rho^i-1}{\rho-1}-\left(\frac{\rho^{(i-1)}-1}{\rho-1}\right)\left(\frac{q}{p}\right)\right]$ $\mathbb{Z}_q$-linear code.
Now, we are going to prove the theorem for k. By Theorem 8,

$$d(S_k(q,z)) = min\left\{\rho d,\ n\rho+1-n\left(\frac{q}{p}\right),\ \rho n+1-\left(n-wt(c')+\frac{q}{p}wt(c')\right)\right\}$$

where $n=\frac{\rho^{k-1}-1}{\rho-1}$ and $wt(c')=d=\frac{\rho^{(k-1)}-1}{\rho-1}-\left(\frac{\rho^{(k-2)}-1}{\rho-1}\right)\left(\frac{q}{p}\right)$ are length and minimum distance of the code $S_{k-1}(q,z)$, respectively. Therefore, by induction hypothesis,
$d(S_k(q,u)) = min\left\{\frac{\rho^k-1}{\rho-1}-1-\left(\frac{\rho^{(k-1)}-1}{\rho-1}-1\right)\left(\frac{q}{p}\right),\ \ \frac{\rho^k-1}{\rho-1}-\left(\frac{\rho^{(k-1)}-1}{\rho-1}\right)\left(\frac{q}{p}\right),\right.$
$\left.\frac{\rho^k-1}{\rho-1}-\left(\frac{\rho^{(k-2)}-1}{\rho-1}\right)\left(\frac{q}{p}\right)\left(1-\frac{q}{p}\right)-\left(\frac{\rho^{(k-1)}-1}{\rho-1}\right)\right\}$ and hence,
$d(S_k(q,u)) = \frac{\rho^k-1}{\rho-1}-\left(\frac{\rho^{(k-1)}-1}{\rho-1}\right)\left(\frac{q}{p}\right).$ □

The following example illustrates Theorem 9.

Example 3. Let $q=12$. Then $\{2,\ 3,\ 4,\ 6,\ 8,\ 9,\ 10\}$ are zero divisors in $\mathbb{Z}_{12}$ and the generator matrix of $S_2(q,z)$ is

$$G_2(q,z)=\begin{bmatrix} 1 & 0 & 2 & 3 & 4 & 6 & 8 & 9 & 10 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The Matrix

Table 3. **Parameters of zero divisor $\mathbb{Z}_q$-Simplex codes.**

Generator matrix $G_k(12,z)$ and its parameters	
$G_2(12,z)$	[9, 2, 3]
$G_3(12,z)$	[73, 3, 19]
$G_4(12,z)$	[585, 4, 147]
$G_5(12,z)$	[4681, 5, 1171]
$G_6(12,z)$	[37449, 6, 9363]
$G_7(12,z)$	[299593, 7, 74899]
$G_8(12,z)$	[2396745, 8, 599187]
$G_9(12,z)$	[19173961, 9, 4793491]

$$G_k(q,z) = \left[\begin{array}{c|c|c|c|c|c} 1 & 00\cdots0 & z_1z_1\cdots z_1 & z_2z_2\cdots z_2 & \cdots & z_sz_s\cdots z_s \\ \hline 0 & & & & & \\ \vdots & G_{k-1}(q,z) & G_{k-1}(q,z) & G_{k-1}(q,z) & \cdots & G_{k-1}(q,z) \\ 0 & & & & & \end{array}\right]$$

for $k > 2$. By Theorem 9, the parameters of the code generated by the matrix $G_2(12,z)$, $G_3(12,z)$, $G_4(12,z)$, $G_5(12,z)$, $G_6(12,z)$, $G_7(12,z)$, $G_8(12,z)$ and $G_9(12,z)$ are listed in Table 3.

