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Research Article

On local antimagic chromatic number of various join graphs

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Abstract: A local antimagic edge labeling of a graph $G = (V, E)$ is a bijection $f : E \to \{1, 2, \dots, |E|\}$ such that the induced vertex labeling $f^+ : V \to \mathbb{Z}$ given by $f^+(u) = \sum f(e)$, where the summation runs over all edges e incident to u, has the property that any two adjacent vertices have distinct labels. A graph G is said to be locally antimagic if it admits a local antimagic edge labeling. The local antimagic chromatic number $\chi_{la}(G)$ is the minimum number of distinct induced vertex labels over all local antimagic labelings of G. In this paper we obtain sufficient conditions under which $\chi_{la}(G \vee H)$, where H is either a cycle or the empty graph $O_n = \overline{K_n}$, satisfies a sharp upper bound. Using this we determine the value of $\chi_{la}(G \vee H)$ for many wheel related graphs G.

Keywords: Local antimagic chromatic number, join product, wheels, fans.

AMS Subject classification: 05C78, 05C69

1. Introduction

A connected graph $G = (V, E)$ is said to be *local antimagic* if it admits a *local antimagic edge labeling*, i.e., a bijection $f : E \to \{1, \dots, |E|\}$ such that the induced

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vertex labeling $f^+ : V \to \mathbb{Z}$ given by $f^+(u) = \sum f(e)$ (with e ranging over all the edges incident to u) has the property that any two adjacent vertices have distinct induced vertex labels (see [1, 12]). Thus, f^+ is a coloring of G. Clearly, the order of G must be at least 3. The vertex label $f^+(u)$ is called the *induced color* of u under f (the *color* of u, for short, if no ambiguity occurs). The number of distinct induced colors under f is denoted by $c(f)$, and is called the *color number* of f. Such an f is also called a *local antimagic* $c(f)$*-coloring* of G. The *local antimagic chromatic number* of G, denoted by $\chi_{la}(G)$, is $\min\{c(f) \mid f$ is a local antimagic labeling of $G\}$. In [9] and [15], further results on local antimagic chromatic number are given. Local antimagic chromatic number of some join graphs and disconnected graphs are presented in [14] and [2] respectively. A conjecture on local antimagic labeling was proposed in [1] and Haslegrave [5] proved this conjecture. Local antimagic labeling is a relaxation of antimagic labeling. Several types of antimagic labeling have been extensively investigated and in [6] the authors investigated the existence of one type of antimagic labeling for the Cartesian product of a path and a wheel.

Throughout this paper, we let P_m be the path of order $m \geq 2$, C_n be the cycle of order $n \geq 3$, and $O_n = \overline{K_n}$ be the null graph of order $n \geq 1$ with vertices v_j, $1 \leq j \leq n$. For any two graphs G and H, the join graph $G \vee H$ is defined by $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$. For $m \geq 3$, the wheel graph of order $m+1$ is $W_m = C_m \vee K_1$ and the fan graph is $F_m = P_m \vee K_1$. Note that F_m is also the graph W_m with an edge of C_m deleted. For integers $a < b$, $[a, b]$ denotes the set of integers between a and b. For notations and concepts not defined in this paper we refer to the book [3].

Let G be a graph of order $m \geq 3$. In [8, Theorem 3], the authors gave sufficient conditions for $\chi_{la}(G \vee O_n) = \chi_{la}(G) + 1$ in terms of m and n as follows.

Theorem 1. [8] *Suppose G is of order $m \geq 3$ with $m \equiv n \pmod 2$ and $\chi(G) = \chi_{la}(G)$. If (i) $n \geq m$, or (ii) $m \geq n^2/2$ and $n \geq 4$, then $\chi_{la}(G \vee O_n) = \chi_{la}(G) + 1$.*

Note that condition (ii) above is not applicable for sufficiently small m that is greater than n. Motivated by this, in this paper, we obtained new sufficient conditions for sharp upper bounds of $\chi_{la}(G \vee O_n)$. This then allows us to determine the local antimagic chromatic number of $G \vee O_n$ for m and n not satisfying condition (ii) above. Further, we obtained sufficient conditions for sharp upper bounds of $\chi_{la}(G \vee C_n)$ for $n \geq 3$. Consequently, we obtained $\chi_{la}(G \vee H)$ for many wheel related graphs G and $H \in \{O_n, C_n\}$ where $|V(G)| \equiv |V(H)| \pmod 2$. Interested readers may refer to [7] for more results with $|V(G)| \not\equiv |V(H)| \pmod 2$.

If G is a graph with $\chi_{la}(G) = t \geq 2$ and f is a local antimagic labeling of G that induced t distinct vertex colors, then $V_f = \{V_1, \ldots, V_t\}$ is the partition of $V(G)$ such that every vertex in each V_i has the same induced color under f. For $t \geq 2$, consider the following conditions for a graph G:

(i) $\chi_{la}(G) = t$ and f is a local antimagic labeling of G that induces a t-independent partition $\bigcup_{i=1}^{t} V_i$ of $V(G)$.

(ii) For each $x \in V_k$, $1 \leq k \leq t$, $\deg(x) = d_k$ satisfying $f^+(x) - d_a \neq f^+(y) - d_b$, where $x \in V_a$ and $y \in V_b$ for $1 \leq a \neq b \leq t$.

(iii) There exist two non-adjacent vertices u, v with $u \in V_i, v \in V_j$ for some $1 \leq i \neq j \leq t$ such that

(a) $|V_i| = |V_j| = 1$ and $\deg(x) = d_k$ for $x \in V_k$, $1 \leq k \leq t$; or

(b) $|V_i| = 1$, $|V_j| \geq 2$ and $\deg(x) = d_k$ for $x \in V_k$, $1 \leq k \leq t$ except that $\deg(v) = d_j - 1$; or

(c) $|V_i| \geq 2$, $|V_j| \geq 2$ and $\deg(x) = d_k$ for $x \in V_k$, $1 \leq k \leq t$ except that $\deg(u) = d_i - 1$, $\deg(v) = d_j - 1$,

each satisfying $f^+(x) + d_a \neq f^+(y) + d_b$, where $x \in V_a$ and $y \in V_b$ for $1 \leq a \neq b \leq t$.

Lemma 1. [11] *Let e be an edge of G. If G satisfies Conditions (i) and (ii) and $f(e) = 1$, then $\chi_{la}(G - e) \leq t$.*

2. Graphs join with null graphs

The following lemma is obvious.

Lemma 2. *Let A be a $p \times r$ magic rectangle using integers in $[1, rp]$. Let R and C be the row sum and column sum of A, respectively. Then $R - C = \frac{1}{2}(r - p)(rp + 1)$.*

It was shown in [4] that a $p \times r$ magic rectangle exists whenever p and r have the same parity, except for the impossible cases where exactly one of p and r is 1, and for $p = r = 2$.

Theorem 2. *Let G be a connected graph of order p and size q. Suppose G admits a local antimagic t-coloring f. Without loss of generality, let $f^+(x_1) \leq f^+(x_2) \leq \cdots \leq f^+(x_{p-1}) \leq f^+(x_p)$, where x_i for $i \in [1, p]$ are vertices of G. Let $r \geq 2$ and $p \equiv r \pmod 2$. Then $\chi_{la}(G \vee O_r) \leq t + 1$ if either when $r - p \geq 0$ or when $p - r \geq 2$ and f satisfies the following two conditions:*

(a) $f^+(x_{p-1}) \leq 4p - 2$, and

(b) $2f^+(x_p) \neq (p - r)(rp + 2q + 1)$.

Proof. Let $V(O_r) = \{v_j \mid 1 \leq j \leq r\}$. Define $g : E(G \vee O_r) \to [1, rp + q]$ by

$$g(e) = \begin{cases} f(e) & \text{if } e \in E(G); \\ a_{ij} + q & \text{if } e = x_i v_j, i \in [1,p], j \in [1,n], \end{cases}$$

where (a_{ij}) is a $p \times r$ magic rectangle with $a_{ij} \in [1, rp]$ and whose row sum and column sum are R and C, respectively. So, $g^+(x_i) = f^+(x_i) + R + rq$ and $g^+(v_j) = C + pq$ for $i \in [1,p]$ and $j \in [1,r]$. Thus $g^+(x_i) = g^+(x_i')$ if and only if $f^+(x_i) = f^+(x_i')$. From Lemma 2 we have

$$\begin{aligned} g^+(x_i) - g^+(v_j) &= f^+(x_i) + rq - pq + R - C \\ &= f^+(x_i) + \frac{1}{2}(r-p)(rp + 2q + 1). \end{aligned} \tag{1}$$

Suppose $r - p \geq 0$. It is clear that $g^+(v_j) < g^+(x_1) \leq g^+(x_2) \leq \cdots \leq g^+(x_{p-1}) \leq g^+(x_p)$ for $j \in [1,r]$.
Suppose $p - r \geq 2$. Since G is connected, $q \geq p - 1$. From (1) and condition (a),

$$\begin{aligned} g^+(x_{p-1}) - g^+(v_j) &= f^+(x_{p-1}) + \frac{1}{2}(r-p)(rp + 2q + 1) \\ &\leq f^+(x_{p-1}) - rp - 2p + 1 \leq f^+(x_{p-1}) - 4p + 1 < 0. \end{aligned}$$

From (1) and condition (b), $g^+(x_p) - g^+(v_j) = f^+(x_p) + \frac{1}{2}(r-p)(rp + 2q + 1) \neq 0$. So, g is a local antimagic labeling of $G \vee O_r$ inducing $t + 1$ colors. Hence we have the theorem. □

By a similar proof of Theorem 2 we have:

Theorem 3. *Let G be a connected graph of order p and size q. Let $r \geq 2$ and $p \equiv r \pmod 2$. Suppose G admits a local antimagic t-coloring f. Then $\chi_{la}(G \vee O_r) \leq t+1$ if either when $r - p \geq 0$ or when $p - r \geq 2$ and $2f^+(x) \neq (p-r)(rp + 2q + 1)$ for each $x \in V(G)$.*

Corollary 1. *For $m \geq 2$ and $n \geq 1$, $\chi_{la}(W_{2m} \vee O_{2n-1}) = 4$.*

Proof. When $n = 1$, then $G = W_{2m} \vee O_1 = C_{2m} \vee K_2$ and the result follows from Theorem 3.10 in [11]. Thus we only consider $n \geq 2$.

