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Research Article

Signless Laplacian eigenvalues of the zero divisor graph associated to finite commutative ring $\mathbb{Z}_{p^{M_1}q^{M_2}}$

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Abstract: For a commutative ring R with identity $1 \neq 0$, let the set $Z(R)$ denote the set of zero-divisors and let $Z^*(R) = Z(R) \setminus \{0\}$ be the set of non-zero zero divisors of R. The zero divisor graph of R, denoted by $\Gamma(R)$, is a simple graph whose vertex set is $Z^*(R)$ and two vertices $u, v \in Z^*(R)$ are adjacent if and only if $uv = vu = 0$. In this article, we find the signless Laplacian spectrum of the zero divisor graphs $\Gamma(\mathbb{Z}_n)$ for $n = p^{M_1}q^{M_2}$, where $p < q$ are primes and M_1, M_2 are positive integers.

Keywords: Signless Laplacian matrix; zero divisor graph, finite commutative ring, Eulers's totient function

AMS Subject classification: 05C50, 05C12, 15A18

1. Introduction

All graphs considered in this article are connected, undirected, simple and finite graphs. A graph is denoted by $G(V(G), E(G))$ (or simply by G), where $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the vertex set and $E(G)$ is the edge set of G. The *order* and the *size* of G are the cardinalities of $V(G)$ and $E(G)$, respectively. The *degree* of a vertex v in G is the number of edges incident with v and is denoted by $d_G(v)$ (or simply by d_v if it is clear from the context). The *neighbourhood* of a vertex v, denoted by $N(v)$, is the set of vertices of G adjacent to v, so that $d_v = |N(V)|$. A graph is called *regular* if every vertex is of same degree. The adjacency matrix

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$A = (a_{ij})$ of G is a square matrix of order n, whose (i,j)-entry is equal to 1, if v_i is adjacent to v_j and equal to 0, otherwise. Let $D(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees $d_i = d_G(v_i)$, $i = 1, 2, \ldots, n$ associated to G. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are respectively the Laplacian and the signless Laplacian matrices and their spectrum are respectively the Laplacian spectrum and signless Laplacian spectrum of G. These matrices are real symmetric and positive semi-definite having real eigenvalues which can be ordered as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$, respectively. More about Laplacian and signless Laplacian matrices can be seen in [8, 9, 14] and the references therein.

Let R be a commutative ring with multiplicative identity $1 \neq 0$. A non-zero element $x \in R$ is called a zero divisor of R if there exists a non-zero element $y \in R$ such that $xy = 0$. The zero divisor graphs of commutative rings were first introduced by Beck [5], in the definition he included the additive identity and was interested mainly in coloring of commutative rings. Later Anderson and Livingston [4] modified the definition of zero divisor graphs and excluded the additive identity of the ring in the zero divisor set. For a commutative ring R with identity denoted by 1, let the set $Z(R)$ denote the set of zero-divisors and let $Z^*(R) = Z(R) \setminus \{0\}$ be the set of non-zero zero divisors of R. The zero divisor graph of R, denoted by $\Gamma(R)$, is a simple graph whose vertex set is $Z^*(R)$ and two vertices $u, v \in Z^*(R)$ are adjacent if and only if $uv = vu = 0$. We denote the ring of integers modulo n by $\mathbb{Z}_n$. The order of the zero divisor graph $\Gamma(\mathbb{Z}_n)$ is $n - \phi(n) - 1$, where ϕ is Euler's totient function. The adjacency and the Laplacian spectral analysis was done in [7, 11, 17]. The normalized Laplacian and the signless Laplacian spectra were discussed in [1, 15]. More literature about zero divisor graphs can be found in [2–4, 13] and the references therein.

For any graph G, we write $Spec(G)$ for the spectrum of G which contains its eigenvalues including multiplicities. if vertices x and y are adjacent in G, then we write $x \sim y$. We use the standard notation, K_n and $K_{a,b}$ for the complete graph and the bipartite graph, respectively. Other undefined notations and terminology can be seen in [8, 12].

The rest of the paper is organized as follows. In Section 2, we start with some basic and useful results and then apply them to prove our main results.

2. Signless Laplacian eigenvalues of the zero divisor graph $\Gamma(\mathbb{Z}_{p^{M_1}q^{M_2}})$

We start the section with some definitions and known results which are used to prove the main results of the section.

Definition 1. Let $G(V, E)$ be a graph of order n having vertex set $\{1, 2, \ldots, k\}$ and $G_i = G_i(V_i, E_i)$ be disjoint graphs of order n_i, $1 \leq i \leq k$. The graph $G[G_1, G_2, \ldots, G_n]$ is formed by taking the graphs $G_1, G_2, \ldots, G_n$ and joining each vertex of G_i to every vertex of G_j whenever i and j are adjacent in G.

This graph operation $G[G_1, G_2, \ldots, G_n]$ is called the generalized join graph operation in [6] and G-join operation in [8]. Herein we follow the later name with the notation $G[G_1, G_2, \ldots, G_n]$ and call it G-join.
The signless Laplacian spectrum of G-join of graphs is given by the following result.