7. Weight distribution of zero divisor $\mathbb{Z}_q$-Simplex code of dimension 3 for $q = p^m$ where p is a prime integer and $m \geq 2$

Now, we are going to discuss the coordinates of the code $S_2(q,z)$. If $k = 2$, then the matrix $G_2(q,z)$ generates the code

$$S_2(q,z) = \{\beta_1(1\,0\,z_1\,z_2\ldots z_s) + \beta_2(0\,1\,1\,1\ldots1) \mid \beta_1, \beta_2 \in \mathbb{Z}_q\}.$$

Case i. Clearly, if $\beta_1,\ \beta_2 \in B_1$, then $S_2(q,z)$ has the codeword of the form $(\rho+1,\ B_1)$, that is, $wt(c) = 0$.

Case ii. It is easy to see that, if $\beta_1 \in B_1$ and $\beta_2 \in B_{p^i}$ for $1 \leq i \leq m$, then $S_2(q,z)$ has the codewords of the form $\left(\rho,\ B_{p^i}\right)$. That is, $wt(c) = \rho$.

Case iii. It is easy to see that, if $\beta_2 \in B_1$ and $\beta_1 \in B_{p^i}$ for $1 \leq i \leq m$, then $S_2(q,z)$ has the codewords of the form $\left(\frac{q}{p^i}(p^j - p^{j-1}),\ B_{p^i}\right)$ for $1 \leq j \leq i-1$ and $\left(\frac{q}{p^i},\ B_1\right)$ and $\left(1,\ B_{p^i}\right)$ That is, $wt(c) = \rho + 1 - \frac{q}{p^i}$.

Case iv. Choose the scalars $\beta_1 \in B_{p^i}$ and $\beta_2 \in B_{p^j}$ for $1 \leq j < i \leq m$. Then $S_2(q,z)$ has the codewords of the form $\left(\frac{q}{p^i}\left(p^e - p^{e-1}\right), \; B_{p^e}\right)$ for $1 \leq e \leq i-1$, $(1, \; B_i)$ and $\left(\frac{q}{p^i}, \; B_1\right)$. That is, $wt(c) = \rho + 1 - \frac{q}{p^i}$.

Case v. Choose the scalars $\beta_1 \in B_{p^i}$ and $\beta_2 \in B_{p^j}$ for $1 \leq i \leq j \leq m$. Then $c_2 \in S_2(q,z)$ has the codewords of the form $\left(\rho, \; B_{p^j}\right)$ and $\left(1, B_{p^i}\right)$. That is, $wt(c) = \rho + 1$.

By the above discussion, the number of distinct weight codewords in $S_2(q,z)$ is $m+2$.

In the same way, we are going to discuss the coordinates of the code $S_3(q,z)$. If $k=3$, then the matrix $G_3(q,z)$ generates the code

$$S_3(q,z) = \{\beta_3\,(1\;\mathbf{0}\;\boldsymbol{z_1}\;\boldsymbol{z_2}\cdots\boldsymbol{z_s}) + (0\;c\;c\cdots c) \mid \beta_3 \in \mathbb{Z}_q, c \in S_2(q,z),$$
$$\boldsymbol{z_i} = (z_i z_i \cdots z_i), \mathbf{0} = (00\cdots 0) \in \mathbb{Z}_q^{\rho+1}\}.$$

Case i. If we fix the scalar β_2 or $\beta_3 \in B_1$ then this case is similar to the previous case $k=2$, that is, if $c_2 \in S_2(q,z)$ has the weight w, then $c_3 \in S_3(q,z)$ has the weight ρw.

Case ii. It is easy to see that, if $\beta_1, \; \beta_2 \in B_1$ and $\beta_3 \in B_{p^i}$ for $1 \leq i \leq m$, then $S_2(q,z)$ has the codewords of the form $\left(\frac{q}{p^i}(\rho+1)(p^j - p^{j-1}), \; B_{p^j}\right)$ for $1 \leq j \leq i-1$ and $\left(\frac{q}{p^i}(\rho+1), \; B_1\right)$. That is, $wt(c) = \frac{\rho^3-1}{\rho-1} - \frac{q}{p^i}(\rho+1)$.