Let $V(W_{2m}) = \{v\} \cup \{u_i \,|\, 1 \leq i \leq 2m\}$ and $E(W_{2m}) = \{vu_i, u_i u_{i+1} \,|\, 1 \leq i \leq 2m\}$, where $u_{2m+1} = u_1$.

Suppose $m = 2k$. Let f_1 be the local antimagic 3-coloring of W_{4k} defined in the proof of [8, Theorem 5], in which $f_1^+(v) = 20$, $f_1^+(u_{2l}) = 15$ and $f_1^+(u_{2l-1}) = 11$ for $l = 1, 2$ when $k = 1$; and $f_1^+(v) = 2k(12k+1)$, $f_1^+(u_{2l}) = 11k + 1$ and $f_1^+(u_{2l-1}) = 9k + 2$ for $1 \leq l \leq 2k$ when $k \geq 2$. For $m \geq 4$, it is easy to check that W_{4k} admits a local antimagic 3-coloring $h_1 = 8k + 1 - f_1$ with induced vertex colors $h_1^+(v) = 2k(4k+1)$, $h_1^+(u_{2l}) = 13k + 2$ and $h_1^+(u_{2l-1}) = 15k + 1$. Moreover, label 1 is assigned to a spoke of W_{4k}.

Suppose $m = 2k+1$. Let f_2 be the local antimagic 3-coloring of W_{4k+2} defined in the proof of [1, Theorem 2.14], in which $f_2^+(v) = (2k+1)(12k+7)$, $f_2^+(u_{2l}) = 11k+7$ and $f_2^+(u_{2l-1}) = 9k+6$ for $1 \le l \le 2k+1$. It is easy to check that W_{4k+2} admits a local antimagic 3-coloring $h_2 = 8k+5-f_2$ with induced vertex colors $h_2^+(v) = (2k+1)(4k+3)$, $h_2^+(u_{2l}) = 13k+8$ and $h_2^+(u_{2l-1}) = 15k+9$. Moreover, label 1 is assigned to a spoke of W_{4k+2}.

In order to show $\chi_{la}(W_{2m} \vee O_{2n-1}) \le 4$, by Theorem 2 we only need to consider $p-r = 2(m-n+1) \ge 2$, i.e., $m \ge n$.

We denote f_1 (of W_4) or h_1 or h_2 by f. It is easy to check that f satisfies condition (a) of Theorem 2. We are going to check the condition (b) of Theorem 2. It is easy to see that $f^+(v) = m(2m+1)$ when $m \ge 3$.

$$\begin{aligned}\frac{1}{2}(p-r)(rp+2q+1) - f^+(v) &= (m-n+1)(4mn+6m+2n) - m(2m+1)\\ &= 4mn(m-n) + 4m^2 + 5m - 2n^2 + 2n > 0.\end{aligned}$$

When $m = 2$, then $f^+(v) = 20$ and $n = 2$. Thus $\frac{1}{2}(p-r)(rp+2q+1) - f^+(v) = (4mn+6m+2n) - 20 = 12$. Thus condition (b) holds.

By Theorem 2, $\chi_{la}(W_{2m} \vee O_{2n-1}) \le 4$. Since $\chi(W_{2m} \vee O_{2n-1}) = 4$, $\chi_{la}(W_{2m} \vee O_{2n-1}) = 4$. □

In this paper, we shall keep the notation related to W_s defined above for $s \ge 3$.

Example 1. The labeling matrix of $W_4 \vee O_5$ under g is given below.

	u_1	u_2	u_3	u_4	v	v_1	v_2	v_3	v_4	v_5	$f^+(u_i)$
u_1	$*$	7	$*$	3	1	31	13	15	22	24	116
u_2	7	$*$	2	$*$	6	12	14	21	28	30	120
u_3	$*$	2	$*$	4	5	18	20	27	29	11	116
u_4	3	$*$	4	$*$	8	19	26	33	10	17	120
v	1	6	5	8	$*$	25	32	9	16	23	125
$f^+(v_j)$	$*$	$*$	$*$	$*$	$*$	105	105	105	105	105	

Note that $W_{2m} \vee O_1 = C_{2m} \vee K_2$. Suppose e is an edge of the K_2, then $(W_{2m} \vee O_1) - e = C_{2m} \vee O_2$. By Theorem 3.3 [11], we have $\chi_{la}((W_{2m} \vee O_1) - e) = 3$. For $n \ge 2$, if e is an edge of the C_{2m} subgraph of W_{2m}, then $(W_{2m} \vee O_{2n-1}) - e = F_{2m} \vee O_{2n-1}$ as in Corollaries 6 and 7.

Corollary 2. *Suppose $m, n \ge 2$. If e is a spoke of W_{2m}, then $\chi_{la}((W_{2m} \vee O_{2n-1}) - e) = 4$.*

Proof. Note that $(W_{2m} \vee O_{2n-1}) - e = (W_{2m} - e) \vee O_{2n-1}$. Since $\chi((W_{2m} \vee O_{2n-1}) - e) = 4$, we only need to show that $\chi_{la}((W_{2m} \vee O_{2n-1}) - e) \le 4$.
From Corollary 1 we know that there is a local antimagic 3-coloring η for W_{2m} such that $\eta(e) = 1$. Let $F = \eta - 1$ be a labeling for $W_{2m} - e$. Then $F^+(x) = \eta^+(x) - \deg_{W_{2m}}(x)$, $x \in V(W_{2m})$.
For this case, the labeling η is f_1 or h_1 or h_2 corresponding to $m = 2$ or $m = 2k \ge 4$ or $m = 2k + 1 \ge 3$, which are described in the proof of Corollary 1. According to Theorem 2, $p = 2m + 1$, $q = 4m - 1$, $r = 2n - 1$, $A = (p - r)(rp + 2q + 1) = 2(m - n + 1)(4mn + 2n + 6m - 2)$ for the graph $W_{2m} - e$. We only need to check $A \neq 2F^+(x_p)$ if F is a local antimagic labeling when $p - r \ge 2$, i.e., $m \ge n$. We have the following cases:

1. $m = 2$, $F^+(v) = 16$, $F^+(u_{j_e}) = 12$ and $F^+(u_{j_o}) = 8$, where j_e is even and j_o is odd. So F is a local antimagic 3-coloring. Here $p = 5$, $q = 7$, $4p - 2 = 18$ and $x_p = v$.

$$A - 2F^+(v) = 2(3 - n)(10n + 10) - 32 = 20n(2 - n) + 28 > 0.$$

2. $m = 4$. We cannot use F defined before, because it is not local antimagic. We use the following labeling F for $W_8 - e$ given in Figure 1, which was defined in [11].

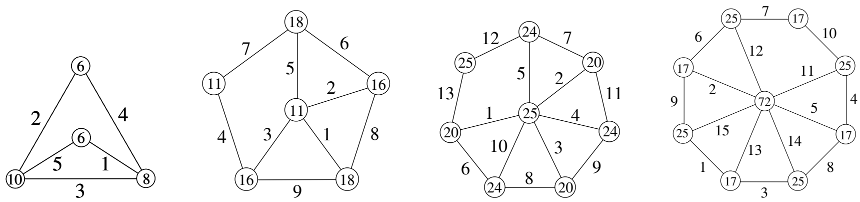

Figure 1. **Local antimagic 3-colorings for $W_3 - e$, $W_5 - e$, $W_7 - e$ and $W_8 - e$.**

Now, the F^+-values are $17, 25, 72$. Here $p = 9$, $q = 15$, $4p - 2 = 34$ and $x_p = v$. $A - 2F^+(v) = 2(5 - n)(18n + 22) - 144 = 36n(4 - n) - 8n + 76 > 0$, since $2 \le n \le 4$.

