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Research Article

Some properties of the essential annihilating-ideal graph of commutative rings

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Abstract: Let $\mathcal{S}$ be a commutative ring with unity and $A(\mathcal{S})$ denotes the set of annihilating-ideals of $\mathcal{S}$. The essential annihilating-ideal graph of $\mathcal{S}$, denoted by $\mathcal{EG}(\mathcal{S})$, is an undirected graph with $A^*(\mathcal{S})$ as the set of vertices and for distinct $\mathcal{I}, \mathcal{J} \in A^*(\mathcal{S})$, $\mathcal{I} \sim \mathcal{J}$ is an edge if and only if $Ann(\mathcal{I}\mathcal{J}) \leq_e \mathcal{S}$. In this paper, we classify the Artinian rings $\mathcal{S}$ for which $\mathcal{EG}(\mathcal{S})$ is projective. We also discuss the coloring of $\mathcal{EG}(\mathcal{S})$. Moreover, we discuss the domination number of $\mathcal{EG}(\mathcal{S})$.

Keywords: Annihilating-ideal graph; essential annihilating-ideal graph; crosscap of a graph; domination number of a graph

AMS Subject classification: 13A15, 05C10, 05C12, 05C25

1. Introduction

Throughout the paper $\mathcal{S}$ is a commutative ring with unit element $1 \neq 0$. $\mathbb{I}(\mathcal{S})$ denotes the set of ideals of $\mathcal{S}$ and $\mathbb{I}^*(\mathcal{S}) = \mathbb{I}(\mathcal{S}) \setminus \{0\}$. An ideal $\mathcal{I}$ of $\mathcal{S}$ is called an *annihilating-ideal* if $\mathcal{I}\mathcal{J} = 0$ for some $\mathcal{J} \in \mathbb{I}^*(\mathcal{S})$. $A(\mathcal{S})$ denotes the set of annihilating-ideals of $\mathcal{S}$ and $A^*(\mathcal{S}) = A(\mathcal{S}) \setminus \{0\}$. If $\mathcal{I} \cap \mathcal{J} \neq 0$ for every $\mathcal{J} \in \mathbb{I}^*(\mathcal{S})$, then $\mathcal{I}$ is said to be an *essential ideal* of $\mathcal{S}$. An essential ideal $\mathcal{I}$ of $\mathcal{S}$ is denoted by $\mathcal{I} \leq_e \mathcal{S}$. The set of zero-divisors and the set of maximal ideals of $\mathcal{S}$ is denoted by $Z(\mathcal{S})$ and $Max(\mathcal{S})$, respectively. If $a^n = 0$ for some positive integer n, then a is called nilpotent element of $\mathcal{S}$. The least positive integer m such that $a^m = 0$, is called the *nilpotency index* of a denoted by η_a. We denote the set of nilpotent elements of $\mathcal{S}$ by $Nil(\mathcal{S})$ and

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$Nil^*(\mathcal{S}) = Nil(\mathcal{S}) \setminus \{0\}$. We refer the reader to [5] for any ambiguous notation or vocabulary in ring theory.

Let $G(V, E)$ be a graph with vertex set V and edge set E. Two adjacent vertices u and v in G are denoted by $u \sim v$. The *distance* between two vertices u and v, denoted by $d(u, v)$, is the length of shortest path between u and v. If there is no such path, then $d(u, v) = \infty$. If all the vertices of G are adjacent with each other, then G is called a *complete graph*, which is denoted by K_n for $|V| = n$. If V can be partitioned in to two nonempty disjoint sets V_1 and V_2 such that every edge of G has one end in V_1 and other end in V_2, then G is called a *bipartite graph*. Also, if each vertex of V_1 is adjacent with every vertex of V_2, then G is called a *complete bipartite graph*. Moreover, if $|V_1| = n$ and $|V_2| = m$, then it is denoted by $K_{n,m}$. A maximal complete subgraph of G is called a *clique* of G. The number of vertices in a clique of G is called the *clique number* of G denoted by $\omega(G)$. The *coloring* of G is an assignment of color to the vertices of G such that no two adjacent vertices have the same color. The minimum number of colors required for the coloring of G is called the chromatic number of G denoted by $\chi(G)$. A set $S \subseteq V$ is called a *dominating set* of G, if every element of $V \setminus S$ is adjacent with at least one element of S. The cardinality of a minimum dominating set of G is called the *domination number* of G denoted by $\lambda(G)$. A dominating set S of minimum cardinality in G is called $\lambda(G)$-set of G. For more details on graph theory, we refer the reader to [14, 15].

Several authors have intensively examined the graphs created from algebraic structures during the last three decades and it has become a prominent subject of research. In the study of ring structure, assigning graphs to rings has been widely used. Studying these graphs has the advantage of revealing knowledge about algebraic structures and vice versa.