Theorem 1. ([16]) *Let G be a graph with $V(G) = \{1, 2, \ldots, t\}$, and G_i's be r_i-regular graphs of order n_i $(i = 1, 2, \ldots, t)$. If $G = G[G_1, G_2, \cdots, G_t]$, then the signless Laplacian spectrum of G can be computed as follows:*

$$Spec_Q(G) = \left(\bigcup_{i=1}^{t} \Big(N_i + \Big(Spec_Q(G_i) \backslash \{2r_i\} \Big) \Big) \right) \bigcup Spec(C_Q(G)),$$

where $N_i = \sum_{j \in N_G(i)} n_j$ and

$$C_Q(G) = (c_{ij})_{t \times t} = \begin{cases} 2r_i + N_i, & i = j, \\ \sqrt{n_i n_j}, & ij \in E(G), \\ 0 & \text{otherwise.} \end{cases} \qquad (1)$$

Let n be a positive integer and let $\tau(n)$ denotes the number of positive factors of n, that is

$$\tau(n) = \sum_{d|n} 1,$$

where $d|n$ denotes d divides n.
The *Euler's totient function* $\phi(n)$ denotes the number of positive integers less or equal to n and relatively prime to n.
We say n is in *canonical decomposition* if $n = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$, where $r, n_1, n_2, \ldots, n_r$ are positive integers and $p_1, p_2, \ldots, p_r$ are distinct primes.
The following result counts the values of $\tau(n)$.

Lemma 1. ([10]) *Let n be a positive integer with canonical decomposition $n = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$. Then*

$$\tau(n) = (n_1 + 1)(n_2 + 1) \ldots (n_r + 1)$$

The following result gives some properties of Euler's totient function.

Theorem 2. ([10]) *Let ϕ be the Euler's totient function. Then following hold.*

- **(i)** *ϕ is multiplicative, that is $\phi(st) = \phi(s)\phi(t)$, whenever s and t are relatively prime.*
- **(ii)** *Let n be a positive integer. Then $\sum_{d|n} \phi(d) = n$.*
- **(iii)** *Let p be a prime. Then $\sum_{i=1}^{l} \phi(p^l) = p^l - 1$.*

An integer d dividing n is called a proper divisor of n if and only if $1 < d < n$. Let Υ_n be the simple graph with vertex set as proper divisor set $\{d_1, d_2, \ldots, d_t\}$ of n, in which two distinct vertices are adjacent if and only if n divides $d_i d_j$. It is easy to see that Υ_n is a connected graph [7]. Let $p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$ be the canonical decomposition of n, then by Lemma 1, order of Υ_n is given by

$$|V(\Upsilon_n)| = (n_1 + 1)(n_2 + 1) \ldots (n_r + 1) - 2.$$

For $1 \leq i \leq t$, let

$$A_{d_i} = \{x \in \mathbb{Z}_n : (x, n) = d_i\},$$

where (x, n) denotes the greatest common divisor of x and n. We observe that $A_{d_i} \cap A_{d_j} = \phi$, when $i \neq j$, implying that the sets $A_{d_1}, A_{d_2}, \ldots, A_{d_t}$ are pairwise disjoint and partitions the vertex set of $\Gamma(\mathbb{Z}_n)$ as

$$V(\Gamma(\mathbb{Z}_n)) = A_{d_1} \cup A_{d_2} \cup \cdots \cup A_{d_t}.$$

From the definition of A_{d_i}, a vertex of A_{d_i} is adjacent to the vertex of A_{d_j} in $\Gamma(\mathbb{Z}_n)$ if and only if $n | d_i d_j$, for $i, j \in \{1, 2, \ldots, t\}$ [7]. The following result can be found in [17], which gives the cardinality of A_{d_i}.

Lemma 2. *Let d_i be the divisor of n. Then $|A_{d_i}| = \phi\left(\frac{n}{d_i}\right)$, for $1 \leq i \leq t$.*

The next lemma [7] says that the induced subgraphs $\Gamma(A_{d_i})$ of $\Gamma(\mathbb{Z}_n)$ are either cliques or null graphs.

Lemma 3. ([7]) *Let n be the positive integer and d_i be its proper divisor. Then the following hold.*

- **(i)** *For $i \in \{1, 2, \ldots, t\}$, the induced subgraph $\Gamma(A_{d_i})$ of $\Gamma(\mathbb{Z}_n)$ on the vertex set A_{d_i} is either the complete graph $K_{\phi\left(\frac{n}{d_i}\right)}$ or its complement $\overline{K}_{\phi\left(\frac{n}{d_i}\right)}$. Also, $\Gamma(A_{d_i})$ is $K_{\phi\left(\frac{n}{d_i}\right)}$ if and only $n | d_i^2$.*
- **(ii)** *For $i, j \in \{1, 2, \ldots, t\}$ with $i \neq j$, a vertex of A_{d_i} is adjacent to either all or none of the vertices in A_{d_j} of $\Gamma(\mathbb{Z}_n)$.*

The following lemma says that $\Gamma(\mathbb{Z}_n)$ is a G-join of certain complete graphs and null graphs.

Lemma 4. ([7]) *Let $\Gamma(A_{d_i})$ be the induced subgraph of $\Gamma(\mathbb{Z}_n)$ on the vertex set A_{d_i} for $1 \leq i \leq t$. Then $\Gamma(\mathbb{Z}_n) = \Upsilon_n[\Gamma(A_{d_1}), \Gamma(A_{d_2}), \ldots, \Gamma(A_{d_t})]$.*

Now, we will find the signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$, for $n = p^{M_1}q^{M_2}$, where p and q, $p < q$, are primes. This generalizes the results obtained in [1, 15]. We prove the case when M_1 and M_2, $M_1 \leq M_2$, are positive even integers and the odd case can be similarly proved.