3. $m = 2k \ge 6$. $F^+(v) = 2k(4k - 1)$, $F^+(u_e) = 13k - 1$ and $F^+(u_o) = 15k - 2$. Clearly F is a local antimagic 3-coloring. Here $p = 4k + 1$, $q = 8k - 1$, $4p - 2 = 16k + 2$ and $x_p = v$.

$$\begin{aligned} A - 2F^+(v) &= 32k^2 + 32k^2 n + 20k - 16kn^2 - 4n^2 + 8n - 4 \\ &= 4(4k^2 - n^2) + 16k^2 + 20k + 16kn(2k - n) + 8n - 4 > 0. \end{aligned}$$

4. $m = 2k + 1 \ge 3$. $F^+(v) = (2k + 1)(4k + 1)$, $F^+(u_e) = 13k + 5$ and $F^+(u_o) = 15k + 6$. Here $p = 4k + 3$, $q = 8k + 3$, $4p - 2 = 16k + 10$. Now $x_p = v$ if $k \ge 2$ and $x_p = u_o$ if $k = 1$. Clearly F is a local antimagic 3-coloring.

For $k \geq 2$,

$$\begin{aligned} A - 2F^+(v) &= 32k^2 + 32k^2n + 32kn + 52k - 16kn^2 - 12n^2 + 16n + 14 \\ &= 16kn(2k+1-n) + 4(4k+3n)(2k+1-n) \\ &+ 8kn + 36k + 4n + 14 > 0. \end{aligned}$$

For $k = 1$, $A - 2F^+(u_o) = 2(4-n)(14n+16) - 42 = 28n(3-n) - 4n + 86 > 0$ since $n = 2, 3$.

By Theorem 2 we have $\chi_{la}(W_{2m} \vee O_{2n-1} - e) \leq 4$ and hence the corollary holds. □

Corollary 3. *For $m, n \geq 2$, $\chi_{la}(W_{2m-1} \vee O_{2n}) = 5$.*

Proof. In [1, Theorem 2.14], the authors provided a local antimagic 4-coloring f of W_k for odd k (there is a typo on the induced vertex label in the original paper). Namely,

when $k \equiv 3 \pmod 4$: $f^+(u_i) = \begin{cases} \frac{9k+9}{4} & \text{if } i \text{ is odd and } i \neq 1; \\ \frac{11k+7}{4} & \text{if } i \text{ is even;} \\ 2k+2 & \text{if } i = 1. \end{cases}$ and $f^+(v) = \frac{(3k+1)k}{2}$,

when $k \equiv 1 \pmod 4$: $f^+(u_i) = \begin{cases} \frac{11k+17}{4} & \text{if } i \text{ is odd and } i \neq 1; \\ \frac{9k+11}{4} & \text{if } i \text{ is even;} \\ \frac{5k+11}{4} & \text{if } i = 1. \end{cases}$ and $f^+(v) = \frac{6k^2+k+1}{4}$.

Let $G = W_{2m-1}$. According to the notation in Theorem 2, $p = 2m$, $q = 4m-2$ and $r = 2n$. We only need to consider when $p - r \geq 2$, i.e., $m - n \geq 1$. Clearly condition (a) of Theorem 2 holds for both cases. For condition (b), we need to have $(p-r)(rp+2q+1) - 2f^+(x_p) \neq 0$ when $m - n \geq 1$.

Suppose $2m - 1 \equiv 3 \pmod 4$.

$$\begin{aligned} (p-r)(rp+2q+1) - 2f^+(x_p) &= (2m-2n)(4mn+8m-3) - [3(2m-1)^2 + (2m-1)] \\ &= 4m^2 + 8m^2n + 4m - 8mn^2 - 16mn + 6n - 2 \\ &= 8mn(m-n-2) + 4m^2 + 4m + 6n - 2. \end{aligned}$$

The last expression is greater than 0 when $m - n \geq 2$. So we only need to consider $m - n = 1$. For this case, $8mn(m-n-2) + 4m^2 + 4m + 6n - 2 = -8m^2 + 18m - 8 \neq 0$, since the discriminant is not a perfect square.

Suppose $2m - 1 \equiv 1 \pmod 4$.

$$\begin{aligned} (p-r)(rp+2q+1) - 2f^+(x_p) &= (2m-2n)(4mn+8m-3) - [3(2m-1)^2 + m] \\ &= 4m^2 + 8m^2n + 5m - 8mn^2 - 16mn + 6n - 3 \\ &= 8mn(m-n-2) + 4m^2 + 5m + 6n - 3. \end{aligned}$$

Same as the previous case, the last expression is greater than 0 when $m-n\geq 2$. When $n=m-1$, the last expansion is $-8m^2+19m-8\neq 0$, since the discriminant is not a perfect square.
By Theorem 2 we have $\chi_{la}(W_{2m-1}\vee O_{2n})\leq 5$. Since $\chi(W_{2m-1})=4$, $\chi(W_{2m-1}\vee O_{2n})=5$. Hence we have the corollary. □

Corollary 4. *For $2\leq m\leq 4$ and $n\geq 1$, $\chi_{la}((W_{2m-1}\vee O_{2n})-e)=4$, where e is a spoke of W_{2m-1}.*

Proof. Using the local antimagic 3-colorings of W_k-e for $k=3,5,7$ (see Fig.1), which were shown in the proof of [11, Theorem 3.7], we can easily show the conditions of Theorem 2 are met.
Since $\chi((W_{2m-1}\vee O_{2n})-e)=4$, we have the corollary. □

Corollary 5. *Suppose $m\geq 5$ and $n\geq 1$. If e is a spoke of W_{2m-1}, then*

$$4\leq \chi_{la}((W_{2m-1}\vee O_{2n})-e)\leq 5.$$

Proof. Since $\chi((W_{2m-1}\vee O_{2n})-e)=4$, it suffices to show $\chi_{la}((W_{2m-1}\vee O_{2n})-e)\leq 5$. We rewrite the f^+ values of the local antimagic 4-coloring f of W_{2m-1} used in the proof of Corollary 3 as:

when $m=2k$: $f^+(u_i)=\begin{cases}9k & \text{if } i \text{ is odd and } i\neq 1;\\ 11k-1 & \text{if } i \text{ is even;}\\ 8k & \text{if } i=1.\end{cases}$ and $f^+(v)=(6k-1)(4k-1)$,

when $m=2k+1$: $f^+(u_i)=\begin{cases}11k+7 & \text{if } i \text{ is odd and } i\neq 1;\\ 9k+5 & \text{if } i \text{ is even;}\\ 5k+4 & \text{if } i=1.\end{cases}$ and $f^+(v)=24k^2+13k+2$.

When $m=2k\geq 6$. Let $h_1=8k-1-f$. Then h_1 is a local antimagic 4-coloring of W_{4k-1} with induced vertex colors $h_1^+(v)=2k(4k-1)$, $h_1^+(u_{2l})=13k-2$, $h_1^+(u_{2l-1})=15k-3$, for $l\neq 1$ and $h_1^+(u_1)=16k-3$.

When $m=2k+1\geq 5$. Let $h_2=8k+3-f$. Then h_2 is a local antimagic 4-coloring of W_{4k+1} with induced vertex colors $h_2^+(v)=8k^2+15k+1$, $h_2^+(u_{2l})=15k+4$, $h_2^+(u_{2l-1})=13k+2$, for $l\neq 1$ and $h_2^+(u_1)=19k+5$.

So we have a local antimagic 4-coloring η for W_{2m-1} such that $\eta(e)=1$, here η is h_1 or h_2 according to $m=2k$ or $m=2k+1$. Same as the proof of Corollary 2, let $F=\eta-1$ be a labeling for $W_{2m}-e$. Then $F^+(x)=\eta^+(x)-\deg_{W_{2m}}(x)$, $x\in V(W_{2m})$. According to Theorem 2, $p=2m$, $q=4m-3$, $r=2n$, $A=(p-r)(rp+2q+1)=2(m-n)(4mn+8m-5)$ for the graph $W_{2m-1}-e$. We only need to check $A\neq 2F^+(x_p)$ when $p-r\geq 2$, i.e., $2m\geq 2n+1$. We have the following two cases:

1. $m=2k$, where $k\geq 3$. Now $F^+(v)=(2k-1)(4k-1)$, $F^+(u_{2l})=13k-5$, $F^+(u_{2l-1})=15k-6$, for $l\neq 1$ and $F^+(u_1)=16k-6$. Here $p=4k$, $4p-2=16k-2$

and $x_p = v$. Note that $4k \geq 2n+1$ implies $4k \geq 2n+2$, i.e., $2k \geq n+1$. Now

$$\begin{aligned} A - 2F^+(v) &= 2(2k-n)(8kn+16k-5) - 2(2k-1)(4k-1) \\ &= 48k^2 + 32k^2n - 8k - 16kn^2 - 32kn + 10n - 2 \\ &= 16kn(2k-n-2) + 48k^2 - 8k + 10n - 2 > 0. \end{aligned}$$

2. $m = 2k+1$, where $k \geq 2$. Now $F^+(v) = 8k^2+11k$, $F^+(u_{2l}) = 15k+1$, $F^+(u_{2l-1}) = 13k-1$, for $l \neq 1$ and $F^+(u_1) = 19k+2$. Here $p = 4k+2$, $4p-2 = 16k+6$ and $x_p = v$. Note that $4k+1 \geq 2n+1$, i.e., $2k \geq n$. Now

$$\begin{aligned} A - 2F^+(v) &= 2(2k+1-n)(8kn+4n+16k+3) - 2(8k^2+11k) \\ &= 48k^2 + 32k^2n + 22k + 2n - 16kn^2 - 8n^2 + 6 \\ &= 16kn(2k-n) + 48k^2 - 8n^2 + 22k + 2n + 6 > 0. \end{aligned}$$

