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Double Roman domination in graphs: algorithmic complexity

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Abstract: Let $G = (V, E)$ be a graph. A double Roman dominating function (DRDF) of G is a function $f : V \to \{0, 1, 2, 3\}$ such that, for each $v \in V$ with $f(v) = 0$, there is a vertex u adjacent to v with $f(u) = 3$ or there are vertices x and y adjacent to v such that $f(x) = f(y) = 2$ and for each $v \in V$ with $f(v) = 1$, there is a vertex u adjacent to v with $f(u) > 1$. The weight of a DRDF f is $f(V) = \sum_{v \in V} f(v)$. Let n and k be integers such that $3 \leq 2k + 1 \leq n$. The generalized Petersen graph $GP(n, k) = (V, E)$ is the graph with $V = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ and $E = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : 1 \leq i \leq n\}$, where addition is taken modulo n.

In this paper, we firstly prove that the decision problem associated with double Roman domination is NP-complete even restricted to planar bipartite graphs with maximum degree at most 4. Next, we give a dynamic programming algorithm for computing a minimum DRDF (i.e., a DRDF with minimum weight along all DRDFs) of $GP(n, k)$ in $O(n81^k)$ time and space and so a minimum DRDF of $GP(n, O(1))$ can be computed in $O(n)$ time and space.

Keywords: Double Roman dominating function, Algorithm, Dynamic programming, Generalized Petersen graph

AMS Subject classification: 05C78, 05C76

1. Introduction

Let $G = (V, E)$ be a graph with the vertex set V and the edge set E. Here, we study finite, simple and undirected graphs. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of $v \in V$, denoted by $\deg_G(v)$, is the cardinality of $N_G(v)$. For any $S \subseteq V$ the *induced subgraph* $G[S]$ is the graph whose vertex set is S and whose edge set consists of all edges in E that have both endpoints in S. If $\deg_G(v) = 1$, then v is called a *pendant* vertex of G.

The graph G is called a *bipartite* graph if V can be partitioned into two subsets X and Y such that each edge in E has one end in X and one end in Y, denoted by

$G = (X, Y, E)$. A *tree* is a connected graph with no cycles. A tree $T = (V, E)$ is called a *star* if $|V| = 2$ or $|V| \geq 3$ and T contains exactly one vertex that is not pendant that is called the *central vertex* of the star. A *path* is a tree with exactly two pendants and a *triad* is three paths with a common end.

A function $f : V \to \{0, 1, 2\}$ is called a *Roman dominating function* of G if for every vertex $v \in V$ with $f(v) = 0$, there is a vertex $u \in N(v)$ with $f(u) = 2$. Roman domination was initially motivated by the defence of the Roman empire. In the main problem, a city may be defended by using one of the two legions from a neighboring city. Beeler et al. [5] first initiated the study of double Roman dominating functions, a stronger version of Roman domination functions that can defend any attack by at least two legions. A *double Roman dominating function* (DRDF) of G is a function $f : V \to \{0, 1, 2, 3\}$ such that:

(i) for each $v \in V$ with $f(v) = 0$, there is a vertex $u \in N_G(v)$ with $f(u) = 3$ or there are vertices $x, y \in N_G(v)$ with $f(x) = f(y) = 2$, and

(ii) for each $v \in V$ with $f(v) = 1$, there is a vertex $u \in N_G(v)$ with $f(u) > 1$.

For a DRDF f of G, we use the notation $f = (V_0, V_1, V_2, V_3)$, where V_i is the set of all vertices of G with label i under f for each $i \in \{0, 1, 2, 3\}$. The *weight* of a DRDF f, denoted by $w(f)$, is $f(V) = \sum_{v \in V} f(v)$. The *double Roman domination number* of G, denoted by $\gamma_{dR}(G)$, is the minimum weight of a DRDF of G between all DRDFs of G. A minimum DRDF of G is a DRDF f of G with $w(f) = \gamma_{dR}(G)$.

Variants of double Roman domination of graphs have been studied extensively in the literature, for example [2, 3, 6, 8, 9, 13]. The decision problem associated with the double Roman domination is NP-complete even when restricted to bipartite graphs and chordal graphs [1, 7], star convex bipartite graphs and tree convex bipartite graphs [11] and undirected path graphs, chordal bipartite graphs and circle graphs [4]. There are linear time algorithms for computing the double Roman domination number of special classes of graphs such as proper interval graphs and block graphs [4], trees [15], and unicyclic graphs [12].

Let n and k be integers such that $3 \leq 2k + 1 \leq n$. Watkins [14] has introduced the *generalized Petersen* graph $GP(n, k) = (V, E)$ as the graph with the vertex set $V = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ and the edge set $E = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : 1 \leq i \leq n\}$, where the subscripts are added modulo n.

In this paper, we first prove that the decision problem associated with the double Roman domination is NP-complete even when restricted to planar bipartite graphs with maximum degree at most 4. Then, we propose an algorithm to compute a minimum DRDF of $GP(n, k)$ in $O(n81^k)$ time and space. For this purpose we first propose an algorithm based on a dynamic programming approach to compute $\gamma_{dR}(GP(n, k))$ and then using a backtracking search algorithm we find a minimum DRDF of $GP(n, k)$ in $O(n81^k)$ time and space. As a result, we can compute a minimum DRDF of $GP(n, O(1))$ in $O(n)$ time and space.

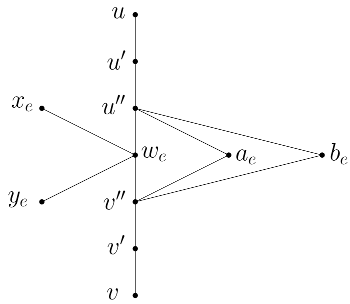


Figure 1. **Illustrating the gadget G_e, where $e = uv$.**

2. Hardness

In this section we study the computational complexity of the following decision problem.

Double Roman Domination (DRD) problem:
Instance: A graph G and a positive integer t.
Question: Is there a DRDF f on G with $w(f) \leq t$?

We prove that the DRD problem is NP-complete even when restricted to planner bipartite graphs with maximum degree at most 4. We introduce a reduction from the vertex cover (VC) problem to the DRD problem, where VC is the problem of deciding whether given a graph $G = (V, E)$ and a positive integer k, there is a vertex cover (i.e., a set $S \subseteq V$ such that each edge has at least one endpoint in S) in G with cardinality at most k. The VC problem is NP-complete even when restricted to 2-connected planar cubic graphs [10].
For a given 2-connected planar cubic graph $G = (V, E)$, let H be a graph constructed from G by replacing each edge $e = uv$ in G by a gadget G_e illustrated in Figure 1. The graph H is a planar bipartite graph with maximum degree at most 4 that can be constructed in polynomial time of $|E|$. Let $\beta(G)$ denote the vertex covering number of a graph G. In the rest of the paper, we need the following result of Beeler et. al. [5].