Case iii. If β_2 in B_1 and others in B_{p^i} for $1 \leq i \leq m$, then $c_3 \in S_3(q,z)$ has the codewords of the form $(\rho, \; B_{p^i})$, $\left(\left(\frac{\rho q}{p^i}\right)(p^j - p^{j-1}), \; B_{p^j}\right)$ for $1 \leq j \leq i-1$ and $\left(\frac{\rho q}{p^i}, \; B_1\right)$. That is, $wt(c) = \frac{\rho^3-1}{\rho-1} - \frac{q\rho}{p^i}$.

Case iv. If β_2 in B_1, β_3 in B_{p^j} and β_1 in B_{p^i} for $1 \leq j < i \leq m$, then $c_3 \in S_3(q,z)$ has the codewords of the form $(\rho, \; B_{p^i})$, $\left(\frac{q\rho}{p^i}(p^e - p^{e-1}), \; B_{p^e}\right)$ for $1 \leq e \leq i-1$ and $\left(\frac{q\rho}{p^i}, \; B_1\right)$. That is, $wt(c) = \frac{\rho^3-1}{\rho-1} - \frac{q\rho}{p^i}$.

Case v. If β_2 in B_1, β_3 in B_{p^i} and β_1 in B_{p^j} for $1 \leq j < i \leq m$, then $c_3 \in S_3(q,z)$ has the codewords of the form $(1, \; B_{p^i})$, $\left((\rho+1)(p^e - p^{e-1}), \; B_{p^e}\right)$ for $j < e < i$, $\left(\frac{q^2 p^j}{p^i p^e}(p^e - p^{e-1}), \; B_{p^e}\right)$ for $1 \leq e \leq j$ and $\left(\frac{q}{p^i}(\rho+1), \; B_1\right)$. That is, $wt(c) = \frac{\rho^3-1}{\rho-1} - \frac{q}{p^i}(\rho+1)$.

Case vi. Choose the scalars $\beta_1 \in B_{p^i}$, $\beta_2 \in B_{p^j}$ and $\beta_3 \in B_{p^k}$ for $1 \leq j < k \leq i \leq m$. Then $S_3(q,z)$ has the codewords of the form $\left(\rho, \; B_{p^i}\right)$,

$(1,\ B_{p^k})$, $\left(\frac{q\rho}{p^i}\left(p^e - p^{e-1}\right),\ B_{p^e}\right)$ for $1 \le e \le i-1$ and $\left(\frac{\rho q}{p^i},\ B_1\right)$. That is, $wt(c) = \frac{\rho^3-1}{\rho-1} - \frac{\rho q}{p^i}$.

Case vii. Choose the scalars $\beta_1 \in B_{p^i}$, $\beta_2 \in B_{p^j}$ and $\beta_3 \in B_{p^k}$ for $1 \le j < i < k \le m$. Then $c_3 \in S_3(q,z)$ has the codewords of the form $(1,\ B_{p^i})$, $\left((\rho+1)(p^e - p^{e-1}),\ B_{p^e}\right)$ for $i < e < k$, $\left(\frac{q^2 p^i}{p^k p^e}(p^e - p^{e-1}),\ B_{p^e}\right)$ for $1 \le e \le i$ and $\left(\frac{q}{p^i}(\rho+1),\ B_1\right)$. That is, $wt(c) = \frac{\rho^3-1}{\rho-1} - \frac{q}{p^i}(\rho+1)$.

Case viii. Choose the scalars $\beta_1 \in B_{p^i}$, $\beta_2 \in B_{p^j}$ and $\beta_3 \in B_{p^k}$ for $1 \le j, i < k \le m$ and $i = j$. Then $c_3 \in S_3(q,z)$ has the codewords of the form $(1,\ B_{p^k})$, $\left(\left(\rho - \frac{q}{p^k}(p^{i-1} - p^{i-2})\right)(\rho+1),\ B_{p^j}\right)$, $\left(\frac{q}{p^k}\frac{q}{p^{i-1}}(p^{i-1} - p^{i-2})(p^e - p^{e-1}),\ B_{p^e}\right)$ for $1 \le e \le j-1$ and $\left(\frac{q}{p^i}(\rho+1),\ B_1\right)$. That is, $wt(c) = \frac{\rho^3-1}{\rho-1} - \frac{q}{p^i}(\rho+1)$.