By Theorem 2, we have the corollary. □

Theorem 4. *Suppose $m \geq 2$, $n \geq 1$ and either $8mn^2-2m^2+12mn-4n^2+11m-6n-8 < 0$ or $-12n^2+16n^2m+24nm-20n-2m^2+15m-14 < 0$. If e is a spoke of W_{2m-1}, then $\chi_{la}((W_{2m-1} \vee O_{2n}) - e) = 5$.*

Proof. Without loss of generality we may let $e = vu_1$. Let $W = (W_{2m-1} \vee O_{2n}) - e$. Note that $\chi_{la}(W) \geq \chi(W) = 4$. We are going to find a necessary condition for W admitting a local antimagic 4-coloring, say f. Then we must have $f^+(v) = f^+(u_1)$. Since $\deg(v) = 2m+2n-2$ and $\deg(u_1) = 2n+2$, we have

$$(m+n-1)(2m+2n-1) = \sum_{i=1}^{2m+2n-2} i \leq f^+(v) = f^+(u_1) \leq \sum_{j=1}^{2n+2} (q-j+1) = (n+1)(2q-2n-1),$$

where $q = 4mn+4m-3$, the size of G. Thus

$$L_1 = (m+n-1)(2m+2n-1) \leq f^+(v) = f^+(u_1) \leq (n+1)(8nm-2n+8m-7) = U_1.$$

Since the edges incident to v are different from those to u_1, $(m+2n)(2m+4n+1) \leq f^+(v) + f^+(u_1)$. So

$$L_2 = \frac{1}{2}(m+2n)(2m+4n+1) \leq f^+(v) = f^+(u_1).$$

By using $U_1 - L_1$, we have $8mn^2 - 2m^2 + 12mn - 4n^2 + 11m - 6n - 8 \geq 0$.
By using $2(U_1 - L_2)$, we have $-12n^2 + 16n^2m + 24nm - 20n - 2m^2 + 15m - 14 \geq 0$.
This means $\chi_{la}(W) \geq 5$ if $8mn^2 - 2m^2 + 12mn - 4n^2 + 11m - 6n - 8 < 0$ or $-12n^2 + 16n^2m + 24nm - 20n - 2m^2 + 15m - 14 < 0$. Combining with Corollary 5 we have the theorem. □

Conjecture 5. Let $m \geq 5$, $n \geq 1$ and e be a spoke of W_{2m-1}. Then $8mn^2 - 2m^2 + 12mn - 4n^2 + 11m - 6n - 8 \geq 0$ and $-12n^2 + 16n^2m + 24nm - 20n - 2m^2 + 15m - 14 \geq 0$ is a sufficient condition for $\chi_{la}((W_{2m-1} \vee O_{2n}) - e) = 4$.

Note that $(W_3 \vee O_{2n}) - e = K_{1,1,2,2n}$. We also conjecture that

Conjecture 6. $\chi_{la}(K_{p,q,r,s}) = 4$ for all $p \geq q \geq r \geq s \geq 1$.

More general, we propose

Conjecture 7. For any complete t-partite graph K, $\chi_{la}(K) = t$, $t \geq 4$.

Corollary 6. *For $m \geq 3$ and $n \geq 1$, $\chi_{la}(F_{2m} \vee O_{2n-1}) = 4$.*

Proof. When $n = 1$, $F_{2m} \vee O_1 \cong P_{2m} \vee K_2$. The result was proved by Yang *al et.* [13, Theorem 2.2]. So we assume $n \geq 2$.

Keep the local antimagic labeling of W_{2m} in the proof of Corollary 1. Note that the label 1 is assigned to u_1u_2 under f (see the proofs of [8, Theorem 5] and [1, Theorem 2.14]). One may easily check that f satisfies the conditions of Lemma 1. From the proof of Lemma 1 in [11, Lemma 2.4] we know that the restriction of $f - 1$ on F_{2m}, denoted by h, is a local antimagic 3-coloring of F_{2m}, $m \geq 3$. In this case, $p = 2m + 1$, $r = 2n - 1$, $q = 4m - 1$. By Theorem 2 we only consider $p - r = 2m - 2n + 2 \geq 2$, i.e., $m \geq n$.

Now $h^+(u_i) = f^+(u_i) - 3$ and $h^+(v) = f^+(v) - 2m = m(6m + 1) - 2m$. So $h^+(v) > h^+(u_{2l}) > h^+(u_{2l-1})$ for $l \in [1, m]$. It is easy to check that h satisfies Condition (a) of Theorem 2.

For Condition (b),

$$\begin{aligned}\frac{1}{2}(p-r)(rp+2q+1) - h^+(v) &= (m-n+1)(4mn+6m+2n-2) - m(6m+1) + 2m \\ &= 4mn(m-n) + 5m - 2n^2 + 4n - 2.\end{aligned}$$

Similar to the proof of Corollary 1, the above expression is not zero. Hence by Theorem 2 we have $\chi_{la}(F_{2m} \vee O_{2n-1}) \leq 4$. Since $\chi(F_{2m} \vee O_{2n-1}) = 4$, $\chi_{la}(F_{2m} \vee O_{2n-1}) = 4$. □

Corollary 7. *For $n \geq 1$, $4 \leq \chi_{la}(F_4 \vee O_{2n-1}) \leq 5$.*

Proof. When $n = 1$, $\chi_{la}(F_4 \vee O_1) = \chi_{la}(P_4 \vee K_2) = 4$ was proved by Yang *al et.* [13, Theorem 2.2]. So we assume $n \geq 2$.

Since $\chi(F_4 \vee O_{2n-1}) = 4$. So $4 \leq \chi_{la}(F_4 \vee O_{2n-1})$. Let g be the corresponding local antimagic 4-coloring of F_4 defined in the proof of [10, Theorem 2.3 (b)]. We see that

$g^+(v) = 20$ and the other induced vertex weights are 8, 9, 11 from [10, Theorem 3.3]. By Theorem 3 we only need to consider $p - r \geq 2$, i.e., $p = 5$ and $r = 3$. Now $(p-r)(rp + 2q + 1) = 60$ which does not equal to $2g^+(x)$ for any $x \in V(F_4)$. So by Theorem 3, we have $4 \leq \chi_{la}(F_4 \vee O_{2n-1}) \leq 5$. □

Corollary 8. *For $m \geq 2$ and $n \geq 2$, $\chi_{la}(F_{2m-1} \vee O_{2n}) = 4$.*

Proof. Now $p = 2m$, $q = 4m-3$ and $r = 2n$. From [10, Corollary 3.3] we know that $\chi_{la}(F_{2m-1}) = 3$. Let g be the corresponding local antimagic 3-coloring defined in the proof [10, Theorem 2.3 (b)]. From the proof of [10, Corollary 3.3] we have

$$g^+(u_{j_o}) = \begin{cases} 10k+1 & \text{if } m = 2k+1; k \geq 2 \\ 11k+7 & \text{if } m = 2k+2; k \geq 1 \\ 10 & \text{if } m = 3 \\ 6 & \text{if } m = 2 \end{cases} = \begin{cases} 5m-4 & \text{for odd } m \geq 5 \\ \frac{11m}{2} - 4 & \text{for even } m \geq 4 \\ 10 & \text{if } m = 3 \\ 6 & \text{if } m = 2 \end{cases}$$

$$g^+(u_{j_e}) = \begin{cases} 11k & \text{if } m = 2k+1; k \geq 2 \\ 13k+10 & \text{if } m = 2k+2; k \geq 1 \\ 14 & \text{if } m = 3 \\ 8 & \text{if } m = 2 \end{cases} = \begin{cases} \frac{11(m-1)}{2} & \text{for odd } m \geq 5 \\ \frac{13m}{2} - 3 & \text{for even } m \geq 4 \\ 14 & \text{if } m = 3 \\ 8 & \text{if } m = 2 \end{cases} \tag{2}$$

$$g^+(v) = \begin{cases} 22k^2+12k+1 & \text{if } m = 2k+1; k \geq 2 \\ 16k^2+19k+6 & \text{if } m = 2k+2; k \geq 1 \\ 32 & \text{if } m = 3 \\ 10 & \text{if } m = 2 \end{cases} = \begin{cases} \frac{11m^2+1}{2} - 5m & \text{for odd } m \geq 5 \\ 4m^2 - \frac{13m}{2} + 3 & \text{for even } m \geq 4 \\ 32 & \text{if } m = 3 \\ 10 & \text{if } m = 2 \end{cases}$$

where j_o is odd and j_e is even. Thus $g^+(v) > g^+(u_e) > g^+(u_o)$ and $g^+(u_e) < 4p-2 = 8m-2$ for even e. Similar to the proof of Corollary 1 we consider $p - r \geq 2$, i.e., $m - n \geq 1$. Clearly Condition (a) of Theorem 2 holds. Now we are going to look at Condition (b) of Theorem 2.

Let $B = (p-r)(rp+2q+1) - 2g^+(v) = 2(m-n)(4mn+8m-5) - 2g^+(v)$.

(1) $m = 2$. No case to check.