Theorem 3. *Let $\Gamma(\mathbb{Z}_n)$ be the zero divisor graph of order N, where $n = p^{M_1}q^{M_2}$ and $M_1 = 2m_1 \leq 2m_2 = M_2$. The signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ consists of the eigenvalues*

$$\begin{aligned}
\mu_i =& p^i - 1, \text{ for } i = 1, 2, \dots, m_1, \dots, 2m_1,\\
\mu_i =& q^j - 1, \text{ for } j = 1, 2, \dots, 2m_2 \text{ and } i = M_1 + 1, M_1 + 2, \dots, M_1 + M_2,\\
\mu_i =& pq^j - 1, \text{ for } j = 1, 2, \dots, 2m_2 \text{ and } i = M_1 + M_2 + 1, \dots, M_1 + 2M_2,\\
& \vdots\\
\mu_i =& p^{m_1}q^j - 1, \text{ for } j = 1, 2, \dots, m_2 - 1 \text{ and } i = M_1 + m_1M_2 + 1, \dots, M_1 + m_1M_2 + m_2 - 1,\\
\mu_i =& p^{m_1}q^j - 3, \text{ for } j = m_2, \dots, 2m_2 \text{ and } i = M_1 + m_1M_2 + m_2, \dots, M_1 + (m_1 + 1)M_2,\\
& \vdots\\
\mu_i =& p^{2m_1}q^j - 1, \text{ for } j = 1, 2, \dots, m_2 - 1 \text{ and } i = M_1 + M_1M_2 + 1, \dots, M_1 + M_1M_2 + m_2 - 1,\\
\mu_i =& p^{2m_1}q^j - 3, \text{ for } j = m_2, \dots, 2m_2 \text{ and } i = M_1 + M_1M_2 + m_2, \dots, M_1 + M_1M_2 - 1,
\end{aligned}$$

with multiplicities

$$\phi(p^{M_1-i}q^{M_2}) - 1, \phi(p^{M_1}q^{M_2-j}) - 1, \phi(p^{M_1-1}q^{M_2-j}) - 1, \dots, \phi(p^{m_1}q^{M_2-k}) - 1, \phi(p^{m_1}q^{M_2-l}) - 1, \dots, \phi(q^{M_2-k}) - 1, \phi(q^{M_2-l}) - 1,$$

respectively, where $i = 1, \dots, M_1$, $j = 1, \dots, M_2$, $k = 1, \dots, m_2 - 1$ and $l = m_2, m_2 + 1, \dots, M_2 - 1$. The remaining signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the eigenvalues of the matrix given in (1).

Proof. Let $n = p^{M_1}q^{M_2}$, where p and q, $2 < p < q$, are primes and M_1 and M_2, $2 \leq M_1 = 2m_1 \leq 2m_2 = M_2$, are positive even integers. Then the proper divisors of n are

$$\Big\{p, p^2, \dots, p^{m_1}, \dots, p^{M_1}, q, q^2, \dots, q^{m_2}, \dots, q^{M_2}, pq, pq^2, \dots, pq^{m_2}, \dots, pq^{M_2}, \dots, p^{m_1}q, p^{m_1}q^2, \dots, p^{m_1}q^{m_2-1}, p^{m_1}q^{m_2}, \dots, p^{m_1}q^{M_2}, \dots, p^{M_1}q, p^{M_1}q^2, \dots, p^{M_1}q^{m_2-1}, p^{M_1}q^{m_2}, \dots, p^{M_1}q^{M_2-1}\Big\}$$

and the size of Υ_n is $(M_1+1)(M_2+1)-2 = M_1M_2+M_1+M_2-1$. By the definition of Υ_n, we see that

$$\begin{aligned}
p &\sim p^{M_1-1}q^{M_2},\\
p^2 &\sim p^{M_1-2}q^{M_2}, p^{M_1-1}q^{M_2},\\
p^3 &\sim p^{M_1-3}q^{M_2}, p^{M_1-2}q^{M_2}, p^{M_1-1}q^{M_2},\\
&\vdots\\
p^{m_1} &\sim p^{m_1}q^{M_2}, p^{m_1+1}q^{M_2}, \dots, p^{M_1-1}q^{M_2},\\
&\vdots\\
p^{M_1} &\sim q^{M_2}, pq^{M_2}, p^2q^{M_2}, \dots, p^{m_1}q^{M_2}, \dots, p^{M_1-1}q^{M_2}.
\end{aligned}$$

That is,

$$p^i \sim p^j q^{M_2},\ i+j \geq M_1,\ \text{for } i = 1,2,\dots,M_1.$$

Now, following the similar procedure, we have

$$\begin{aligned}
q^i &\sim p^{M_1}q^j,\ i+j \geq M_2,\ \text{for } i = 1,2,\dots,M_2,\\
pq^i &\sim p^k q^j,\ i+j \geq M_2,\ \text{for } i = 1,2,\dots,M_2 \text{ and } k \geq 2m_1-1,\\
&\vdots\\
p^{m_1}q^i &\sim p^k q^j,\ i+j \geq M_2,\ \text{for } i = 1,2,\dots,M_2 \text{ and } k \geq m_1\\
&\vdots\\
p^{M_1}q^i &\sim p^k q^j,\ i+j \geq M_2,\ \text{for } i = 1,2,\dots,M_2-1 \text{ and } k \geq 0.
\end{aligned}$$

By Lemma 2, for $i = 1,2,\dots,M_1$ and $j = 1,2,\dots,M_2$, we see that $|A_{p^i}| = \phi(p^{M_1-i}q^{M_2})$, $|A_{q^j}| = \phi(p^{M_1}q^{M_2-j})$, $|A_{pq^j}| = \phi(p^{M_1-1}q^{M_2-j}), \dots,$ $|A_{p^{m_1}q^j}| = \phi(p^{m_1}q^{M_2-j}), \dots, |A_{p^{M_1-1}q^j}| = \phi(pq^{M_2-j})$ and

$$|A_{p^{M_1}q^k}| = \phi(q^{M_2-k}),\ \text{for } k = 1,2,\dots,M_2-1.$$

Also, by Lemma 3, we have

$$G_i = \begin{cases}
\Gamma\big(A_{d_{p^i}}\big) = \overline{K}_{\phi(p^{M_1-i}q^{M_2})}, & 1 \leq i \leq M_1,\\
\Gamma\big(A_{d_{q^j}}\big) = \overline{K}_{\phi(p^{M_1}q^{M_2-j})}, & 1 \leq j \leq M_2,\\
\Gamma\big(A_{d_{p^iq^j}}\big) = \overline{K}_{\phi(p^{M_1-i}q^{M_2-j})}, & 1 \leq i \leq m_1-1 \text{ and } 1 \leq j \leq M_2\\
& \text{or } 1 \leq i \leq M_1 \text{ and } 1 \leq j \leq m_2-1,\\
\Gamma\big(A_{d_{p^iq^j}}\big) = K_{\phi(p^{M_1-i}q^{M_2-j})}, & m_1 \leq i \leq M_1 \text{ and } m_2 \leq j \leq M_2.
\end{cases} \tag{2}$$