Case ix. Choose the scalars $\beta_1 \in B_{p^i}$, $\beta_2 \in B_{p^j}$ and $\beta_3 \in B_{p^k}$ for $1 \le k, i \le j \le m$. Then $c_3 \in S_3(q,z)$ has the codewords of the form $\left(\rho^2 + \rho - 1,\ B_{p^j}\right)$ $\left(1, B_{p^i}\right)$ and $\left(1, B_{p^k}\right)$. That is, $wt(c) = \frac{\rho^3-1}{\rho-1}$.

From the above, we conclude that, the number of distinct weight codewords in $S_3(q,z)$ is $4m$ for $m \ge 2, p \ge 2$. Thus, we have

Theorem 10. *Let $q = p^m$ where p is a prime integer and $m \ge 2$. Then the weight distribution of $\mathbb{Z}_q$-linear code generated by the matrix $G_3(q,z)$ is*

$$
\begin{aligned}
A_0 &= 1,\\
A_{\rho^2} &= q-1,\\
A_{\rho(\rho+1-\frac{q}{p^i})} &= 2\phi(p^i) + 2\phi(p^i)\phi(p^j),\ \textit{for}\ 1 \le i \le m\ \textit{and}\ 1 \le j < i \le m,\\
A_{\rho(\rho+1)} &= 2\phi(p^i)\phi(p^j)\ \textit{for}\ 1 \le i \le j \le m,\\
A_{\frac{\rho^3-1}{\rho-1} - \frac{q}{p^i}(\rho+1)} &= \phi(p^i) + \phi(p^i)\phi(p^j) + \phi(p^i)\phi(p^j)\phi(p^k)\ \textit{for}\ 1 \le i \le m,\\
&\quad 1 \le j < i \le m\ \textit{and}\ 1 \le j, i < k \le m,\\
A_{\frac{\rho^3-1}{\rho-1} - \frac{q\rho}{p^i}} &= \phi(p^i)^2 + \phi(p^i)\phi(p^j) + \phi(p^i)\phi(p^j)\phi(p^k)\ \textit{for}\ 1 \le i \le m,\\
&\quad 1 \le j < i \le m, 1 \le j < k \le i \le m,\\
A_{\frac{\rho^3-1}{\rho-1}} &= \phi(p^i)\phi(p^j)\phi(p^k)\ \textit{for}\ 1 \le k, i \le j \le m.
\end{aligned}
$$

Here, we support Theorem 10 by the following example.

Example 4. Let

$$G_2(4,z) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

Table 4. **Weight distribution of zero divisor $\mathbb{Z}_q$-Simplex code of dimension 3.**

weight distribution of $G_3(q,z)$	
$G_3(4,z)$	$A_0=1,\ A_1=1,\ A_2=1,\ A_3=1,\ A_4=15,$ $A_5=15,\ A_6=15$ and $A_7=15$
$G_3(9,z)$	$A_0=1,\ A_1=2,\ A_3=2,\ A_4=4,\ A_9=80,$ $A_{10}=160,\ A_{12}=160$ and $A_{13}=320$
$G_3(25,z)$	$A_0=1,\ A_1=4,\ A_5=4,\ A_6=16,\ A_{25}=624,$ $A_{26}=2496,\ A_{30}=2496$ and $A_{31}=9948$

$$G_2(9,z)=\begin{bmatrix}1&0&3&6\\0&1&1&1\end{bmatrix}$$

$$G_2(25,z)=\begin{bmatrix}1&0&5&10&15&20\\0&1&1&1&1&1\end{bmatrix}$$

where $\{2\}$, $\{3,\ 6\}$ and $\{5,\ 10,\ 15,\ 20\}$ are zerodivisors of $\mathbb{Z}_4$, $\mathbb{Z}_9$ and $\mathbb{Z}_{25}$, respectively. By Theorem 10, using matlab, the weight distribution of the code generated by the matrices $G_3(4,z)$, $G_3(9,z)$ and $G_3(25,z)$ are listed in Table 4.