(2) $m = 2k+1$. Thus $2k - n \geq 0$ and

$$\begin{aligned} B &= 42k^2 + 32k + 32k^2n + 2n - 16kn^2 - 8n^2 + 5 \\ &= 10k^2 + 8(4k^2 - n^2) + 32k + 2n + 16kn(2k-n) + 5 > 0. \end{aligned}$$

(3) $m = 2k+2$. Thus $2k - n + 1 \geq 0$ and

$$\begin{aligned} B &= 48k^2 + 89k + 32k^2n + 32kn + 10n - 16kn^2 - 16n^2 + 38 \\ &= 48k^2 + 16kn - 16n^2 + 89k + 10n + 16kn(2k-n+1) + 38 \\ &\geq 48k^2 + 16nk - 16n^2 + 89k + 10n + 38 \\ &\geq 48\left(\frac{(n-1)^2}{4}\right) + 16n\left(\frac{n-1}{2}\right) - 16n^2 + \frac{89(n-1)}{2} + 10n + 38 \\ &= 4n^2 + \frac{45n+11}{2} > 0. \end{aligned}$$

By Theorem 2 we have the corollary. □

3. Graphs join with cycles

We shall apply the following local antimagic labeling of $C_r = v_1v_2\cdots v_rv_1$ with $r \ge 3$, which was provided in [1], to prove Theorem 8. Let $e_i = v_iv_{i+1}$, $1 \le i \le r-1$ and $e_r = v_rv_1$. Define $\phi : E(C_r) \to [1, r]$ by

$$\phi(e_i) = \begin{cases} r - \frac{i-1}{2} & \text{if } i \text{ is odd;} \\ \frac{i}{2} & \text{if } i \text{ is even} \end{cases} \tag{3}$$

so that

$$\phi^+(v_i) = \begin{cases} r & \text{if } i \text{ is odd; } i \neq 1; \\ r+1 & \text{if } i \text{ is even;} \\ 2r - \lfloor \frac{r}{2} \rfloor & \text{if } i = 1. \end{cases}$$

Theorem 8. *Let G be a connected graph of order p and size q. Suppose G admits a local magic t-coloring f. Without loss of generality, let $f^+(x_1) \le f^+(x_2) \le \cdots \le f^+(x_{p-1}) \le f^+(x_p)$, where x_i for $i \in [1, p]$ are vertices of G. Let $r \ge 3$, $p \ge 3$ and $p \equiv r \pmod 2$. Then $\chi_{la}(G \vee C_r) \le t + 3$ if one of the following condition holds:*

(a) $r - p \ge 6$;

(b) $r - p = 4$ and $f^+(x_1) \ge 6$;

(c) $r - p \le 2$, $f^+(x_{p-1}) \le 6p$ and

$$2f^+(x_p) + (r-p)(rp + 2q + 1) - 4rp - 4q - 2r \notin \{2r - 2\left\lfloor \frac{r}{2} \right\rfloor, 2, 0\}. \tag{*}$$

Proof. Keeping all notation defined in the proof of Theorem 2. Let $H = G \vee C_r$ be obtained from $G \vee O_r$ by adding the edges v_jv_{j+1} for $1 \le j \le r$ where $v_{r+1} = v_1$.

Now $|E(H)| = rp + q + r$. We define a bijection $\psi : E(H) \to [1, rp + q + r]$ by $\psi(e) = g(e)$ if $e \in E(G \vee O_r)$ and $\psi(v_jv_{j+1}) = \phi(v_jv_{j+1}) + rp + q$. Thus,

$$\begin{aligned}
\psi^+(x_i) &= g^+(x_i) = f^+(x_i) + R + rq \text{ for } i \in [1, p]; \\
\psi^+(v_1) &= g^+(v_1) + 2r - \left\lfloor \frac{r}{2} \right\rfloor + 2rp + 2q = C + pq + 2r - \left\lfloor \frac{r}{2} \right\rfloor + 2rp + 2q; \\
\psi^+(v_{j_e}) &= g^+(v_{j_e}) + r + 1 + 2rp + 2q = C + pq + r + 1 + 2rp + 2q \text{ for even } j_e \in [2, r]; \\
\psi^+(v_{j_o}) &= g^+(v_{j_o}) + r + 2rp + 2q = C + pq + r + 2rp + 2q \text{ for odd } j_o \in [2, r].
\end{aligned}$$

Clearly $\psi^+(x_i)$ is a constant translation of $f^+(x_i)$, and $\psi^+(v_1) > \psi^+(v_{j_e}) > \psi^+(v_{j_o})$ for even $j_e \in [2, r]$ and odd $j_o \in [3, r]$.

For $i \in [1, p]$, we have

$$\begin{aligned}
\psi^+(x_i) - \psi^+(v_1) &= f^+(x_i) + R - C + rq - pq - 2rp - 2q - 2r + \left\lfloor \frac{r}{2} \right\rfloor \\
&= f^+(x_i) + \frac{1}{2}(r-p)(rp + 2q + 1) - 2rp - 2q - 2r + \left\lfloor \frac{r}{2} \right\rfloor \\
&= f^+(x_i) + \frac{1}{2}(r - p - 4)(rp + 2q + 1) + 2q + 2 - 2r + \left\lfloor \frac{r}{2} \right\rfloor.
\end{aligned} \tag{4}$$

(a) Suppose $r-p\geq 6$. From (4) we have

$$\psi^+(x_1)-\psi^+(v_1) > (rp+2q+1)+2q+2-2r+\left\lfloor\frac{r}{2}\right\rfloor > rp-2r+\left\lfloor\frac{r}{2}\right\rfloor > 0.$$

Thus, $\psi^+(x_p)\geq\cdots\geq\psi^+(x_1)>\psi^+(v_1)>\psi^+(v_{j_e})>\psi^+(v_{j_o})$.

(b) When $r-p=4$, we have $r=p+4\geq 7$. From (4) we have

$$\begin{aligned}\psi^+(x_1)-\psi^+(v_1) &= f^+(x_1)+2q+2-2r+\left\lfloor\frac{r}{2}\right\rfloor \geq f^+(x_1)+2(p-1)+2-2r+\left\lfloor\frac{r}{2}\right\rfloor\\ &= f^+(x_1)+2(r-5)+2-2r+\left\lfloor\frac{r}{2}\right\rfloor\\ &= f^+(x_1)-8+\left\lfloor\frac{r}{2}\right\rfloor \geq f^+(x_1)-5>0. \qquad \text{(by assumption)}\end{aligned}$$

Thus, $\psi^+(v_{j_o})<\psi^+(v_{j_e})<\psi^+(v_1)<\psi^+(x_1)\leq\psi^+(x_2)\leq\cdots\leq\psi^+(x_{p-1})\leq\psi^+(x_p)$.

(c) Suppose $r-p\leq 2$. By assumption $p\equiv r(mod\ 2)$ and hence $r-p\neq 1$.

When $r-p\leq 0$, similar to (4), we have

$$\begin{aligned}\psi^+(x_{p-1})-\psi^+(v_{j_o}) &= f^+(x_{p-1})+\frac{1}{2}(r-p)(rp+2q+1)-2rp-2q-r\\ &\leq f^+(x_{p-1})-2rp-2q-r\\ &\leq f^+(x_{p-1})-6p-2q-3<0. \qquad \text{(by assumption)}\end{aligned}$$

When $r-p=2$, then

$$\begin{aligned}\psi^+(x_{p-1})-\psi^+(v_{j_o}) &= f^+(x_{p-1})-rp+1-r = f^+(x_{p-1})-p^2-3p-1\\ &\leq f^+(x_{p-1})-6p-1<0. \qquad \text{(by assumption)}\end{aligned}$$

Thus, $\psi^+(x_1)\leq\psi^+(x_2)\leq\cdots\leq\psi^+(x_{p-1})<\psi^+(v_{j_o})<\psi^+(v_{j_e})<\psi^+(v_1)$ when $r-p\leq 2$.

(*) guarantees that $\psi^+(x_p)$ is different from $\psi^+(v_{j_o})$, $\psi^+(v_{j_e})$ and $\psi^+(v_1)$.

Thus, for each case, ψ is a local antimagic $(t+3)$-coloring. This completes the proof. □

Corollary 9. *For $n,m\geq 2$, $\chi_{la}(W_{2m}\vee C_{2n-1})=6$.*

Proof. Use the same notation in the proof of Theorem 2 and Corollary 1. Now $p=2m+1$, $r=2n-1$. It is easy to see that $6\leq f^+(x_1)$ and $f^+(x_{p-1})\leq 6p$. So we only need to check (*) as follows:

Now, $f^+(v) = m(2m+1)$, $2r - 2\left\lfloor \frac{r}{2} \right\rfloor = 2n$ and $r - p \le 2$ implies that $n - m \le 2$. We have

$$\begin{aligned} & 2f^+(v) + (r-p)(rp+2q+1) - 4rp - 4q - 2r \\ = & 2m(2m+1) + (2n-2m-2)(4mn+2n-2m-1+8m+1) \\ & - 4(4mn+2n-2m-1) - 16m - (4n-2) \\ = & 8mn(n-m-2) + 4n^2 - 16n - 8m^2 - 18m + 6 \\ \le & 4(m^2+4m+4) - 16n - 8m^2 - 18m + 6 < 0. \end{aligned}$$

By Theorem 8, we have $\chi_{la}(W_{2m} \vee C_{2n-1}) \le 6$. Since $\chi(W_{2m} \vee C_{2n-1}) = 6$, $\chi(W_{2m} \vee C_{2n-1}) = 6$. □

Corollary 10. *Suppose $m, n \ge 2$. If e is a spoke of W_{2m}, then $\chi_{la}((W_{2m} \vee C_{2n-1}) - e) = 6$.*

Proof. Keep the local antimagic 3-coloring F of $W_{2m} - e$ defined in the proof of Corollary 2. Clearly, $6 \le F^+(x_1)$ and $F^+(x_{p-1}) \le 6p$. So we only need to check (*) under the condition $n \le m+2$. Let $D = (r-p)(rp+2q+1) - 4rp - 4q - 2r = 2(n-m-1)(4mn+2n+6m-2) - 16mn - 12n - 8m + 10$.