By using Lemma 4, the joined union of the zero divisor graph $\Gamma(\mathbb{Z}_n)$ is given by

$$\begin{aligned}\Gamma(\mathbb{Z}_n) =&\Upsilon_n\,[\overline{K}_{\phi(p^{M_1-1}q^{M_2})},\dots,\overline{K}_{\phi(p^{m_1}q^{M_2})},\dots,\overline{K}_{\phi(q^{M_2})},\overline{K}_{\phi(p^{M_1}q^{M_2-1})},\dots,\overline{K}_{\phi(p^{M_1}q^{m_2})},\dots,\\ &\overline{K}_{\phi(p^{M_1})},\overline{K}_{\phi(p^{M_1-1}q^{M_2-1})},\dots,\overline{K}_{\phi(p^{M_1-1}q^{m_2})},\dots,\overline{K}_{\phi(p^{M_1-1})},\dots,\overline{K}_{\phi(p^{m_1}q^{M_2-1})},\dots,\\ &K_{\phi(p^{m_1}q^{m_2-1})},K_{\phi(p^{m_1}q^{m_2})},\dots,K_{\phi(p^{m_1})},\dots,K_{(\phi q^{M_2-1})},\dots,K_{\phi(q^{m_2-1})},K_{\phi(q^{m_2})},\dots,\\ &K_{\phi(q)}].\end{aligned}$$

Now, we use Theorem 1, to calculate the signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$. For that we first need to know the values of N_i's. It is well known that the zero divisor graphs are of diameter at most three, so that $p^i \sim q^i$ if and only if $i = j = n$, otherwise $p^i \sim p^k q^n,\ i + k \geq n$ and $q^j \sim p^n q^h,\ j + h \geq n$ and finally $p^k q^n \sim p^n q^h,\ k \geq 1, h \geq 1$. This implies that $d(p^i, q^j) = 3$, if $1 \leq i, j \leq n - 1$ in Υ_n. Similarly the distance between other vertices is at most 2. Now, by Theorems 1 and 2, we have

$$\begin{aligned}N_1 &= \phi(p) = p - 1\\ N_2 &= \phi(p) + \phi(p^2) = p^2 - 1\\ &\ \vdots\\ N_{m_1} &= \phi(p^{m_1}) + \phi(p^{m_1-1}) + \cdots + \phi(p) = p^{m_1} - 1\\ &\ \vdots\\ N_{M_1} &= \phi(p^{M_1}) + \phi(p^{M_1-1}) + \cdots + \phi(p) = p^{M_1} - 1,\end{aligned}$$

that is,

$$N_i = p^i - 1,\ \text{for } i = 1, 2, \dots, M_1.$$

By proceeding in the similar manner, other N_i's are given by

$$\begin{aligned}N_i =&q^j - 1,\ \text{for } i = M_1 + 1, \dots, M_1 + M_2,\ \text{and } j = 1, 2, \dots, m_2, \dots, M_1,\\ N_i =&pq^j - 1 \text{ for } i = M_1 + M_2 + 1, \dots, M_1 + 2M_2 \text{ and } j = 1, 2, \dots, m_2, \dots, M_1,\\ &\vdots\\ N_i =&p^{m_1}q^j - 1,\ \text{for } i = M_1 + m_1M_2 + 1, \dots, M_1 + m_1M_2 + m_2 - 1 \text{ and } j = 1, 2, \dots, m_2 - 1,\\ N_i =&p^{m_1}q^j - 1 - -\phi(p^{m_1}q^j),\ \text{for } i = M_1 + m_1M_2, \dots, M_1 + (m_1 + 1)M_2 \text{ and } j = m_2, \dots, N_2,\\ &\vdots\\ N_i =&p^{M_1}q^j - 1,\ \text{for } i = M_1 + M_1M_2 + 1, \dots, M_1 + M_1M_2 + m_2 - 1 \text{ and } j = 1, 2, \dots, m_2 - 1,\\ N_i =&p^{N_1}q^j - 1 - \phi(q^{N_2-j}),\ \text{for } i = M_1 + M_1M_2 + m_2, \dots, M_1 + M_1M_2 + M_2 - 1\\ &\qquad\text{and } j = m_2, \dots, M_2 - 1.\end{aligned}$$

Thus, by Theorem 1 and Equation (2), the signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are

$$\begin{aligned}
&\mu_i = N_i \text{ for } i = 1, 2, \ldots, M_1 + 2M_2,\\
&\quad\vdots\\
&\mu_i = N_i \text{ for } i = M_1 + m_1M_2 + 1, \ldots, M_1 + m_1M_2 + m_2 - 1,\\
&\mu_i = N_i + \phi(p^{m_1}q^j) - 2 = p^{m_1}q^j - 3 \text{ for } i = M_1 + m_1M_2 + m_2, \ldots, M_1 + (m_1 + 1)M_2\\
&\qquad\qquad\qquad j = m_2, \ldots, M_2,\\
&\quad\vdots\\
&\mu_i = N_i \text{ for } i = M_1 + M_1M_2 + 1, \ldots, M_1 + M_2M_2 + m_2 - 1,\\
&\mu_i = N_i + \phi(q^{N_2-j}) - 2 = p^{M_1}q^j - 3 \text{ for } i = M_1 + M_1M_2 + m_2, \ldots, M_1 + M_1M_2 - 1\\
&\qquad\qquad\qquad j = m_2, \ldots, M_2,
\end{aligned}$$

with multiplicities as in the statement. By using the adjacency relations, Equation (2) and value of N_i's the remaining signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the eigenvalues of the matrix given in (1). □

In particular, if $q = 1$ in Theorem 3, we get the signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_{p^{2m}})$.