Corollary 1. *For any graph G, there is a minimum DRDF $f = (V_0, V_1, V_2, V_3)$ with $V_1 = \emptyset$.*

Lemma 1. *Given a 2-connected planar cubic graph G, let H be a graph constructed from G by replacing each edge $e = uv$ in G by a gadget G_e illustrated in Figure 1. Then, $\gamma_{dR}(H) = \beta(G) + 2|V(G)| + 6|E(G)|$.*

Proof. Let D be a vertex cover of G with $|D| = \beta(G)$. Let $e = uv$ be an edge in $E(G)$. At least one of vertices u and v is in D. We construct a DRDF f of H as follows. Initially, set D_3' to be the empty set. Add w_e to D_3'. If $v \notin D$, then add v'' to D_3', if $u \notin D$, then add u'' to D_3', and if both u and v are in D, then add u'' to D_3'. Thus, $|D_3'| = 2|E(G)|$. Let $f = (V(H) \setminus (D_3' \cup V(G)), \emptyset, V(G) \setminus D, D_3' \cup D)$. We obtain that f is a DRDF of H with $w(f) = 2|V(G) \setminus D| + 3|D_3' \cup D| = \beta(G) + 2|V(G)| + 6|E(G)|$ and so $\gamma_{dR}(H) \leq w(f)$.
On the other hand, by Corollary 1, let $g = (V_0^g, \emptyset, V_2^g, V_3^g)$ be a minimum DRDF of H. Let $e = uv \in E(G)$. If $g(x_e) + g(y_e) > 0$, then $g(x_e) + g(y_e) + g(w_e) > 3$. By replacing both $g(x_e)$ and $g(y_e)$ by 0 and $g(w_e)$ by 3, we obtain a new DRDF of H with weight less than $w(g)$, contradicting that g is a minimum DRDF of H. Hence, $g(x_e) = g(y_e) = 0$ and $g(w_e) = 3$. Similarly, $g(a_e) = g(b_e) = 0$ and so either at least one of vertices u'' and v'' is in V_3^g or both u'' and v'' are in V_2^g. Assume $g(u'') = g(v'') = 2$. If $g(u) = 3$ (respectively, $g(v) = 3$), then by replacing $g(u'')$ and $g(v'')$ by 0 and 3 (resp., 3 and 0), respectively, we obtain a new DRDF of H with weight less than $w(g)$, a contradiction. So, both u and v are not in V_3^g. If $g(u) = 0$ (respectively, $g(v) = 0$), then $g(u') = a > 1$ and so by replacing $g(u)$ and $g(u')$ (respectively, $g(v)$ and $g(v')$) by a and 0, respectively, we obtain a new DRDF of H with weight less or equal to $w(g)$. Hence, we may assume $g(u) = g(v) = 2$. By replacing $g(u)$, $g(u'')$, and $g(v'')$ by 3, 0, and 3, respectively, we obtain a new DRDF of H with weight less or equal to $w(g)$. Hence, we may assume either $g(u'') = 3$ and $g(v'') = 0$ or $g(u'') = 0$ and $g(v'') = 3$. Let $S = \{u'', v'', w_e : e = uv \in E(G)\}$, let $S_3 = \{x \in S : g(x) = 3\}$, let $V' \subseteq V(H)$ be the set of vertices that are not adjacent to some vertex in S_3, and let H' be the induced subgraph $H[V']$. All vertices $a_e, b_e, x_e, y_e, w_e, u'', v''$ and either u' or v' (not both) are not in V' and so H' is a forest of trees with $|V(G)|$ components that each component is a star whose central vertex is a vertex in $V(G)$. Let T be a component of H'. If T is a single vertex, then $g(z) = 2$, where $V(T) = \{z\}$ and if T is not a single vertex, then $g(z) = 3$, where z is the central vertex of T and so at least one of two vertices u and v is in V_3. Let $D = V(G) \cap V_3^g$. We obtain that D is a VC of G and $w(g) = 3|D| + 2(|V(G)| - |D|) + 6|E(G)| = |D| + 2|V(G)| + 6|E(G)|$. Thus, $\beta(G) \leq |D| = w(g) - 2|V(G)| - 6|E(G)| = \gamma_{dR}(H) - 2|V(G)| - 6|E(G)|$. This completes the proof of the lemma. □ □

By Lemma 1 and the fact that H is a planar bipartite graph with maximum degree at most 4 that can be computed in polynomial time of $|E|$, where $G = (V, E)$ is a given 2-connected planar cubic graph, and the fact that the DRD problem is in NP, we have the following result.

Theorem 1. *The decision version of the double Roman domination problem is NP-complete even when restricted to planar bipartite graphs with maximum degree at most 4.*

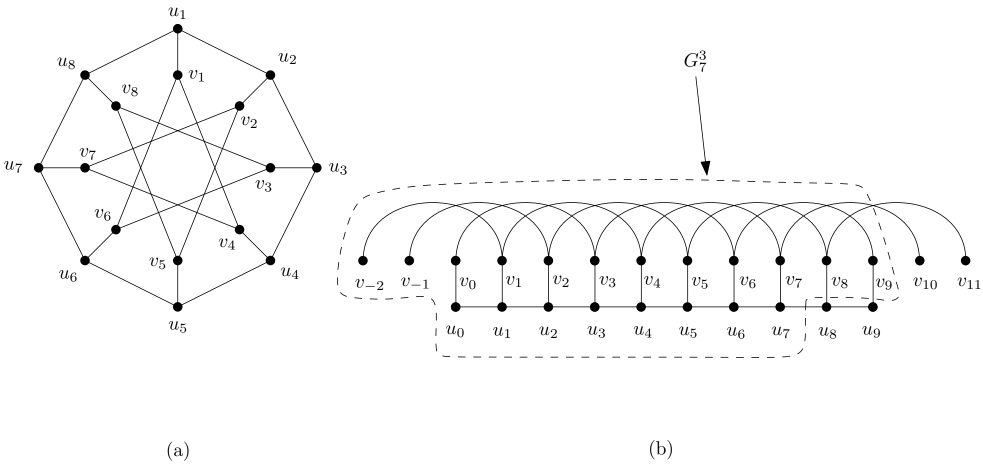


Figure 2. **Illustrating (a)** $GP(8,3)$ **and (b)** $SGP(8,3)$ **and** G_7^3**.**

3. Computing γ_{dR} of generalized Petersen graphs

In this section, we give an algorithm to compute the double Roman domination number of the generalized Petersen graph $GP(n,k)$. Before we begin our algorithm, we need the following notations.