1. $m = 2$. $F^+(x_p) = 16$. Note that $n \le 4$.

$$2F^+(x_p) + D = 32 + (20n^2 - 84n - 66) = 20n^2 - 84n - 34 = 4(5n-1)(n-4) - 50 < 0.$$

2. $m = 4$. $F^+(x_p) = 72$. Note that $n \le 6$.

$$2F^+(x_p) + D = 144 + (36n^2 - 212n - 242) = 36n^2 - 212n - 98 = 4(9n+1)(n-6) - 74 < 0.$$

3. $m = 2k \ge 6$. $F^+(x_p) = 2k(4k-1)$. Now $n - 2k \le 2$.

$$\begin{aligned} 2F^+(x_p) + D &= 4k(4k-1) + 4n^2 + 16n^2k - 20n - 32nk^2 - 48k^2 - 32k - 32nk + 14 \\ &= -32k^2 - 36k + 4n^2 + 16kn^2 - 20n - 32k^2n - 32kn + 14 \\ &= 16nk(n-2k-2) + 4(n-2)^2 - 4n - 32k^2 - 36k - 2 < 0. \end{aligned}$$

4. $m = 2k+1 \ge 3$. Now $n \le 2k+3$.
   When $k = 1$. $F^+(x_p) = 21$. Then $n \le 5$.

$$2F^+(x_p) + D = 42 + (28n^2 - 140n - 142) = 28n^2 - 140n - 100 = 28n(n-5) - 100 < 0.$$

   When $k \ge 2$. $F^+(x_p) = (2k+1)(4k+1)$.

Suppose $n \le 2k+2$.

$$\begin{aligned}2F^+(x_p)+D &= 2(2k+1)(4k+1)+12n^2+16n^2k-64nk-44n-32nk^2-48k^2-80k-14\\ &= 12n^2+16n^2k-64nk-44n-32nk^2-32k^2-68k-12\\ &= 16nk(n-2k-3)-16nk+12n^2-44n-32k^2-68k-12\\ &\le -32nk+12n^2-44n-32k^2-68k-12\\ &= 12n(n-2k-2)-8nk-20n-68k-32k^2-12<0.\end{aligned}$$

When $n=2k+3$, $2F^+(x_p)+D=-16k^2-60k-36<0$.

For each case, $2F^+(x_p)+D \notin \{2n,2,0\}$. Since $\chi((W_{2m}\vee C_{2n-1})-e)=6$, we have the corollary by Theorem 8. $\square$

Corollary 11. *For $n \ge 2$ and $m \ge 3$, $\chi_{la}(F_{2m}\vee C_{2n-1})=6$.*

Proof. Keep the notation used in the proof of Corollary 6. Now $p=2m+1$, $q=4m-1$ and $r=2n-1$. We have $h^+(v)=(6m+1)m-2m$, $h^+(u_e)=\frac{11m+3}{2}-3$, $h^+(u_o)=\frac{9m+3}{2}-3$ for odd m; and $h^+(v)=(6m+1)m-2m$, $h^+(u_e)=\frac{11m+2}{2}-3$, $h^+(u_o)=\frac{9m+4}{2}-3$ for even m, where e is even and o is odd. Clearly, $h(u_o)\ge 6$ and $h^+(u_e)\le 6p$. By Theorem 8, we shall need to check (*) under the condition $n\le m+2$. Now

$$\begin{aligned}&2h^+(v)+(r-p)(rp+2q+1)-4rp-4q-2r\\ &=2[(6m+1)m-2m]+(2n-2m-2)[(2n-1)(2m+1)+2(4m-1)+1]\\ &\quad -4(2n-1)(2m+1)-4(4m-1)-2(2n-1)\\ &=-18m+8mn^2+4n^2-20n-8m^2n-16mn+14\\ &=8mn(n-m-2)+4n^2-20n-18m+14.\end{aligned}$$

Suppose $n-m\le 1$. It is easy to see that $8mn(n-m-2)+4n^2-20n-18m+14 < -8mn+4n^2-20n-18m+14=4n(n-m-5)-4mn-18m+14<0$.

Suppose $n-m=2$. Then $8mn(n-m-2)+4n^2-20n-18m+14=4n^2-38n+50=4(n-1)(n-9)+2n+14>2n$ if $n\ge 9$. For $5\le n\le 7$, $4n^2-38n+50=2(2n-3)(n-8)+2\le -2(2n-3)+2<0$. So the condition (*) of Theorem 8 holds when $n=m+2\ge 5$ except $n=8$. When $n=8$ i.e., $m=6$. Condition (*) does not holds. So we need to provide an ad hoc labeling for $F_{12}\vee C_{15}$.

Let $V(F_{12})=\{x_i \mid i\in[1,13]\}$ as shown in Figure 2 and $V(O_{15})=\{v_j \mid j\in[1,15]\}$. We define a labeling f for F_{12} using labels in $[1,11]\cup[207,218]$ as follows:
Let $L=F_{12}\vee O_{15}$. Now $p=13$, $q=23$, $r=15$. Define $g: E(L)\to[1,218]$ by

$$g(e)=\begin{cases} f(e) & \text{if } e\in E(F_{12});\\ a_{ij}+11 & \text{if } e=x_iv_j,\end{cases}$$

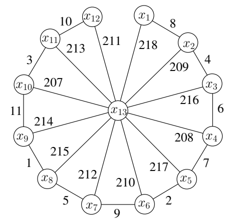

Figure 2. **A labeling f for F_{12}.**

where (a_{ij}) is a 13×15 magic rectangle with $a_{ij} \in [1, 195]$. Note that the row sum and the column sum of this magic rectangle are $R = 1470$ and $C = 1274$, respectively. It is easy to see that g is a local antimagic 4-coloring for L. Namely,

$$g^+(x_i) = f^+(x_i) + R + 11r = f^+(x_i) + 1635 = \begin{cases} 1861 & i \text{ is odd}, i \in [1, 12]; \\ 1856 & i \text{ is even}, i \in [1, 12]; \\ 4185 & i = 13, \end{cases}$$

$$g^+(v_j) = C + 11p = 1417.$$

We use the labeling ψ defined in the proof of Theorem 8. Then we have

$$\psi^+(x_i) = g^+(x_i) = \begin{cases} 1861 & i \text{ is odd}, i \in [1, 12]; \\ 1856 & i \text{ is even}, i \in [1, 12]; \\ 4185 & i = 13, \end{cases}$$

$$\psi^+(v_1) = g^+(v_1) + 2r - \left\lfloor \frac{r}{2} \right\rfloor + 2rp + 2q = 1876,$$

$$\psi^+(v_e) = g^+(v_e) + r + 1 + 2rp + 2q = 1869 \text{ for even } e \in [2, 15],$$

$$\psi^+(v_o) = g^+(v_o) + r + 2rp + 2q = 1868 \text{ for odd } o \in [2, 15].$$

Clearly ψ is a local antimagic 6-coloring for $F_{12} \vee C_{15}$.

Thus, by Theorem 8 or above $\chi_{la}(F_{2m} \vee C_{2n-1}) \le 6$. Since $\chi(F_{2m} \vee C_{2n-1}) = 6$, $\chi_{la}(F_{2m} \vee C_{2n-1}) = 6$. □

By the proof of [11, Theorem 3.3] we have the following theorem which can be used to improve Theorem 8 when $p = 2m$.