8. Comparison of $\mathbb{Z}_q$-Simplex code, unit $\mathbb{Z}_q$-Simplex code and unit $\mathbb{Z}_q$-Simplex code type α

The crucial point in coding theory is to establish the codes that achieve a fixed error correction capability with a minimum amount of redundancy. A code with higher code rate implies a more efficient use of redundancy than a lower code rate. Therefore, these codes are more preferable. In the other point of view, we also consider the error-correcting capabilities of the code. There is a basic trade-off between code rate and minimum distance, when increasing the code rate, it decrease the minimum distance, and vice-versa.

On the basis of the above view, the distinction between $\mathbb{Z}_q$-Simplex code, unit $\mathbb{Z}_q$-Simplex code and unit $\mathbb{Z}_q$-Simplex code type α with respect to the code rate for the particular value of q is shown in the Table 5. The notations R, R' and R'' represents the codes rate of $\mathbb{Z}_q$-Simplex code, unit $\mathbb{Z}_q$-Simplex code, unit $\mathbb{Z}_q$-Simplex code of type α, respectively.

9. Conclusion and Future work

In the recent decades, many punctured versions of $\mathbb{Z}_q$-Simplex codes have been deployed. In [2] and [18], the authors examine the parameters for some of the punctured

Table 5. **Code rates of punctured version of $\mathbb{Z}_9$-Simplex codes**

Code rate	R	R'	R''
$k=2$	$\frac{2}{10}=0.2$	$\frac{2}{8}=0.25$	$\frac{2}{8}=0.25$
$k=3$	$\frac{3}{91}=0.033$	$\frac{3}{57}=0.053$	$\frac{3}{49}=0.061$
$k=4$	$\frac{4}{820}=0.0049$	$\frac{4}{400}=0.01$	$\frac{4}{295}=0.014$
$k=5$	$\frac{5}{7381}=0.00068$	$\frac{5}{2801}=0.0019$	$\frac{5}{1771}=0.003$

version of $\mathbb{Z}_q$-Simplex codes. In this article, we have analysed the parameters for unit $\mathbb{Z}_q$-Macdonald codes, unit $\mathbb{Z}_q$-Simplex codes of type α and zero divisor $\mathbb{Z}_q$-Simplex codes. Discussing the weight distribution and covering radius for higher rank codes is not an easy task. In [18], we have obtained the complete weight distribution of the higher rank unit $\mathbb{Z}_q$-Simplex codes for the particular case $q=p^2$ and also we have attained the partial weight distribution of the higher rank unit $\mathbb{Z}_q$-Simplex codes for the particular case $q=p^m$, where p is a prime number and m is a positive integer. Here, we reach the weight distribution of the unit $\mathbb{Z}_q$-Simplex codes of type α and zero divisor $\mathbb{Z}_q$-Simplex codes for the rank 3. The weight distribution of the higher rank ($k \geq 4$) unit $\mathbb{Z}_q$-Simplex codes of type α and zero divisor $\mathbb{Z}_q$-Simplex codes are still open problems. Also, the bounds on the covering radius of the punctured version of $\mathbb{Z}_q$-Simplex codes remains unknown even for the smaller rank codes.

Acknowledgements

The authors would like to thank the National Board for Higher Mathematics (NBHM), DAE, Govt. of India for providing a financial support under the grant number 02011/7/2019-R&D-II/1975.

References

[1] R.C. Bose and R.C. Burton, *A characterization of flat spaces in a finite geometry and the uniqueness of the Hamming and the Macdonald codes*, J. Combin. Theory Ser. A **1** (1966), no. 1, 96–104.