Beck [6] established the concept of the zero-divisor graph of a commutative ring in 1988, where he was primarily concerned in colorings. Beck proposed that $\chi(\mathcal{S}) = \omega(\mathcal{S})$ for any commutative ring $\mathcal{S}$ in [6]. For some types of rings, such as reduced rings and principal ideal rings, he established the supposition. However, this is not the case in general. This was established in 1993, when Anderson and Naseer presented a convincing counter example (see Theorem 2.1 in [4]) that proved Beck's conjecture for general rings to be false. Anderson and Naseer continued their research into the colorings of a commutative ring. They take the vertex set as the ring elements and define an edge between the vertices a and b if and only if $ab = 0$. In [3], Anderson and Livingston introduced the zero-divisor graph of $\mathcal{S}$, denoted by $\Gamma(\mathcal{S})$, with vertex set $Z^*(\mathcal{S})$ and for distinct $a, b \in Z^*(\mathcal{S})$, the vertices a and b are adjacent if and only if $ab = 0$. In 2011, Behboodi and Rakeei [7, 8] described a new graph, called it *annihilating-ideal graph* $AG(\mathcal{S})$ on $\mathcal{S}$, with the vertex set $A^*(\mathcal{S})$ and two distinct vertices $\mathcal{I}$ and $\mathcal{J}$ are adjacent if and only if $\mathcal{I}\mathcal{J} = 0$. Selvakumar and Subbulakshmi [9] characterize all commutative Artinian non-local rings $\mathcal{S}$ for which $AG(\mathcal{S})$ has genus one. The annihilator-inclusion ideal graph and sum-annihilating essential ideal graph of a commutative ring have been studied [1, 2].

Recently, Nazim and Rehman [10] introduced and studied the *essential annihilating-ideal graph* of a commutative ring $\mathcal{S}$ denoted by $\mathcal{EG}(\mathcal{S})$. It is an undirected graph

with vertex set $A^*(\mathcal{S})$ and two distinct vertices $\mathcal{I}$ and $\mathcal{J}$ are adjacent if and only if $Ann(\mathcal{I}\mathcal{J})$ is an essential ideal of $\mathcal{S}$. They proved that $AG(\mathcal{S})$ is a spanning subgraph of $\mathcal{EG}(\mathcal{S})$. Also, they proved that $\mathcal{EG}(\mathcal{S})$ is connected with $diam(\mathcal{EG}(\mathcal{S})) \leq 3$ and $gr(\mathcal{EG}(\mathcal{S})) \in \{3, 4, \infty\}$. Moreover, they classified the Artinian commutative rings $\mathcal{S}$ for which $\mathcal{EG}(\mathcal{S})$ is a tree, a unicycle graph, a split graph, a planar graph, an outerplanar graph or a toroidal graph.
In this paper, we first classify the Artinian rings $\mathcal{S}$ for which $\mathcal{EG}(\mathcal{S})$ is a projective graph. Then we discuss about the coloring of $\mathcal{EG}(\mathcal{S})$ and prove that $\mathcal{EG}(\mathcal{S})$ is weakly perfect in case of Artinian ring. Finally, we discuss about the domination number of $\mathcal{EG}(\mathcal{S})$ and prove that the domination number of $\mathcal{EG}(\mathcal{S})$ can be any arbitrary number.

2. Crosscap of essential annihilating-ideal graph

For non-negative integer k, let N_k signify a sphere with k crosscap attached to it. For some non-negative integer k, every connected compact surface is homeomorphic to N_k. The least positive integer k such that the graph G can be embedded in N_k is the *crosscap number* of G, denoted by $\overline{\gamma}(G)$. It is obvious that $\overline{\gamma}(H) \leq \overline{\gamma}(G)$ for every subgraph H of G. A graph with crosscap number one is said to be *projective*.
In this section, we classify the Artinian commutative rings $\mathcal{S}$ for which $\mathcal{EG}(\mathcal{S})$ is a projective plane i.e., $\overline{\gamma}(\mathcal{EG}(\mathcal{S})) = 1$. The crosscap of the complete graph and complete bipartite graphs are given in the following results, which are useful for proving the results of this section.

Lemma 1. *[15] (1) Let $n \geq 3$. Then*

$$\overline{\gamma}(K_n) = \begin{cases} \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil & \text{if } n \geq 3 \text{ and } n \neq 7; \\ 3 & \text{if } n = 7. \end{cases}$$

(2) *Let $n, m \geq 2$. Then*

$$\overline{\gamma}(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil.$$

Theorem 1. *Let $(\mathcal{S}, \Im)$ be a commutative Artinian local ring. Then $\overline{\gamma}(\mathcal{EG}(\mathcal{S})) = 1$ if and only if $\mathcal{S}$ have at least five and at most six nonzero proper ideals.*

Proof. The proof follows from [10, Lemma 3.2] and Lemma 1. □

Theorem 2. *Let $\mathcal{S} = \Psi_1 \times \Psi_2 \times \cdots \times \Psi_n$ be a commutative ring, where Ψ_i is a field for each i and $n \geq 4$. Then $\overline{\gamma}(\mathcal{EG}(\mathcal{S})) = 1$ if and only if $n = 4$.*

Proof. Since $\mathcal{S}$ is a reduced ring, thus by [10, Theorem 2.5], $\mathcal{EG}(\mathcal{S}) = AG(\mathcal{S})$. Hence the proof follows from [12, Theorem 3.2].