Corollary 1. *If $n = p^{2m}$ for some positive integer $m \geq 2$, then the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ consists of the eigenvalue $p^i - 1$, with multiplicity $\phi(p^{2m-i})$, for $i = 1, 2, \ldots, m-1$, the eigenvalue $p^i - 3$, with multiplicity $\phi(p^{2m-i})$, for $i = m, m+1, \ldots, 2m-1$ and the remaining signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the zeros of the characteristic polynomial of the matrix given in (3).*

Proof. The proper divisors of n are $\{p, p^2, \ldots, p^{2m-1}\}$ and so by definition of $\Upsilon_{p^{2m}}$, the vertex p^i is adjacent to the vertex p^j if and only if $j \geq 2m - i$ with $1 \leq i \leq 2m-1$ and $i \neq j$. For $i = 1, 2, \ldots, 2m-2, 2m-1$, it is easy to see that $N_i = \sum_{i=1}^{m-1} \phi(p^i)$, and using the fact that $\sum_{i=1}^{r} \phi(p^r) = p^r - 1$, we have

$$N_i = p^i - 1, \text{ for } i = 1, 2, \ldots, m-2, m-1.$$

Similarly, for $i = m, m+1, \ldots, 2m-2, 2m-1$, we have

$$N_i = \sum_{j=1}^{i} \phi(p^j) - \phi(p^{2m-i}) = p^i - 1 - \phi(p^{2m-i}).$$

Since n does not divide $(p^i)^2$, for $i = 1, 2, \ldots, m-1$ and n divides $(p^i)^2$, for $i = m, m+1, \ldots, 2m-2, 2m-1$, therefore, we have

$$G_i = \begin{cases} \overline{K}_{\phi(p^{2m-i})} & \text{for } i = 1, 2, 3, \ldots, m-1,\\ K_{\phi(p^{2m-i})} & \text{for } i = m, m+1, \ldots, 2m-2, 2m-1. \end{cases}$$

Also, $2r_i + N_i = p^i - 1$ for $i = 1, 2 \ldots, m-1$, and $2r_i + N_i = p^i + \phi(p^{2m-i}) - 3$ for $i = m, \ldots, 2m-2, 2m-1$. Further, order of G_i's are $n_i = \phi(p^{2m-i})$, and using Theorem 1, we have

$$Spec_Q(\Gamma(\mathbb{Z}_n)) = \Big\{(p-1)^{[\phi(p^{2m-1})-1]}, (p^2-1)^{[\phi(p^{2m-2})-1]}, \ldots, (p^{m-2}-1)^{[\phi(p^{m+2})-1]}, (p^{m-1}-1)^{[\phi(p^{m+1})-1]}\Big\} \left(\bigcup_{i=m}^{2m-1} \Big(N_i + \Big(Spec\Big(K_{\phi(p^{2m-i})}\Big) \setminus \{2r_i\}\Big)\Big)\right)$$

and the eigenvalues of matrix (3).

$$\begin{pmatrix} N_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & b_{1,2m-1} \\ 0 & N_2 & \cdots & 0 & 0 & 0 & \cdots & b_{2,2m-2} & b_{2,2m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & N_{m-1} & 0 & b_{m-1,m+1} & \cdots & b_{m-1,2m-2} & b_{m-1,2m-1} \\ 0 & 0 & \cdots & 0 & a_m & b_{m,m+1} & \cdots & b_{m,2m-2} & b_{m,2m-1} \\ 0 & 0 & \cdots & b_{m+1,m-1} & b_{m+1,m} & a_{m+1} & \cdots & b_{m+1,2m-2} & b_{m+1,2m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & b_{2m-2,2} & \cdots & b_{2m-2,m-1} & b_{2m-2,m} & b_{2m-2,m+1} & \cdots & a_{2m-2} & b_{2m-2,2m-1} \\ b_{2m-1,1} & b_{2m-1,2} & \cdots & b_{2m-1,m-1} & b_{2m-1,m} & b_{2m-1,m+1} & \cdots & b_{2m-1,2m-2} & a_{2m-1} \end{pmatrix} \quad (3)$$

where $b_{i,j} = b_{j,i} = \sqrt{n_i n_j}$, for $1 \leq i, j \leq 2m-1$ and $a_i = 2r_i + N_i$, for $i = m, m+1, \ldots, 2m-1$.
We recall that the signless Laplacian spectrum of $K_{\phi(p^{2m-i})}$ is $\Big\{2\phi(p^{2m-i}) - 2, (\phi(p^{2m-i})-2)^{[\phi(p^{2m-i})-1]}\Big\}$ and using $N_i = p^i - 1 - \phi(p^{2m-i})$ for $i = m, \ldots, 2m-1$, it easily follows that

$$\bigcup_{i=m}^{2m-1} \Big(N_i + \Big(Spec\Big(K_{\phi(p^{2m-i})}\Big) \setminus \{2r_i\}\Big)\Big) = \Big\{(p^m-3)^{[\phi(p^m)-1]}, (p^{m+1}-3)^{\left[\phi\left(p^{m-1}\right)-1\right]}, \ldots, (p^{2m-2}-3)^{\left[\phi\left(p^2\right)-1\right]}, (p^{2m-1}-3)^{[\phi(p)-1]}\Big\}.$$

□

If $m_1 = 1$ and $q = 1$ in Theorem 3, we have $\Gamma(\mathbb{Z}_n) = K_{\phi(p^2)}$ and its signless Laplacian spectrum is given by the following observation.