[2] Durairajan C. and J. Mahalakshmi, *On codes over integers modulo q*, Adv. Appl. Discrete Math. **15** (2015), no. 2, 125–143.

[3] K. Chatouh, K. Guenda, T.A. Gulliver, and L. Noui, *Simplex and Macdonald codes over* $\mathbb{R}_q$, J. Appl. Math. Comput. **55** (2017), no. 1, 455–478.

[4] P. Chella Pandian and C. Durairajan, *On $\mathbb{Z}_q$-linear and $\mathbb{Z}_q$-Simplex codes and its related parameters for q is a prime power*, J. Discrete Math. Sci. Cryptogr.

18 (2015), no. 1-2, 81–94.

[5] C.J. Colbourn and M.K. Gupta, *On quaternary macdonald codes*, Proceedings ITCC 2003. International Conference on Information Technology: Coding and Computing, IEEE, 2003, pp. 212–215.

[6] A. Dertli and Y. Cengellenmis, *Macdonald codes over the ring* $\mathbb{F}_2 + v\mathbb{F}_2$, Int. J. Algebra **5** (2011), no. 20, 985–991.

[7] C. Ding and C. Li, *Infinite families of 2-designs and 3-designs from linear codes*, Discrete Math. **340** (2017), no. 10, 2415–2431.

[8] C. Ding and J. Yang, *Hamming weights in irreducible cyclic codes*, Discrete Math. **313** (2013), no. 4, 434–446.

[9] C. Durairajan, J. Mahalakshmi, and P.C. Pandian, *On the* $\mathbb{Z}_q$*-Simplex codes and its weight distribution for dimension 2*, Discrete Math. Algorithms Appl. **7** (2015), no. 03, 1550030.

[10] R.A. Fisher, *The theory of confounding in factorial experiments in relation to the theory of groups*, Annals of Eugenics **11** (1941), no. 1, 341–353.

[11] ______, *A system of confounding for factors with more than two alternatives, giving completely orthogonal cubes and higher powers*, Annals of Eugenics **12** (1943), no. 1, 283–290.

[12] M.K. Gupta, *On some linear codes over* $\mathbb{Z}_{2s}$, Ph.D. thesis, Indian Institute of Technology, Kanpur, 1999.

[13] M.K. Gupta, D.G. Glynn, and T.A. Gulliver, *On senary simplex codes*, International Symposium on Applied Algebra, Algebraic Algorithms, and Error-Correcting Codes, Springer, 2001, pp. 112–121.

[14] Y. Liu, C. Ding, and C. Tang, *Shortened linear codes over finite fields*, IEEE Transactions on Information Theory (2021).

[15] J.E. MacDonald, *Design methods for maximum minimum-distance error-correcting codes*, IBM Journal of Research and Development **4** (1960), no. 1, 43–57.

[16] J. Mahalakshmi and C. Durairajan, *On the* $\mathbb{Z}_q$*-Macdonald code and its weight distribution of dimension 3*, J. Math. Comput. Sci. **6** (2016), no. 2, 173–187.

[17] A. Patel, *Maximal q-nary linear codes with large minimum distance (Corresp.)*, IEEE Transactions on Information Theory **21** (1975), no. 1, 106–110.

[18] J. Prabu, J. Mahalakshmi, C. Durairajan, and S. Santhakumar, *On some punctured codes of* $\mathbb{Z}_q$*-Simplex codes*, Discrete Math. Algorithms Appl. (2021), 2250012.

[19] Y. Wang and J. Gao, *Macdonald codes over the ring* $\mathbb{F}_p + v\mathbb{F}_p + v^2\mathbb{F}_p$, Comput. Appl. Math. **38** (2019), no. 4, 1–15.

[20] A. Yardi and R. Pellikaan, *On shortened and punctured cyclic codes*, arXiv preprint arXiv:1705.09859 (2017).