3.1. Notations needed for the Algorithm

In the rest of the paper, we fix integers n and k such that $3 \leq 2k+1 \leq n$. Let $GP(n,k) = (V_{GP}, E_{GP})$ be the generalized Petersen graph with $V_{GP} = \{u_1, \dots, u_n\} \cup \{v_1, \dots, v_n\}$ and $E_{GP} = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : 1 \leq i \leq n\}$, where addition is taken modulo n. The *semi generalized Petersen graph* $SGP(n,k) = (V_s, E_s)$ (corresponding to $GP(n,k)$) is a graph with the vertex set

$$V_s = V_{GP} \cup V_l \cup V_r,$$

where $V_l = \{v_{1-k}, v_{2-k}, \dots, v_0, u_0\}$ and $V_r = \{u_{n+1}, v_{n+1}, v_{n+2}, \dots, v_{n+k}\}$ and the edge set

$$E_s = (E_{GP} \setminus \{u_1 u_n, v_{n-k+i} v_i : 1 \leq i \leq k\}) \cup E_l \cup E_r,$$

where $E_l = \{v_{1-k}v_1, v_{2-k}v_2, \dots, v_0 v_k, u_0 u_1, u_0 v_0\}$ and $E_r = \{u_{n+1}v_{n+1}, u_n u_{n+1}, v_{n-k+1}v_{n+1}, v_{n-k+2}v_{n+2}, \dots, v_n v_{n+k}\}$. See Fig. 2.

Remark 1. We have $\deg_{SGP(n,k)}(v) = 3$ for every vertex $v \in V_{GP}$ and $\deg_{SGP(n,k)}(v) < 3$ for every vertex $v \in V_l \cup V_r$.

Let $G' = (V', E')$ be a connected subgraph of $SGP(n, k)$. A function $f : V' \to \{0, 1, 2, 3\}$ is a *semi double Roman dominating function* (SDRDF) of G' such that for each vertex $v \in V'$ with $\deg_{G'}(v) = 3$,

(i) if $f(v) = 0$, there is a vertex $u \in N_{G'}(v)$ with $f(u) = 3$ or there are vertices $x, y \in N_{G'}(v)$ with $f(x) = f(y) = 2$, and

(ii) if $f(v) = 1$, there is a vertex $u \in N_{G'}(v)$ with $f(u) > 1$.

Let G_i^k be the subgraph of $SGP(n, k)$ induced by $\mathtt{V}_i = V_l \cup \{u_1, \ldots, u_i\} \cup \{v_1, \ldots, v_{i+k-1}\}$ for each $1 \leq i \leq n+1$. We obtain that $G_{n+1}^k = SGP(n, k)$. See Fig. 2(b). Let $b_1, b_2, \ldots, b_{2k+2} \in \{0, 1, 2, 3\}$ and let $i \in \{1, 2, \ldots, n+1\}$. In the following, we define $\gamma_{dR}^{b_{2k+2}b_{2k+1}\cdots b_1}(G_i^k)$. Here, $b_{2k+2}, b_{2k+1}, \ldots, b_1$ are corresponding to vertices $v_{i-k}, v_{i-k+1}, \ldots, v_{i-1}, u_{i-1}, u_i, v_i, v_{i+1}, \ldots, v_{i+k-1}$, respectively, of G_i^k. Let $j \in \{1, \ldots, 2k+2\}$. The value $\gamma_{dR}^{b_{2k+2}\cdots b_1}(G_i^k)$ is the weight of a minimum SDRDF $f = (V_0, V_1, V_2, V_3)$ of G_i^k such that if $b_j = x \in \{0, 1, 2, 3\}$, then the corresponding vertex of b_j is in V_x. Let S be the set of vertices corresponding to b_j for all $j \in \{1, \ldots, 2k+2\}$. Note that each vertex $w \in S$ with $d_{G_i^k}(w) = 3$ is adjacent to at least one vertex in $V(G_i^k)$ that is not in S and so $\gamma_{dR}^{b_{2k+2}\cdots b_1}(G_i^k)$ is well-defined. Since there are $4^{2k+2} = 16^{k+1}$ different cases for defining $\gamma_{dR}^{b_{2k+2}\cdots b_1}(G_i^k)$, in the following we give the complete formal definition of some cases.

- $\gamma_{dR}^{0\cdots 0}(G_i^k) = \min\{w(f) : f = (V_0, V_1, V_2, V_3)$ is a SDRDF of G_i^k, $v_{i-k} \in V_0$, $v_{i-k+1} \in V_0, \ldots, v_{i-1} \in V_0$, $u_{i-1} \in V_0$, $u_i \in V_0$, $v_i \in V_0$, $v_{i+1} \in V_0, \ldots, v_{i+k-1} \in V_0\}$,
- $\gamma_{dR}^{1\cdots 12}(G_i^k) = \min\{w(f) : f = (V_0, V_1, V_2, V_3)$ is a SDRDF of G_i^k, $v_{i-k} \in V_1$, $v_{i-k+1} \in V_1, \ldots, v_{i-1} \in V_1$, $u_{i-1} \in V_1$, $u_i \in V_1$, $v_i \in V_1$, $v_{i+1} \in V_1, \ldots, v_{i+k-1} \in V_2\}$, and
- $\gamma_{dR}^{3\cdots 3}(G_i^k) = \min\{w(f) : f = (V_0, V_1, V_2, V_3)$ is a SDRDF of G_i^k, $v_{i-k} \in V_3$, $v_{i-k+1} \in V_3, \ldots, v_{i-1} \in V_3$, $u_{i-1} \in V_3$, $u_i \in V_3$, $v_i \in V_3$, $v_{i+1} \in V_3, \ldots, v_{i+k-1} \in V_3\}$.

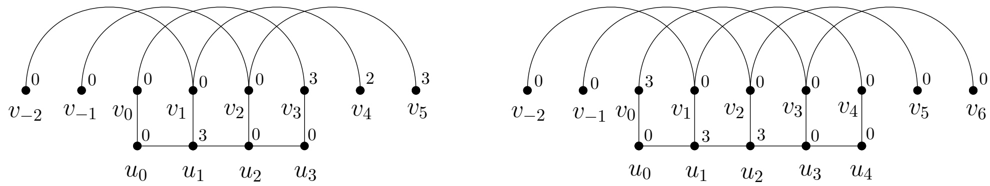
A $\gamma_{dR}^{0\cdots 0}(G_i^k)$-function is a minimum SDRDF $f = (V_0, V_1, V_2, V_3)$ of G_i^k such that $v_{i-k} \in V_0$, $v_{i-k+1} \in V_0, \ldots, v_{i-1} \in V_0$, $u_{i-1} \in V_0$, $u_i \in V_0$, $v_i \in V_0$, $v_{i+1} \in V_0, \ldots, v_{i+k-1} \in V_0$. Similarly, we define the others. See Fig. 3. Let $X_{n,k}$ be the set of all minimum SDRDF $f = (V_0, \emptyset, V_2, V_3)$ of $SGP(n, k)$ such that

(i) $f(u_j) = f(u_{n+j})$ for each $j \in \{0, 1\}$, and

(ii) $f(v_j) = f(v_{n+j})$ for each $j \in \{-k+1, -k+2, \ldots, k\}$.

The following proposition is clear.