Corollary 2. *If $n = p^2$, then the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ is*

$$\{2p-4, (p-3)^{[p-2]}\}.$$

For $n = p^3$, zero divisor graph is

$$\Gamma(\mathbb{Z}_{p^3}) = \Upsilon_{p^3}[\Gamma(A_p), \Gamma(A_{p^2})] = K_2[\overline{K}_{\phi(p^2)}, \overline{K}_{\phi(p)}] = \overline{K}_{p(p-1)} \nabla K_{p-1}$$

and its signless Laplacian spectrum is given by the following observation.

Corollary 3. *If $n = p^3$, then the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ is*

$$\left\{(p-1)^{[p^2-p-1]}, (p^2-3)^{[p-2]}, \frac{1}{2}\left(p^2-3 \pm \sqrt{p^4-6p^2+8p+1}\right)\right\}.$$

The following result gives the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^{M_1}q^{M_2}})$, when both M_1 and M_2 are odd. Its proof is similar to that of Theorem 3.

Theorem 4. *Let $\Gamma(\mathbb{Z}_n)$ be the zero divisor graph of order N, where $n = p^{M_1}q^{M_2}$ and $M_1 = 2m_1+1 \le 2m_2+1 = M_2$. The signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ consists of the eigenvalues*

$$\begin{aligned}
\mu_i =& p^i - 1, \text{ for } i = 1,2,\ldots,m_1+1,\ldots,2m_1+1,\\
\mu_i =& q^j - 1, \text{ for } j = 1,2,\ldots,2m_2+1 \text{ and } i = M_1+1, M_1+2,\ldots,M_1+M_2,\\
\mu_i =& pq^j - 1, \text{ for } j = 1,2,\ldots,2m_2+1 \text{ and } i = M_1+M_2+1,\ldots,M_1+2M_2,\\
&\vdots\\
\mu_i =& p^{m_1+1}q^j - 1, \text{ for } j = 1,2,\ldots,m_2 \text{ and } i = N_1+(m_2+1)N_2+1,\ldots,N_1+(m_2+1)N_2+m_2,\\
\mu_i =& p^{m_1+1}q^j - 3, \text{ for } j = m_2+1,\ldots,M_2 \text{ and } i = M_1+(m_2+1)M_2+m_2,\ldots,M_1+(m_2+2)M_2,\\
&\vdots\\
\mu_i =& p^{2m_1+1}q^j - 1, \text{ for } j = 1,2,\ldots,m_2 \text{ and } i = M_1+M_1M_2+1,\ldots,M_1+M_1M_2+m_2,\\
\mu_i =& p^{2m_1+1}q^j - 3, \text{ for } j = m_2+1,\ldots,2m_2 \text{ and } i = M_1+M_1M_2+m_2+1,\ldots,M_1+M_1M_2-1,
\end{aligned}$$

with multiplicities

$$\phi(p^{M_1-i}q^{M_2})-1, \phi(p^{M_1}q^{M_2-j})-1, \phi(p^{M_1-1}q^{M_2-j})-1,\ldots,\phi(p^{m_1}q^{M_2-j})-1,\\
\phi(p^{m_1}q^{M_2-k})-1,\ldots,\phi(q^{M_2-j})-1, \phi(q^{M_2-k})-1,$$

respectively, where $i = 1,\ldots,M_1$, $j = 1,\ldots,m_2$ and $k = m_2+1, m_2+1,\ldots,M_2-1$. The remaining signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the eigenvalues of the matrix given in (1).

In particular, if $q = 1$ in Theorem 4, we have the following observation.

Corollary 4. *If $n = p^{2m+1}$ for some positive integer $m \ge 2$, then the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ consists of the eigenvalue $p^i - 1$, with multiplicity $\phi(p^{2m+1-i})$, for $i = 1,2,\ldots,m$, the eigenvalue $p^i - 3$, with multiplicity $\phi(p^{2m+1-i})$, for $i = m+1, m+2,\ldots,2m$ and the remaining signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the zeros of the characteristic* *polynomial of the following matrix*

$$\begin{pmatrix} N_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & b_{1,2m} \\ 0 & N_2 & \cdots & 0 & 0 & 0 & \cdots & b_{2,2m-1} & b_{2,2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & N_m & 0 & b_{m,m+1} & \cdots & b_{m,2m-1} & b_{m,2m} \\ 0 & 0 & \cdots & 0 & a_{m+1} & b_{m+1,m+1} & \cdots & b_{m+1,2m-1} & b_{m,2m} \\ 0 & 0 & \cdots & b_{m+2,m} & b_{m+2,m+1} & a_{m+2} & \cdots & b_{m+2,2m-1} & b_{m+2,2m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & b_{2m-1,2} & \cdots & b_{2m-1,m} & b_{2m-1,m+1} & b_{2m-1,m+2} & \cdots & a_{2m-1} & b_{2m-1,2m} \\ b_{2m,1} & b_{2m,2} & \cdots & b_{2m,m} & b_{2m,m+1} & b_{2m,m+2} & \cdots & b_{2m,2m-1} & a_{2m} \end{pmatrix},$$

where, $b_{i,j} = b_{j,i} = \sqrt{n_i n_j}$, *for* $1 \leq i, j \leq 2m$ *and* $a_i = 2r_i + N_i$, *for* $i = m+1, m+2 \ldots, 2m$.