Corollary 12. *For $n \ge 2$, $6 \le \chi_{la}(F_4 \vee C_{2n-1}) \le 7$.*

Proof. Use the local antimagic 4-coloring g for F_4 in Corollary 7. Recall that g^+ values are 8, 9, 11, 20. Clearly, we only need to check (*) of Theorem 8 for $3 \le r = 2n - 1 \le p + 2 = 7$. One may easily check that $2f^+(x_p) + (r - p)(rp + 2q + 1) - 4rp - 4q - 2r = 5r^2 - 32r - 63 < 0$ when $3 \le r \le 7$. Hence the corollary holds by $\chi(F_4 \vee C_{2n-1}) = 6$ and Theorem 8. □

Theorem 9. [11, Theorem 3.2] *Let $C_{2n} = v_1v_2\cdots v_{2n}v_1$ and O_{2m} with $V(O_{2m}) = \{x_k \mid k \in [1, 2m]\}$, $n \geq 2$, $m \geq 1$. There is a local antimagic 3-coloring g of $O_{2m} \vee C_{2n}$ such that $g^+(x_k) = 4mn^2 + 4n^2 + n$, $g^+(v_{2i-1}) = 4m^2n - 4mn + 2m + 10n - 1$ and $g^+(v_{2i}) = 4m^2n + 12mn - 6n + 3$, $k \in [1, 2m]$ and $i \in [1, n]$.*

Theorem 10. *Let G be a connected graph of order $2m$ and size q admitting a local antimagic t-coloring f. Then $\chi_{la}(G \vee C_{2n}) \leq t + 2$ if one of the following condition holds, where $m, n \geq 2$.*

(a) $n \geq m + 2$;

(b) $n = m + 1$ and $f^+(x_k) \neq 4m^2 - 7m - 8$ for $k \in [1, 2m]$;

(c) $n = m = 2$;

(d) $n = m \geq 3$, and $f^+(x_k) \neq 8m^2 - 7m + 3 + 2q$ for $k \in [1, 2m]$;

(e) $n = m - 1$ and $f^+(x_k) \notin \{-4m^2 + 19m - 14 + 4q, 12m^2 - 15m + 6 + 4q\}$ for each k;

(f) $n \leq m - 2$, $f^+(x_{2m-1}) \leq 2m + 6q$ and $f^+(x_{2m}) + (4mn + 4n + 2q)(m - n) \notin \{8mn - 7n + 3 + 2q, -8mn + 9n + 2m - 1 + 2q\}$.

Proof. Keep the notation described in Theorem 9. Let $V(G) = \{x_k \mid k \in [1, 2m]\}$. Without loss of generality, we assume $f^+(x_1) \leq f^+(x_2) \leq \cdots \leq f^+(x_{2m})$. Thus $G \vee C_{2n} = G \cup (O_{2m} \vee C_{2n})$, the union of G with $O_{2m} \vee C_{2n}$. Define $h : E(G \vee C_{2n}) \to [1, q + 4mn + 2n]$ by

$$h(e) = \begin{cases} f(e) & \text{if } e \in E(G); \\ g(e) + q & \text{if } e \in E(O_{2m} \vee C_{2n}). \end{cases}$$

Thus, $h^+(x_k) = f^+(x_k) + g^+(x_k) + 2nq$, $h^+(v_1) = h^+(v_{2i-1}) = g^+(v_{2i-1}) + (2m+2)q$ and $h^+(v_2) = h^+(v_{2i}) = g^+(v_{2i}) + (2m+2)q$. Therefore, $h^+(x_k) = h^+(x_{k'})$ if and only if $g^+(x_k) = g^+(x_{k'})$. Also $h^+(v_2) - h^+(v_1) = g^+(v_2) - g^+(v_1) = 16mn - 2m - 16n + 4 = 16(n-1)(m-1) + 14m - 12 > 0$.
Now, for each $k \in [1, 2m]$,

$$\begin{aligned} h^+(x_k) - h^+(v_2) &= [f^+(x_k) + g^+(x_k) + 2nq] - [g^+(v_2) + (2m+2)q] \\ &= [f^+(x_k) + 4mn^2 + 4n^2 + n + 2nq] - [4m^2n + 12mn - 6n + 3 + (2m+2)q] \\ &= f^+(x_k) - 8mn + 7n - 3 - 2q + (4mn + 4n + 2q)(n - m). \end{aligned} \tag{5}$$

Similar to (5) and $q \leq m(2m-1)$ we have

$$\begin{aligned} h^+(x_k) - h^+(v_1) &= [f^+(x_k) + g^+(x_k) + 2nq] - [g^+(v_1) + (2m+2)q] \\ &= f^+(x_k) + 4mn + 4n^2 - 9n - 2m + 1 - 2q + (4mn + 2q)(n - m) \\ &= f^+(x_k) + 8mn - 9n - 2m + 1 - 2q + (4mn + 4n + 2q)(n - m) \qquad (6) \\ &\geq f^+(x_k) + 8mn - 9n - 2m + 1 - 2m(2m-1) + (4mn + 4n + 2q)(n - m) \\ &= f^+(x_k) + (4m - 8)n + (4mn + 4n + 4m + 2q)(n - m) - n + 1. \qquad (7) \end{aligned}$$

(a) Suppose $n \ge m+2$. Then (5) $\ge f^+(x_k) + 15n - 3 + 2q > 0$. Thus, $h^+(v_1) < h^+(v_2) < h^+(x_1) \le h^+(x_2) \le \cdots \le h^+(x_{2m})$. Thus h induces $t+2$ vertex labels.

(b) Suppose $n = m+1$. (7) implies that $h^+(x_k) > h^+(v_1)$. Now (5) becomes $f^+(x_k) - 4m^2 + 7m + 8$. So $f^+(x_k) \ne 4m^2 - 7m - 8$ ensures that h induces $t+2$ vertex labels.

(c) Suppose $n = m = 2$. Since G is a subgraph of K_4, $f^+(x_k) \le 6+5+4 = 15$. Now (5) becomes $f^+(x_k) - 21 - 2q < 0$. That is, $h^+(x_k) < h^+(v_2)$ for $k \in [1,4]$.

Next, (6) becomes $f^+(x_k) + 11 - 2q$. If $G = K_4$, then $f^+(x_k) \ge 6$. Hence $f^+(x_k) + 11 - 2q > 0$. If $G \ne K_4$, then $q \le 5$. Hence $f^+(x_k) + 11 - 2q > 0$. Thus, $h^+(x_k) > h^+(v_1)$ for $k \in [1,4]$.

So, h induces $t+2$ vertex labels.

(d) Suppose $n = m \ge 3$. Then (7) becomes $f^+(x_k) + (4n-8)n - n + 1 > 0$. Now (5) becomes $f^+(x_k) - 8m^2 + 7m - 3 - 2q$. So $f^+(x_k) \ne 8m^2 - 7m + 3 + 2q$ ensures that h induces $t+2$ vertex labels.

(e) Suppose $n = m-1$. From (5) and (6), we have

$$h^+(x_k) - h^+(v_2) = f^+(x_k) - 12mn + 3n - 3 - 4q = f^+(x_k) - 12m^2 + 15m - 6 - 4q,$$
$$h^+(x_k) - h^+(v_1) = f^+(x_k) + 4mn - 2m - 13n + 1 - 4q = f^+(x_k) + 4m^2 - 19m + 14 - 4q,$$

for $k \in [1,2m]$. The assumption ensures that $h^+(x_k)$, $h^+(v_1)$ and $h^+(v_2)$ are distinct for $k \in [1,2m]$.

(f) Suppose $n \le m-2$. From (5) and (6), we have

$$h^+(x_k) - h^+(v_2) \le f^+(x_k) - 16mn - n - 3 - 6q$$
$$h^+(x_k) - h^+(v_1) \le f^+(x_k) - 2m - 17n + 1 - 6q.$$

for $k \in [1,2m]$. Since $f^+(x_{2m-1}) \le 2m + 6q$, $h^+(x_{2m-1}) - h^+(v_1) < 0$. By (5), (6) and the requirement of $f^+(x_{2m})$ imply that $h^+(x_{2m})$, $h^+(v_1)$ and $h^+(v_2)$ are distinct.

Thus we have $\chi_{la}(G \vee C_{2n}) \le t+2$. □

Corollary 13. *For $m \ge 2$ and $n \ge 2$, $\chi_{la}(W_{2m-1} \vee C_{2n}) = 6$.*

Proof. Now $p = 2m$ and $q = 4m-2$. Keep the local antimagic 4-coloring f of W_{2m-1} described in Corollary 3. Then $x_{2m} = v$, $x_1 = u_1$. To show this corollary, we only need to check the conditions (b), (d), (e) and (f) of Theorem 10.

(1) Consider m is even so that $k = 2m-1 \equiv 3 \pmod 4$. The 4 induced vertex labels are $f^+(x_1) = 4m$, $f^+(x_2) = \frac{9m}{2}$, $f^+(x_{2m-1}) = \frac{11m-2}{2}$ and $f^+(x_{2m}) = 6m^2 - 5m + 1$.

(b) Suppose $n = m + 1$. Clearly $f^+(x_k) < 4m^2 - 7m - 8$ for $1 \le k \le 2m - 1$ and $f^+(x_{2m}) - (4m^2 - 7m - 8) = 2m^2 + 2m + 9 > 0$. So the condition (b) of Theorem 10 holds.

(d) Suppose $n = m$. Clearly $f^+(x_{2m}) < 8m^2 - 7m + 3 + 2q$. The condition (d) of Theorem 10 holds.

(e) Suppose $n = m - 1$. Clearly $12m^2 - 15m + 6 + 4q = 12m^2 + m - 2 > f^+(x_k)$ for all k. Next $-4m^2 + 19m - 14 + 4q = -4m^2 + 35m - 22$.

$$\begin{aligned}
4m - (-4m^2 + 35m - 22) &= 4m^2 - 31m + 22; \qquad \triangle = 609,\\
\frac{9m}{2} - (-4m^2 + 35m - 22) &= \frac{1}{2}(8m^2 - 61m + 44);\ 4\triangle = 2313,\\
\frac{11m-2}{2} - (-4m^2 + 35m - 22) &= \frac{1}{2}(8m^2 - 59m + 42);\ 4\triangle = 2137,\\
(6m^2 - 5m + 1) - (-4m^2 + 35m - 22) &= 10m^2 - 40m + 23; \qquad \triangle = 680.
\end{aligned}$$

Since all discriminants $\triangle$ are not perfect squares, the condition (e) of Theorem 10 holds.