Proposition 1. $|X_{n,k}| = 9^{k+1}$.



(a) (b)

Figure 3. **Illustrating (a) a $\gamma_{dR}^{00000323}(G_3^3)$-function and (b) a $\gamma_{dR}^{00000000}(G_4^3)$-function.**

Now, we can present our algorithm (Algorithm 3.1) for computing the double Roman domination number of the generalized Petersen graph $GP(n,k)$. The main idea of our algorithm is as follows. We first show that by using every function in $X_{n,k}$, we get a DRDF on $GP(n,k)$. The algorithm tries to find all $\gamma_{dR}^{b_{2k+2}\cdots b_2 b_1}(SGP(n,k))$-functions in $X_{n,k}$ for all $b_1, b_2, \ldots, b_{2k+2} \in \{0,2,3\}$. A minimum DRDF on $GP(n,k)$ exists between these functions. The algorithm uses a dynamic programming approach to compute the weight of these functions.

Algorithm 3.1: DRDN($GP(n,k)$)

Input: The generalized Petersen graph $GP(n,k) = (V,E)$.
Output: The duoble Roman domination number of $GP(n,k)$.

Let $SGP(n,k)$ be the semi generalized Petersen graph corresponding to $GP(n,k)$.
for $b_1, \ldots, b_{2k+2} \in \{0,2,3\}$ **do**
  Initialize $\gamma_{dR}^{b_{2k+2}\cdots b_1}(G_1^k)$ to be $b_1 + \cdots + b_{2k+2}$;
  for $(x_1, \ldots, x_{2k+2} \in \{0,2,3\}) \wedge (x_{2k+2}\cdots x_1 \neq b_{2k+2}\cdots b_1)$ **do**
    Initialize $\gamma_{dR}^{x_{2k+2}\cdots x_1}(G_1^k)$ to be ∞;
  for $i = 1$ **to** n **do**
    for $x_1, \ldots, x_{2k+2} \in \{0,2,3\}$ **do**
      Compute $\gamma_{dR}^{x_{2k+2}\cdots x_1}(G_{i+1}^k)$ by Lemma 3.
  $\gamma'_{b_{2k+2}\cdots b_1} = \gamma_{dR}^{b_{2k+2}\cdots b_1}(G_{n+1}^k)$;
return $\min\{\gamma'_{b_{2k+2}\cdots b_1} - (b_1 + \cdots + b_{2k+2}) : b_1, \ldots, b_k \in \{0,2,3\}\}$;

3.2. Correctness of Algorithm 3.1

In order to prove Algorithm 3.1 works correctly, we need the following lemmas. The next lemma is the main idea of our algorithm.

Lemma 2. *Let $GP(n,k) = (V,E)$ and let f be a function of $X_{n,k}$ such that $w(f_V) \leq w(g_V)$ for every function $g \in X_{n,k}$, where f_V and g_V are restrictions of f and g, respectively, to V. Then, f_V is a minimum DRDF of $GP(n,k)$.*

Proof. Recall $V_l = \{v_{1-k}, \ldots, v_0, u_0\}$ and $V_r = \{u_{n+1}, v_{n+1}, \ldots, v_{n+k}\}$. Let $f =$

$(V_0^f, \emptyset, V_2^f, V_3^f)$ and let f_V be the restriction of f to V. We first prove that f_V is a DRDF of $GP(n,k)$. By Note 1, we have $\deg_{SGP(n,k)}(w) = 3$ for every vertex $w \in V$. Let v be a vertex of V with label 0 under f. Since f is a SDRDF of $SGP(n,k)$, there is a vertex $u \in V_3^f$ adjacent to v or there are vertices $x, y \in V_2^f$ adjacent to v. We first assume that there is a vertex $u \in V_3^f$ adjacent to v. If $u \in V$, then there is nothing to be proven. Assume that $u \notin V$. So, $u \in V_l \cup V_r$. Assume without loss of generality that $u \in V_l$, that is, $u = v_j$ for some $1 - k \le j \le 0$ (respectively, $u = u_0$). By the definition of $SGP(n,k)$, $N_{SGP(n,k)}(v_j) = \{v_{j+k}\}$ if $j \neq 0$ and $N_{SGP(n,k)}(v_0) = \{v_k, u_0\}$ (respectively, $N_{SGP(n,k)}(u_0) = \{v_0, u_1\}$) and so $v = v_{j+k}$ (respectively, $v = u_1$) because of $v \in V$. Since $f \in X_{n,k}$ and $v_j \in V_3^f$ (respectively, $u_0 \in V_3^f$), we deduce that $v_{n+j} \in V_3^f$ (respectively, $u_n \in V_3^f$). Because $v_{n+j} \in N_{GP(n,k)}(v_{j+k})$ (respectively, $u_n \in N_{GP(n,k)}(u_1)$), hence, f_V is a DRDF of $GP(n,k)$. Similarly, if we assume that there are vertices $x, y \in V_2^f$ adjacent to v, then we deduce that f_V is a DRDF of $GP(n,k)$.

Now, we prove that f_V is a minimum DRDF of $GP(n,k)$. Suppose for a contradiction that f_V is a not a minimum DRDF of $GP(n,k)$. By Corollary 1, assume that $h = (V_0^h, \emptyset, V_2^h, V_3^h)$ is a minimum DRDF of $GP(n,k)$ with $w(h) < w(f_V)$. We construct h' as a SDRDF of $SGP(n,k)$ as follows. We set $h'(v)$ to $h(v)$ for each $v \in V$, $h'(u_{n+1})$ to $h(u_1)$, $h'(u_0)$ to $h(u_n)$, $h'(v_{n+j})$ to $h(v_j)$ for each $j \in \{1, 2, \ldots, k\}$ and $h'(v_{j-n})$ to $h(v_j)$ for each $j \in \{n-k+1, n-k+2, \ldots, n\}$. So, $h' \in X_{n,k}$. Clearly, h is the restriction of $h' \in X_{n,k}$ to V with $w(h) < w(f_V)$, a contradiction. This completes the proof of the lemma. □ □

In order to compute $w(f)$ of all functions $f \in X_{n,k}$ we need the following lemma.