If $m_1 = m_2 = 0$, then $n = pq$. So, by Lemmas 3 and 4, we have

$$\Gamma(\mathbb{Z}_{pq}) = \Upsilon_{pq}[\Gamma(A_p), \Gamma(A_q)] = K_2[\overline{K}_{\phi(p)}, \overline{K}_{\phi(q)}] = \overline{K}_{\phi(p)} \nabla \overline{K}_{\phi(q)} = K_{\phi(p),\phi(q)}. \quad (4)$$

The next consequence of Theorem 4 gives the signless Laplacian spectrum of the bipartite graph $\Gamma(\mathbb{Z}_{pq})$.

Corollary 5. *The signless Laplacian spectrum of* $\Gamma(\mathbb{Z}_{pq})$ *is*

$$\{0, (q-1)^{[p-2]}, (p-1)^{[q-2]}, p+q-2\}.$$

Now, consider the case when one of M_i's is even and other is odd, say M_1 is even and M_2 is odd or M_1 is odd and M_2 is even. In the following result, we discuss the first case and the second case can be treated similarly.

Theorem 5. *Let* $\Gamma(\mathbb{Z}_n)$ *be the zero divisor graph of order* N, *where* $n = p^{M_1} q^{M_2}$ *and* $m_1 < m_2$ *so that* $M_1 = 2m_1 < 2m_2 + 1 = M_2$. *The signless Laplacian spectrum of* $\Gamma(\mathbb{Z}_n)$ *consists of the eigenvalues*

$$\begin{aligned} \mu_i &= p^i - 1, \text{ for } i = 1, 2, \ldots, m_1, \ldots, M_1, \\ \mu_i &= q^j - 1, \text{ for } j = 1, 2, \ldots, M_2 \text{ and } i = M_1 + 1, M_1 + 2, \ldots, M_1 + M_2, \\ \mu_i &= pq^j - 1, \text{ for } j = 1, 2, \ldots, M_2 \text{ and } i = M_1 + M_2 + 1, \ldots, M_1 + 2M_2, \\ &\vdots \\ \mu_i &= p^{m_1} q^j - 1, \text{ for } j = 1, 2, \ldots, m_2 \text{ and } i = M_1 + m_1 M_2 + 1, \ldots, M_1 + m_1 M_2 + m_2, \\ \mu_i &= p^{m_1} q^j - 3, \text{ for } j = m_2 + 1, \ldots, M_2 \text{ and } i = M_1 + m_1 M_2 + m_2 + 1, \ldots, M_1 + (m_1 + 1) M_2 \\ &\vdots \\ \mu_i &= p^{M_1} q^j - 1, \text{ for } j = 1, 2, \ldots, m_2 \text{ and } i = M_1 + M_1 M_2 + 1, \ldots, M_1 + M_1 M_2 + m_2, \\ \mu_i &= p^{M_1} q^j - 3, \text{ for } j = m_2 + 1, \ldots, 2m_2 \text{ and } i = M_1 + M_1 M_2 + m_2 + 1, \ldots, M_1 + M_1 M_2 - 1, \end{aligned}$$

with multiplicities

$$\phi(p^{M_1-i}q^{M_2})-1,\phi(p^{M_1}q^{M_2-j})-1,\phi(p^{M_1-1}q^{M_2-j})-1,\dots,\phi(p^{m_1}q^{M_2-k})-1,\\ \phi(p^{m_1}q^{M_2-l})-1,\dots,\phi(q^{M_2-k})-1,\phi(q^{M_2-l})-1, \tag{5}$$

respectively, where $i=1,\dots,M_1,\ j=1,\dots,M_2,\ k=1,2,\dots,m_2$ *and* $l=m_2+1,m_2+2,\dots,M_2-1$*. The remaining signless Laplacian eigenvalues of* $\Gamma(\mathbb{Z}_n)$ *are the eigenvalues of the matrix given in* (1)*.*

Proof. Let $n=p^{M_1}q^{M_2}$, where p and q, $2<p<q$, are primes and $m_1<m_2$ so that $2\leq M_1=2m_1<2m_2+1=M_2$. The proper divisor set of n is $\{p,p^2,\dots,p^{m_1},\dots,p^{M_1},q,q^2,\dots,q^{m_2+1},\dots,q^{M_2},pq,pq^2,\dots,pq^{m_2+1},\dots,pq^{M_2},\dots,p^{m_1}q,p^{m_1}q^2,\dots,p^{m_1}q^{m_2},p^{m_1}q^{m_2+1},\dots,p^{m_1}q^{M_2},\dots,p^{M_1}q,p^{M_1}q^2,\dots,p^{M_1}q^{m_2},p^{M_1}q^{m_2+1},\dots,p^{M_1}q^{M_2-1}\}$ and the size of Υ_n is $(M_1+1)(M_2+1)-2=M_1M_2+M_1+M_2-1$. By the definition of Υ_n, the adjacency relations are

$$\begin{aligned}
p^i &\sim p^jq^{M_2},\ i+j\geq M_1,\ \text{for } i=1,2,\dots,M_1\\
q^i &\sim p^{M_1}q^j,\ i+j\geq M_2,\ \text{for } i=1,2,\dots,M_2,\\
pq^i &\sim p^kq^j,\ i+j\geq M_2,\ \text{for } i=1,2,\dots,M_2 \text{ and } k\geq 2m_1-1,\\
&\vdots\\
p^{m_1}q^i &\sim p^kq^j,\ i+j\geq M_2,\ \text{for } i=1,2,\dots,M_2 \text{ and } k\geq m_1\\
&\vdots\\
p^{M_1}q^i &\sim p^kq^j,\ i+j\geq M_2,\ \text{for } i=1,2,\dots,M_2-1 \text{ and } k\geq 0.
\end{aligned}$$