(f) Suppose $n \le m - 2$, then $f^+(x_{2m-1}) \le 2m + 6q$ is clear.

$$\begin{aligned}
\alpha &= f^+(x_{2m}) + (4mn + 4n + 2q)(m - n)\\
&= 14m^2 - 9m + 4m^2n - 4mn^2 - 4mn - 4n^2 + 4n + 1,\\
\beta &= 8mn - 7n + 3 + 2q = 8mn + 8m - 7n - 1,\\
\gamma &= -8mn + 9n + 2m - 1 + 2q = -8mn + 10m + 9n - 5
\end{aligned}$$

$$\begin{aligned}
\alpha - \beta &= 14m^2 - 17m + 4m^2n - 4mn^2 - 12mn - 4n^2 + 11n + 2\\
&\ge 14m(n+2) - 17m + 4m^2n - 4mn^2 - 12mn - 4n^2 + 11n + 2\\
&= 2mn + 11m + 4mn(m - n) - 4n^2 + 11n + 2\\
&\ge 11m + 10mn - 4n^2 + 11n + 2 > 0. \qquad \text{(by } m \ge n + 2\text{)}
\end{aligned}$$

$$\begin{aligned}
\alpha - \gamma &= 14m^2 - 19m + 4m^2n - 4mn^2 + 4mn - 4n^2 - 5n + 6\\
&\ge 14m(n+2) - 19m + 4mn(m - n) + 4mn - 4n^2 - 5n + 6\\
&= 18mn - 4n^2 + 9m - 5n + 4mn(m - n) + 6 > 0. \qquad \text{(by } m > n\text{)}
\end{aligned}$$

Thus the condition (f) of Theorem 10 holds.

(2) Suppose m is odd, then $k = 2m - 1 \equiv 1 \pmod 4$. The 4 induced vertex labels are $f^+(x_1) = \frac{5m+3}{2}$, $f^+(x_2) = \frac{9m+1}{2}$, $f^+(x_{2m-1}) = \frac{11m+3}{2}$ and $f^+(x_{2m}) = 6m^2 - 6m + \frac{m+3}{2}$.

(b) Suppose $n = m + 1$. Clearly $f^+(x_k) < 4m^2 - 7m - 8$ for $1 \le k \le 2m - 1$ and $f^+(x_{2m}) - (4m^2 - 7m - 8) = \frac{1}{2}(4m^2 + 3m + 19) > 0$. So the condition (b) of Theorem 10 holds.

(d) Suppose $n = m$. Clearly $f^+(x_{2m}) < 8m^2 - 7m + 3 + 2q$. The condition (d) of Theorem 10 holds.

(e) Suppose $n = m - 1$. Clearly $12m^2 - 15m + 6 + 4q = 12m^2 + m - 2 > f^+(x_k)$ for all k. Next $-4m^2 + 19m - 14 + 4q = -4m^2 + 35m - 22$. Similar to the subcase (e) of case (1), we can check that the condition (e) of Theorem 10 holds.

(f) Suppose $n \le m - 2$. $f^+(x_{2m-1}) \le 2m + 6q$ is clear.
Now, $f^+(x_{2m}) + (4mn + 4n + 2q)(m - n) = 14m^2 - 9m + 4m^2n - 4mn^2 - 4mn - 4n^2 + 4n + 1 + \frac{-m+1}{2}$. Similar to the subcase (f) of case (1), we can check that the condition (f) of Theorem 10 holds.

Since $\chi(W_{2m-1} \vee C_{2n}) = 6$, by Theorem 10 we have $\chi_{la}(W_{2m-1} \vee C_{2n}) = 6$. □

Corollary 14. *For $n \ge 2$ and $m \ge 2$, $\chi_{la}(F_{2m-1} \vee C_{2n}) = 5$.*

Proof. Now $p = 2m$ and $q = 4m - 3$. We keep the notation and the local antimagic 3-coloring g of F_{2m-1} used in Corollary 8. Thus, $x_{2m} = v$, $x_{2m-1} = u_2$ and $x_1 = u_1$. Same as the proof of Corollary 13 we only need to check the conditions (b), (d), (e) and (f) of Theorem 10 by using (2).

(b) Suppose $n = m + 1$. Clearly $g^+(x_i) < 4m^2 - 7m - 8$ for $i \in [1, 2m - 1]$.

When m is odd. $g^+(x_{2m}) - (4m^2 - 7m - 8) = \frac{1}{2}(3m^2 + 4m + 17) > 0$ if $m \ge 5$ and $g^+(x_{2m}) - (4m^2 - 7m - 8) = 57$ if $m = 3$.

When m is even. $g^+(x_{2m}) - (4m^2 - 7m - 8) = \frac{1}{2}(m + 22)$ if $m \ge 4$ and $g^+(x_{2m}) - (4m^2 - 7m - 8) = 16$ if $m = 2$.

It is easy to see that both cases are not zero. So the condition (b) of Theorem 10 holds.

(d) Suppose $n = m$. Clearly $g^+(x_{2m}) < 8m^2 - 7m + 3 + 2q$. The condition (d) of Theorem 10 holds.

(e) Suppose $n = m - 1$. Since $n \ge 2$, $m \ge 3$, clearly $12m^2 - 15m + 6 + 4q = 12m^2 + m - 6 > g^+(x_i)$ for all i. Next $-4m^2 + 19m - 14 + 4q = -4m^2 + 35m - 26$.

(1) If $m = 2k + 1$ for $k \ge 2$, then $-4m^2 + 35m - 26 = -16k^2 + 54k + 5$.

$$\begin{aligned}(10k + 1) - (-16k^2 + 54k + 5) &= 16k^2 - 44k - 4; \qquad \triangle = 2192,\\ 11k - (-16k^2 + 54k + 5) &= 16k^2 - 43k - 5; \qquad \triangle = 2169,\\ (22k^2 + 12k + 1) - (-16k^2 + 54k + 5) &= 38k^2 - 42k - 4 > 0.\end{aligned}$$

(2) If $m = 2k + 2$ for $k \ge 1$, then $-4m^2 + 35m - 26 = -16k^2 + 38k + 28$.

$$\begin{aligned}(11k + 7) - (-16k^2 + 38k + 28) &= 16k^2 - 27k - 21; \ \triangle = 2073,\\ (13k + 10) - (-16k^2 + 38k + 28) &= 16k^2 - 25k - 18; \ \triangle = 1777,\\ (16k^2 + 19k + 6) - (-16k^2 + 38k + 28) &= 32k^2 - 19k - 22 \ \ \triangle = 3177.\end{aligned}$$

(3) If $m = 3$, then $-4m^2 + 35m - 26 = 43$.

So the condition (e) of Theorem 10 holds for each cases.

(f) Suppose $n \leq m - 2$. Since $n \geq 2$, $m \geq 4$, $f^+(x_{2m-1}) \leq 2m + 6q$ is clear. Now, $(4mn + 4n + 2q)(m - n) = 8m^2 + 4m^2n - 4mn^2 - 4mn - 6m - 4n^2 + 6n$.

(1) Consider $m = 2k + 1$, for $k \geq 2$. Note that $2k \geq n + 1$.

$$\begin{aligned}
\alpha &= f^+(x_{2m}) + (4mn + 4n + 2q)(m - n) \\
&= 54k^2 + 32k + 16k^2n + 8kn + 6n - 8kn^2 - 8n^2 + 3, \\
\beta &= 8mn - 7n + 3 + 2q = 8mn + 8m - 7n - 3 = 16kn + 16k + n + 5, \\
\gamma &= -8mn + 9n + 2m - 1 + 2q = -8mn + 10m + 9n - 7 = -16kn + 20k + n + 3.
\end{aligned}$$

$$\begin{aligned}
\alpha - \beta &= 54k^2 + 16k + 16k^2n - 8kn + 5n - 8kn^2 - 8n^2 - 2 \\
&= 54k^2 - 8n^2 + 16k + 8kn(2k - n - 1) + 5n - 2 > 0, \\
\alpha - \gamma &= 54k^2 + 12k + 16k^2n + 24kn + 5n - 8kn^2 - 8n^2 \\
&= 54k^2 - 8n^2 + 12k + 8kn(2k - n) + 24kn + 5n > 0.
\end{aligned}$$

Thus the condition (f) of Theorem 10 holds.

(2) Consider $m = 2k + 2$, for $k \geq 1$. Note that $2k \geq n$. Similar to the above, we can check that the condition (f) of Theorem 10 holds.

Since $\chi(F_{2m-1} \vee C_{2n}) = 5$, by Theorem 10 we have $\chi_{la}(F_{2m-1} \vee C_{2n}) = 5$. □

4. Conclusion

In this paper, we successfully obtained sufficient conditions for the upper bounds of $\chi_{la}(G \vee H)$ that depends on the existence of a suitable local antimagic labeling of G for $H \in \{O_n, C_n\}$. Consequently, the local antimagic chromatic number of many join graphs are obtained. Sufficient conditions that give the exact value of the local antimagic chromatic number of the join of circulant graphs with null graph will be reported in a subsequent paper.

Conflict of interest. The authors declare that they have no conflict of interest.

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