Lemma 3. *Let* $b_1, b_2, \ldots, b_{2k+2} \in \{0,2,3\}$*, let* $i \in \{1, 2, \ldots, n+1\}$ *and let* $b_{k+3} + b_{k+2} \in \{3, 5, 6\}$*. Then,*

(a) $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 0}(G_{i+1}^k) = \gamma_{dR}^{3b_{2k+2}\cdots b_{k+4}300b_k\cdots b_2}(G_i^k)$,

(b) $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 2}(G_{i+1}^k) = \min\{\gamma_{dR}^{xb_{2k+2}\cdots b_{k+4}300b_k\cdots b_2}(G_i^k) : x \in \{2,3\}\} + 2$,

(c) $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 3}(G_{i+1}^k) = \min\{\gamma_{dR}^{xb_{2k+2}\cdots b_{k+4}300b_k\cdots b_2}(G_i^k) : x \in \{0,2,3\}\} + 3$,

(d) $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}002b_k\cdots b_2 0}(G_{i+1}^k) = \min\{\gamma_{dR}^{3b_{2k+2}\cdots b_{k+4}x00b_k\cdots b_2}(G_i^k) : x \in \{2,3\}\} + 2$,

(e) $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}002b_k\cdots b_2 2}(G_{i+1}^k) = \min\{\gamma_{dR}^{xb_{2k+2}\cdots b_{k+4}y00b_k\cdots b_2}(G_i^k) : x, y \in \{2,3\}\} + 4$,

(f) $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}002b_k\cdots b_2 3}(G_{i+1}^k) = \min\{\gamma_{dR}^{xb_{2k+2}\cdots b_{k+4}y00b_k\cdots b_2}(G_i^k) : x \in \{0,2,3\}, y \in \{2,3\}\} + 5$,

(g) $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}003b_k\cdots b_2 0}(G_{i+1}^k) = \min\{\gamma_{dR}^{3b_{2k+2}\cdots b_{k+4}x00b_k\cdots b_2}(G_i^k) : x \in \{0,2,3\}\} + 3$,

(h) $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}003b_k\cdots b_2 2}(G_{i+1}^k) = \min\{\gamma_{dR}^{xb_{2k+2}\cdots b_{k+4}y00b_k\cdots b_2}(G_i^k) : x \in \{2,3\}, y \in \{0,2,3\}\} + 5$,

(i) $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}003b_k\cdots b_2 3}(G_{i+1}^k) = \min\{\gamma_{dR}^{xb_{2k+2}\cdots b_{k+4}y00b_k\cdots b_2}(G_i^k) : x, y \in \{0,2,3\}\} + 6$,

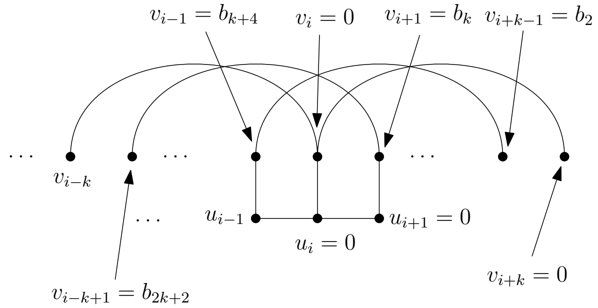


Figure 4. **Illustrating the subgraph G^k_{i+1}.**

(j) $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}02b_{k+1}b_k\cdots b_2 0}(G^k_{i+1}) = \min\{\gamma_{dR}^{xb_{2k+2}\cdots b_{k+4}y20b_k\cdots b_2}(G^k_i) : x \in \{2,3\}, y \in \{0,2,3\}\} + b_{k+1}$,

(k) $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}02b_{k+1}b_k\cdots b_2 x}(G^k_{i+1}) = \min\{\gamma_{dR}^{yb_{2k+2}\cdots b_{k+4}z20b_k\cdots b_2}(G^k_i) : y, z \in \{0,2,3\}\} + b_{k+1} + x$, *where* $x \in \{2,3\}$,

(l) $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}200b_k\cdots b_2 b_1}(G^k_{i+1}) = \min\{\gamma_{dR}^{xb_{2k+2}\cdots b_{k+4}y02b_k\cdots b_2}(G^k_i) : x \in \{0,2,3\}, y \in \{2,3\}\} + b_1$,

(m) $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}20xb_k\cdots b_2 b_1}(G^k_{i+1}) = \min\{\gamma_{dR}^{yb_{2k+2}\cdots b_{k+4}z02b_k\cdots b_2}(G^k_i) : y, z \in \{0,2,3\}\} + x + b_1$, *where* $x \in \{2,3\}$,

(n) $\gamma_{dR}^{b_{2k+2}\cdots b_2 b_1}(G^k_{i+1}) = \min\{\gamma_{dR}^{xb_{2k+2}\cdots b_{k+4}yb_{k+2}b_{k+3}b_k\cdots b_2}(G^k_i) : x, y \in \{0,2,3\}\} + b_{k+1} + b_1$.

Proof. In the rest of the proof assume that $j \in \{2,\ldots,k-1,k,k+4,\ldots,2k+1,2k+2\}$. We first prove (a). Let $f = (V_0, \emptyset, V_2, V_3)$ be a $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 0}(G^k_{i+1})$-function. So, all vertices $v_i, u_i, u_{i+1}, v_{i+k}$ are in V_0, the corresponding vertex to b_j is in V_0 if $b_j = 0$, is in V_2 if $b_j = 2$ and is in V_3 if $b_j = 3$. See Fig. 4. Since $N_{G^k_{i+1}}(v_i) = \{u_i, v_{i-k}, v_{i+k}\}$, $N_{G^k_{i+1}}(u_i) = \{u_{i-1}, u_{i+1}, v_i\}$ and f is a SDRDF of G^k_{i+1}, we deduce that both vertices v_{i-k} and u_{i-1} are in V_3. Let $f' = (V'_0, \emptyset, V'_2, V'_3)$ be the restriction of f to $\mathtt{V}_i = V(G^k_i)$. Hence, f' is a SDRDF of G^k_i such that the corresponding vertex to b_j is in V'_0 if $b_j = 0$, is in V'_2 if $b_j = 2$, is in V'_3 if $b_j = 3$, both vertices v_{i-k} and u_{i-1} are in V'_3 and both vertices v_i and u_i are in V'_0 and so $\gamma_{dR}^{3b_{2k+2}\cdots b_{k+4}300b_k\cdots b_2}(G^k_i) \le w(f') = w(f)$, that is,

$$\gamma_{dR}^{3b_{2k+2}\cdots b_{k+4}300b_k\cdots b_2}(G^k_i) \le \gamma_{dR}^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 0}(G^k_{i+1}). \quad (1)$$

Conversely, let $g = (V^g_0, \emptyset, V^g_2, V^g_3)$ be a $\gamma_{dR}^{3b_{2k+2}\cdots b_{k+4}300b_k\cdots b_2}(G^k_i)$-function. We deduce that $h = (V^h_0 = V^g_0 \cup \{u_{i+1}, v_{i+k}\}, \emptyset, V^h_2 = V^g_2, V^h_3 = V^g_3)$ is a SDRDF of G^k_{i+1} such that the corresponding vertex to b_j is in V^h_0 if $b_j = 0$, is in V^h_2 if $b_j = 2$, is in V^h_3 if $b_j = 3$ and all vertices v_i, u_i, v_{i+k} and u_{i+1} are in V^h_0 and so

$\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 0}(G_{i+1}^k) \leq w(h) = w(g)$, that is, $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 0}(G_{i+1}^k) \leq \gamma_{dR}^{3b_{2k+2}\cdots b_{k+4}300b_k\cdots b_2}(G_i^k)$. This, together with Inequality (1), completes the proof of (a).