By Lemma 2, for $i=1,2,\dots,M_1,\ j=1,2,\dots,M_2$ and $k=1,2,\dots,M_2-1$, we have $|A_{p^i}|=\phi(p^{M_1-i}q^{M_2})$, $|A_{q^j}|=\phi(p^{M_1}q^{M_2-j})$, $|A_{pq^j}|=\phi(p^{M_1-1}q^{M_2-j})$, $\dots$, $|A_{p^{m_1}q^j}|=\phi(p^{m_1}q^{M_2-j})$, $\dots$, $|A_{p^{M_1-1}q^j}|=\phi(pq^{M_2-j})$, $|A_{p^{M_1}q^k}|=\phi(q^{M_2-k})$. Also, by Lemma 3, we have

$$G_i=\begin{cases}\Gamma\big(A_{d_{p^i}}\big)=\overline{K}_{\phi(p^{M_1-i}q^{M_2})}, & 1\leq i\leq M_1,\\ \Gamma\big(A_{d_{q^j}}\big)=\overline{K}_{\phi(p^{M_1}q^{M_2-j})}, & 1\leq j\leq M_2,\\ \Gamma\big(A_{d_{p^iq^j}}\big)=\overline{K}_{\phi(p^{N_1-i}q^{N_2-j})}, & 1\leq i\leq m_1-1 \text{ and } 1\leq j\leq M_2\\ & \text{or } 1\leq i\leq M_1 \text{ and } 1\leq j\leq m_2,\\ \Gamma\big(A_{d_{p^iq^j}}\big)=K_{\phi(p^{N_1-i}q^{N_2-j})}, & m_1\leq i\leq N_1 \text{ and } m_2\leq j\leq N_2.\end{cases} \tag{6}$$

Thus, by Lemma 4, the joined union of $\Gamma(\mathbb{Z}_n)$ is

$$\begin{aligned}\Gamma(\mathbb{Z}_n)=&\Upsilon_n[\overline{K}_{\phi(p^{M_1-1}q^{M_2})},\dots,\overline{K}_{\phi(p^{m_1}q^{M_2})},\dots,\overline{K}_{\phi(q^{M_2})},\overline{K}_{\phi(p^{M_1}q^{M_2-1})},\dots,\overline{K}_{\phi(p^{M_1}q^{m_2})},\dots,\\ &\overline{K}_{\phi(p^{M_1})},\overline{K}_{\phi(p^{M_1-1}q^{M_2-1})},\dots,\overline{K}_{\phi(p^{M_1-1}q^{m_2})},\dots,\overline{K}_{\phi(p^{M_1-1})},\dots,\overline{K}_{\phi(p^{m_1}q^{M_2-1})},\dots,\\ &K_{\phi(p^{m_1}q^{m_2-11})},K_{\phi(p^{m_1}q^{m_2})},\dots,K_{\phi(p^{m_1})},\dots,K_{\phi(q^{M_2-1})},\dots,K_{\phi(q^{m_2-1})},K_{\phi(q^{m_2})},\dots,\\ &K_{\phi(q)}].\end{aligned}$$

By Theorems 1 and 2, we have value of N_i's as

$$\begin{aligned}
N_i =& p^i - 1, \text{ for } i = 1, 2, \dots, M_1\\
N_i =& q^j - 1, \text{ for } i = M_1 + 1, \dots, M_1 + M_2 \text{ and } j = 1, 2, \dots, m_2 + 1, \dots, M_2,\\
N_i =& pq^j - 1 \text{ for } i = M_1 + M_2 + 1, \dots, M_1 + 2M_2 \text{ and } j = 1, 2, \dots, m_2 + 1, \dots, M_2,\\
&\vdots\\
N_i =& p^{m_1} q^j - 1, \text{ for } i = M_1 + m_1 M_2 + 1, \dots, M_1 + m_1 M_2 + m_2 - 1 \text{ and } j = 1, 2, \dots, m_2,\\
N_i =& p^{m_1} q^j - 1 - \phi(p^{m_1} q^j), \text{ for } i = M_1 + m_1 M_2, \dots, M_1 + (m_1 + 1)M_2 \text{ and } j = m_2 + 1, \dots, M_2,\\
&\vdots\\
N_i =& p^{M_1} q^j - 1, \text{ for } i = M_1 + M_1 M_2 + 1, \dots, M_1 + M_1 M_2 + m_2 \text{ and } j = 1, 2, \dots, m_2,\\
N_i =& p^{M_1} q^j - 1 - \phi(q^{M_2 - j}), \text{ for } i = M_1 + M_1 M_2 + m_2 + 1, \dots, M_1 + M_1 M_2 + M_2 - 1\\
& \text{and } j = m_2 + 1, \dots, 2m_2.
\end{aligned}$$

Thus, by Theorem 1 and Equation (6), the signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are

$$\begin{aligned}
&\mu_i = N_i \text{ for } i = 1, 2, \dots, M_1 + 2M_2,\\
&\quad\vdots\\
&\mu_i = N_i \text{ for } i = M_1 + m_1 M_2 + 1, \dots, M_1 + m_1 M_2 + m_2,\\
&\mu_i = N_i + \phi(p^{m_1} q^{M_2 - j}) - 2 = p^{m_1} q^j - 3 \text{ for } i = M_1 + m_1 M_2 + m_2 + 1, \dots, M_1 + (m_1 + 1)M_2\\
&\qquad\qquad j = m_2 + 1, \dots, M_2,\\
&\quad\vdots\\
&\mu_i = N_i \text{ for } i = M_1 + M_1 M_2 + 1, \dots, M_1 + M_2 M_2 + m_2,\\
&\mu_i = N_i + \phi(q^{M_2 - j}) - 2 = p^{M_1} q^j - 3 \text{ for } i = M_1 + M_1 M_2 + m_2 + 1, \dots, M_1 + M_1 M_2 - 1\\
&\qquad\qquad j = m_2 + 1, \dots, M_2.
\end{aligned}$$

with multiplicities as in Equation (5). By using the adjacency relations, Equation (6) and value of N_i's in matrix (1), we can find the remaining signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$. □

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