Now, we prove (b). Let $f = (V_0, \emptyset, V_2, V_3)$ be a $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 2}(G_{i+1}^k)$-function. So, all vertices v_i, u_i, u_{i+1} are in V_0, $v_{i+k} \in V_2$, the corresponding vertex to b_j is in V_0 if $b_j = 0$, is in V_2 if $b_j = 2$ and is in V_3 if $b_j = 3$. Recall that $N_{G_{i+1}^k}(v_i) = \{u_i, v_{i-k}, v_{i+k}\}$ and $N_{G_{i+1}^k}(u_i) = \{u_{i-1}, u_{i+1}, v_i\}$. Because f is a SDRDF of G_{i+1}^k, we deduce that $u_{i-1} \in V_3$ and $v_{i-k} \in V_2 \cup V_3$. In the following we consider these cases. Let $f' = (V_0', \emptyset, V_2', V_3')$ be the restriction of f to $\mathtt{V}_i = V(G_i^k)$. We have $w(f') = w(f) - 2$.

- Assume $u_{i-1} \in V_3$ and $v_{i-k} \in V_2$. So, f' is a SDRDF of G_i^k such that the corresponding vertex to b_j is in V_0' if $b_j = 0$, is in V_2' if $b_j = 2$, is in V_3' if $b_j = 3$, $v_{i-k} \in V_2'$, $u_{i-1} \in V_3'$, $u_i \in V_0'$ and $v_i \in V_0'$ and so $\gamma_{dR}^{2b_{2k+2}\cdots b_{k+4}300b_k\cdots b_2}(G_i^k) \leq w(f') = w(f) - 2$, that is,

$$\gamma_{dR}^{2b_{2k+2}\cdots b_{k+4}300b_k\cdots b_2}(G_i^k) + 2 \leq \gamma_{dR}^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 2}(G_{i+1}^k). \tag{2}$$

- Assume $u_{i-1} \in V_3$ and $v_{i-k} \in V_3$. So,

$$\gamma_{dR}^{3b_{2k+2}\cdots b_{k+4}300b_k\cdots b_2}(G_i^k) + 2 \leq \gamma_{dR}^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 2}(G_{i+1}^k). \tag{3}$$

Conversely, let $g_0 = (V_0^0, \emptyset, V_2^0, V_3^0)$ be a $\gamma_{dR}^{2b_{2k+2}\cdots b_{k+4}300b_k\cdots b_2}(G_i^k)$-function and let $g_1 = (V_0^1, \emptyset, V_2^1, V_3^1)$ be a $\gamma_{dR}^{3b_{2k+2}\cdots b_{k+4}300b_k\cdots b_2}(G_i^k)$-function. We deduce that $h_0 = (V_0^3 = V_0^0 \cup \{u_{i+1}\}, \emptyset, V_2^3 = V_2^0 \cup \{v_{i+k}\}, V_3^3 = V_3^0)$ is a SDRDF of G_{i+1}^k such that the corresponding vertex to b_j is in V_0^3 if $b_j = 0$, is in V_2^3 if $b_j = 2$, is in V_3^3 if $b_j = 3$, all vertices v_i, u_i, u_{i+1} are in V_0^3 and $v_{i+k} \in V_2^3$ and so $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 2}(G_{i+1}^k) \leq w(h_0) = w(g_0) + 2$, that is,

$$\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 2}(G_{i+1}^k) \leq \gamma_{dR}^{2b_{2k+2}\cdots b_{k+4}300b_k\cdots b_2}(G_i^k) + 2. \tag{4}$$

Let $h_1 = (V_0^4 = V_0^1 \cup \{u_{i+1}\}, \emptyset, V_2^4 = V_2^1, V_3^4 = V_3^1 \cup \{v_{i+k}\})$. Similarly, we obtain that $\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 2}(G_{i+1}^k) \leq w(h_1) = w(g_1) + 3$, that is,

$$\gamma_{dR}^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 2}(G_{i+1}^k) \leq \gamma_{dR}^{3b_{2k+2}\cdots b_{k+4}300b_k\cdots b_2}(G_i^k) + 3. \tag{5}$$

Inequalities (2)–(5) complete the proof of (b).

Similarly, we can prove the other cases, i.e., (c)–(n). This completes the proof of the lemma. □ □

Now, we are in a position to compute all functions of $X_{n,k}$. Let $i \in \{1, 2, \ldots, n+1\}$. Note that, in Lemma 3, for all $b_1, b_2, \ldots, b_{2k+2} \in \{0, 2, 3\}$ we compute $\gamma_{dR}^{b_{2k+2}\cdots b_2 b_1}(G_{i+1}^k)$ by using some values $\gamma_{dR}^{a_{2k+2}\cdots a_2 a_1}(G_i^k)$, where $a_1, a_2, \ldots, a_{2k+2} \in \{0, 2, 3\}$.

Lemma 4. *Let $b_1, \ldots, b_{2k+2} \in \{0,2,3\}$ and let $g = (V_0^g, \emptyset, V_2^g, V_3^g) \in X_{n,k}$ be a $\gamma_{dR}^{b_{2k+2}\cdots b_1}(SGP(n,k))$-function. We can compute g in $O(n9^k)$ time and space.*

Proof. Recall that $X_{n,k}$ is the set of all minimum SDRDF $f = (V_0, \emptyset, V_2, V_3)$ of $SGP(n,k)$ such that

(*i*) $f(u_j) = f(u_{n+j})$ for each $j \in \{0,1\}$, and

(*ii*) $f(v_j) = f(v_{n+j})$ for each $j \in \{-k+1, -k+2, \ldots, k\}$.

Since g is in $X_{n,k}$ and a $\gamma_{dR}^{b_{2k+2}\cdots b_1}(SGP(n,k))$-function, $g(u_j) = g(u_{n+j}) = b_{k+2-j}$ for $j \in \{0,1\}$, $g(v_j) = g(v_{n+j}) = b_{k+3-j}$ for $j \in \{-k+1, -k+2, \ldots, 0\}$, and $g(v_j) = g(v_{n+j}) = b_{k+1-j}$ for $j \in \{1, \ldots, k\}$. By Lemma 3 and using a dynamic programming approach, we compute $w(g)$. We initialize $\gamma_{dR}^{b_{2k+2}\cdots b_1}(G_1^k)$ to be $b_1 + \cdots + b_{2k+2}$ and $\gamma_{dR}^{x_{2k+2}\cdots x_1}(G_1^k)$ to be ∞ for each $x_1, \ldots, x_{2k+2} \in \{0,2,3\}$ and $x_{2k+2}\cdots x_1 \neq b_{2k+2}\cdots b_1$. Then, by Lemma 3, compute $\gamma_{dR}^{x_{2k+2}\cdots x_1}(G_{i+1}^k)$ in a constant time for all $x_1, \ldots, x_{2k+2} \in \{0,2,3\}$ and each $i = 1, \ldots, n$, respectively. In the end of this process, we obtain that $w(g) = \gamma_{dR}^{b_{2k+2}\cdots b_1}(G_{n+1}^k)$, where $SGP(n,k) = G_{n+1}^k$.
When we obtain $w(g)$, then using a backtracking search algorithm on values $\gamma_{dR}^{x_{2k+2}\cdots x_1}(G_i^k)$ for all $x_1, \ldots, x_{2k+2} \in \{0,2,3\}$ and $i \in \{1, \ldots, n+1\}$, we can compute g. This process needs $O(n9^k)$ time and space. This completes the proof of the lemma.
□ □

Example 1. In Table 1, we see all steps of the execution of Algorithm 3.1 on $GP(4,1)$ for computing a function $g = (V_0, \emptyset, V_2, V_3) \in X_{4,1}$ such that g is a $\gamma_{dR}^{0003}(SGP(4,1))$-function, i.e., for $(b_4b_3b_2b_1) = (0003)$ in the for-loop of Line 2. As seen from Table 1, $w(g) = 9$ and by using a backtracking search algorithm (circled integers), we get that
$\begin{bmatrix} g(v_0) & g(v_1) & g(v_2) & g(v_3) & g(v_4) & g(v_5) \\ g(u_0) & g(u_1) & g(u_2) & g(u_3) & g(u_4) & g(u_5) \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 & 0 & 0 \end{bmatrix}$.

Theorem 2. *There is an algorithm to compute a minimum DRDF of the generalized Petersen graph $GP(n,k)$ in $O(n81^k)$ time and space.*

Proof. Let $b_1, \ldots, b_{2k+2} \in \{0,2,3\}$ and $GP(n,k) = (V,E)$. By Lemma 4, Algorithm 3.1 on input $GP(n,k)$ in Line 9 computes $w(g^{b_{2k+2}\cdots b_1})$, where $g^{b_{2k+2}\cdots b_1} = (V_0, \emptyset, V_2, V_3) \in X_{n,k}$ is a $\gamma_{dR}^{b_{2k+2}\cdots b_1}(SGP(n,k))$-function. Let $g_V^{b_{2k+2}\cdots b_1}$ be the restriction of $g^{b_{2k+2}\cdots b_1}$ to V. By the definition of $\gamma_{dR}^{b_{2k+2}\cdots b_1}(SGP(n,k))$-function and $X_{n,k}$, we deduce that $w(g_V^{b_{2k+2}\cdots b_1}) = w(g^{b_{2k+2}\cdots b_1}) - (b_1 + \cdots + b_{2k+2})$. By Lemma 2, $\gamma_{dR}(GP(n,k)) = \min\{w(f_V) : f \in X_{n,k}\} = \min\{w(f_V^{x_{2k+2}\cdots x_1}) : x_1, \ldots, x_{2k+2} \in \{0,2,3\}, f \in X_{n,k}\}$. So, Algorithm 3.1 on input $GP(n,k)$ in Line 10 returns the double Roman domination number of $GP(n,k)$.
By Lemma 4, we obtain $g^{b_{2k+2}\cdots b_1}$ in $O(n9^k)$ time and space and so we can compute a minimum DRDF of $GP(n,k)$ in $O(n81^k)$ time and space. This completes the proof of the theorem. □ □

i	1	2	3	4	5	i	1	2	3	4	5	i	1	2	3	4	5
$\gamma_{dR}^{0000}(G_i^1)$	∞	∞	∞	9	9	$\gamma_{dR}^{0020}(G_i^1)$	∞	∞	∞	10	11	$\gamma_{dR}^{0030}(G_i^1)$	∞	∞	⑥	9	11
$\gamma_{dR}^{0002}(G_i^1)$	∞	∞	∞	10	10	$\gamma_{dR}^{0022}(G_i^1)$	∞	∞	∞	11	12	$\gamma_{dR}^{0032}(G_i^1)$	∞	∞	8	11	12
$\gamma_{dR}^{0003}(G_i^1)$	③	∞	∞	9	⑨	$\gamma_{dR}^{0023}(G_i^1)$	∞	∞	∞	10	11	$\gamma_{dR}^{0033}(G_i^1)$	∞	∞	9	11	12
$\gamma_{dR}^{0200}(G_i^1)$	∞	∞	5	7	9	$\gamma_{dR}^{0220}(G_i^1)$	∞	∞	7	9	11	$\gamma_{dR}^{0230}(G_i^1)$	∞	∞	8	10	12
$\gamma_{dR}^{0202}(G_i^1)$	∞	∞	7	9	10	$\gamma_{dR}^{0222}(G_i^1)$	∞	∞	9	11	12	$\gamma_{dR}^{0232}(G_i^1)$	∞	∞	10	12	13
$\gamma_{dR}^{0203}(G_i^1)$	∞	∞	8	10	11	$\gamma_{dR}^{0223}(G_i^1)$	∞	∞	10	12	13	$\gamma_{dR}^{0233}(G_i^1)$	∞	∞	11	13	14
$\gamma_{dR}^{0300}(G_i^1)$	∞	∞	6	⑥	9	$\gamma_{dR}^{0320}(G_i^1)$	∞	∞	8	8	11	$\gamma_{dR}^{0330}(G_i^1)$	∞	∞	9	9	12
$\gamma_{dR}^{0302}(G_i^1)$	∞	∞	8	8	11	$\gamma_{dR}^{0322}(G_i^1)$	∞	∞	10	10	13	$\gamma_{dR}^{0332}(G_i^1)$	∞	∞	11	11	14
$\gamma_{dR}^{0303}(G_i^1)$	∞	∞	9	9	12	$\gamma_{dR}^{0323}(G_i^1)$	∞	∞	11	11	14	$\gamma_{dR}^{0333}(G_i^1)$	∞	∞	12	12	15
$\gamma_{dR}^{2000}(G_i^1)$	∞	∞	∞	7	8	$\gamma_{dR}^{2020}(G_i^1)$	∞	∞	7	9	10	$\gamma_{dR}^{2030}(G_i^1)$	∞	∞	8	10	11
$\gamma_{dR}^{2002}(G_i^1)$	∞	∞	∞	9	10	$\gamma_{dR}^{2022}(G_i^1)$	∞	∞	9	11	12	$\gamma_{dR}^{2032}(G_i^1)$	∞	∞	10	12	13
$\gamma_{dR}^{2003}(G_i^1)$	∞	∞	∞	10	11	$\gamma_{dR}^{2023}(G_i^1)$	∞	∞	10	12	13	$\gamma_{dR}^{2033}(G_i^1)$	∞	∞	11	13	14
$\gamma_{dR}^{2200}(G_i^1)$	∞	∞	7	9	10	$\gamma_{dR}^{2220}(G_i^1)$	∞	∞	9	11	12	$\gamma_{dR}^{2230}(G_i^1)$	∞	∞	10	12	13
$\gamma_{dR}^{2202}(G_i^1)$	∞	∞	9	11	12	$\gamma_{dR}^{2222}(G_i^1)$	∞	∞	11	13	14	$\gamma_{dR}^{2232}(G_i^1)$	∞	∞	12	14	15
$\gamma_{dR}^{2203}(G_i^1)$	∞	∞	10	12	13	$\gamma_{dR}^{2223}(G_i^1)$	∞	∞	12	14	15	$\gamma_{dR}^{2233}(G_i^1)$	∞	∞	13	15	16
$\gamma_{dR}^{2300}(G_i^1)$	∞	∞	8	8	11	$\gamma_{dR}^{2320}(G_i^1)$	∞	∞	10	10	13	$\gamma_{dR}^{2330}(G_i^1)$	∞	∞	11	11	14
$\gamma_{dR}^{2302}(G_i^1)$	∞	∞	10	10	13	$\gamma_{dR}^{2322}(G_i^1)$	∞	∞	12	12	15	$\gamma_{dR}^{2332}(G_i^1)$	∞	∞	13	13	16
$\gamma_{dR}^{2303}(G_i^1)$	∞	∞	11	11	14	$\gamma_{dR}^{2323}(G_i^1)$	∞	∞	13	13	16	$\gamma_{dR}^{2333}(G_i^1)$	∞	∞	14	14	17
$\gamma_{dR}^{3000}(G_i^1)$	∞	③	6	8	9	$\gamma_{dR}^{3020}(G_i^1)$	∞	5	8	10	11	$\gamma_{dR}^{3030}(G_i^1)$	∞	6	9	11	12
$\gamma_{dR}^{3002}(G_i^1)$	∞	5	8	10	11	$\gamma_{dR}^{3022}(G_i^1)$	∞	7	10	12	13	$\gamma_{dR}^{3032}(G_i^1)$	∞	8	11	13	14
$\gamma_{dR}^{3003}(G_i^1)$	∞	6	9	11	12	$\gamma_{dR}^{3023}(G_i^1)$	∞	8	11	13	14	$\gamma_{dR}^{3033}(G_i^1)$	∞	9	12	14	15
$\gamma_{dR}^{3200}(G_i^1)$	∞	∞	8	10	10	$\gamma_{dR}^{3220}(G_i^1)$	∞	∞	10	12	12	$\gamma_{dR}^{3230}(G_i^1)$	∞	∞	11	13	13
$\gamma_{dR}^{3202}(G_i^1)$	∞	∞	10	12	12	$\gamma_{dR}^{3222}(G_i^1)$	∞	∞	12	14	14	$\gamma_{dR}^{3232}(G_i^1)$	∞	∞	13	15	15
$\gamma_{dR}^{3203}(G_i^1)$	∞	∞	11	13	13	$\gamma_{dR}^{3223}(G_i^1)$	∞	∞	13	15	15	$\gamma_{dR}^{3233}(G_i^1)$	∞	∞	14	16	16
$\gamma_{dR}^{3300}(G_i^1)$	∞	∞	9	9	11	$\gamma_{dR}^{3320}(G_i^1)$	∞	∞	11	11	13	$\gamma_{dR}^{3330}(G_i^1)$	∞	∞	12	12	14
$\gamma_{dR}^{3302}(G_i^1)$	∞	∞	11	11	13	$\gamma_{dR}^{3322}(G_i^1)$	∞	∞	13	13	15	$\gamma_{dR}^{3332}(G_i^1)$	∞	∞	14	14	16
$\gamma_{dR}^{3303}(G_i^1)$	∞	∞	12	12	14	$\gamma_{dR}^{3323}(G_i^1)$	∞	∞	14	14	16	$\gamma_{dR}^{3333}(G_i^1)$	∞	∞	15	15	17

Table 1. **Some steps of the execution of Algorithm 3.1.**

By Theorem 2 we have the following result.

Corollary 2. *There is an algorithm to compute a minimum DRDF of the generalized Petersen graph $GP(n, O(1))$ in $O(n)$ time and space.*

References

[1] H. Abdollahzadeh Ahangar, M. Chellali, and S.M. Sheikholeslami, *On the double Roman domination in graphs*, Discrete Appl. Math. **232** (2017), 1–7.

[2] ______, *Outer independent double Roman domination*, Appl. Math. Comput. **364** (2020), ID: 124617.

[3] H. Abdollahzadeh Ahangar, M. Chellali, S.M. Sheikholeslami, and J.C. Valenzuela-Tripodoro, *Maximal double Roman domination in graphs*, Appl. Math. Comput. **414** (2022), ID: 126662.

[4] S. Banerjee, M.A. Henning, and D. Pradhan, *Algorithmic results on double Roman domination in graphs*, J. Comb. Optim. **39** (2020), no. 1, 90–114.

[5] R.A. Beeler, T.W. Haynes, and S.T. Hedetniemi, *Double Roman domination*, Discrete Appl. Math. **211** (2016), 23–29.

[6] G. Hao, L. Volkmann, and D.A. Mojdeh, *Total double Roman domination in graphs*, Commun. Comb. Optim. **5** (2020), no. 1, 27–39.

[7] N. Jafari Rad and H. Rahbani, *Some progress on the double Roman domination in graphs*, Discuss. Math. Graph Theory **39** (2019), no. 1.

[8] R. Khoeilar, H. Karami, M. Chellali, and S.M. Sheikholeslami, *An improved upper bound on the double Roman domination number of graphs with minimum degree at least two*, Discrete Appl. Math. **270** (2019), 159–167.

[9] S. Kosari, Z. Shao, S.M. Sheikholeslami, M. Chellali, R. Khoeilar, and H. Karami, *Double Roman domination in graphs with minimum degree at least two and no c_5-cycle*, Graphs Combin. **38** (2022), no. 2, 1–16.

[10] Bojan Mohar, *Face covers and the genus problem for apex graphs*, J. Combin. Theory Ser. B **82** (2001), no. 1, 102–117.

[11] C. Padamutham and V.S.R. Palagiri, *Complexity of Roman {2}-domination and the double Roman domination in graphs*, AKCE Int. J. Graphs Comb. **17** (2020), no. 3, 1081–1086.

[12] A. Poureidi and N. Jafari Rad, *On algorithmic complexity of double Roman domination*, Discrete Appl. Math. **285** (2020), 539–551.

[13] L. Volkmann, *Double Roman domination and domatic numbers of graphs*, Commun. Comb. Optim. **3** (2018), no. 1, 71–77.

[14] M.E. Watkins, *A theorem on Tait colorings with an application to the generalized Petersen graphs*, J. Combin. Theory **6** (1969), no. 2, 152–164.

[15] X. Zhang, Z. Li, H. Jiang, and Z. Shao, *Double Roman domination in trees*, Inform. Process. Lett. **134** (2018), 31–34.