□

Theorem 3. *Let $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$ be a commutative ring, where $n \geq 2$ and $(\mathcal{S}_i, \Im_i)$ is an Artinian local ring with $\Im_i \neq 0$ for each i, then $\overline{\gamma}(\mathcal{EG}(\mathcal{S})) \neq 1$.*

Proof. Since $AG(\mathcal{S})$ is a subgraph of $\mathcal{EG}(\mathcal{S})$, the proof follows from [12, Theorem 3.3]. □

Theorem 4. *Let $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n \times \Psi_1 \times \Psi_2 \times \cdots \times \Psi_m$ be a commutative ring , where each $(\mathcal{S}_i, \Im_i)$ is an Artinian local ring with $\Im_i \neq 0$, each Ψ_j is a field and $n, m \geq 1$. Let η_i be the nipotency index of $\Im_i$. Then $\overline{\gamma}(\mathcal{EG}(\mathcal{S})) = 1$ if and only if one of the following holds:*

1. *$n = m = 1$, $\eta_1 = 3$ and $\Im_1$ and $\Im_1^2$ are the only nonzero proper ideals of $\mathcal{S}_1$.*
2. *$n = 1$, $m = 2$, $\eta_1 = 2$ and $\Im_1$ is the only nonzero proper ideal of $\mathcal{S}_1$.*

Proof. Suppose $\overline{\gamma}(\mathcal{EG}(\mathcal{S})) = 1$. If $n \geq 2$, then from Theorem 3, $\overline{\gamma}(\mathcal{EG}(\mathcal{S})) \neq 1$, a contradiction. Hence $n = 1$.
Suppose $m \geq 3$, then the set $\{\Im_1 \times (0) \times (0) \times \cdots \times (0), (0) \times \Psi_1 \times (0) \times \cdots \times (0), \Im_1 \times \Psi_1 \times (0) \times \cdots \times (0)\} \cup \{(0) \times (0) \times \Psi_2 \times (0) \times \cdots \times (0), (0) \times (0) \times (0) \times \Psi_3 \times (0) \times \cdots \times (0), (0) \times (0) \times \Psi_2 \times \Psi_3 \times (0) \times \cdots \times (0), \mathcal{S}_1 \times (0) \times (0) \times \cdots \times (0), \mathcal{S}_1 \times (0) \times \Psi_2 \times (0) \times \cdots \times (0)\}$ induces a copy of $K_{3,5}$ in $\mathcal{EG}(\mathcal{S})$. Thus, $\overline{\gamma}(\mathcal{EG}(\mathcal{S})) > 1$ by Lemma 1, a contradiction. Hence $m \leq 2$. Take the following cases into consideration:

Case(i) $m = 2$. Suppose $\eta_1 \geq 3$. Then the set $\{\mathcal{S}_1 \times (0) \times (0), \Im_1 \times (0) \times (0), \Im_1^{\eta_1 - 1} \times (0) \times (0)\} \cup \{(0) \times \Psi_1 \times (0), (0) \times (0) \times \Psi_2, (0) \times \Psi_1 \times \Psi_2, \Im_1 \times \Psi_1 \times (0), \Im_1 \times (0) \times \Psi_2\}$ induces a copy of $K_{3,5}$ as a subgraph of $\mathcal{EG}(\mathcal{S})$, which is a contradiction by Lemma 1. Hence $\eta_1 = 2$.
Now, suppose $\mathcal{I}$ be a nonzero proper ideal of $\mathcal{S}_1$ such that $\mathcal{I} \neq \Im_1$. Then the set $\{\Im_1 \times (0) \times (0), (0) \times \Psi_1 \times (0), \Im_1 \times \Psi_1 \times (0)\} \cup \{\mathcal{I} \times (0) \times (0), \Im_1 \times (0) \times \Psi_2, (0) \times (0) \times \Psi_2, \mathcal{I} \times (0) \times \Psi_2, \mathcal{S}_1 \times (0) \times (0)\}$ induces $K_{3,5}$ as a subgraph of $\mathcal{EG}(\mathcal{S})$, which is a contradiction from Lemma 1. Hence $\Im_1$ is the only nonzero proper ideal of $\mathcal{S}_1$.

Case(ii) $m = 1$. Suppose $\eta_1 \geq 4$, then the set $\{\Im_1^{\eta_1 - 1} \times (0), \Im_1^{\eta_1 - 2} \times (0), \Im_1^{\eta_1 - 3} \times (0)\} \cup \{\Im_1^{\eta_1 - 1} \times \Psi_1, \Im_1^{\eta_1 - 2} \times \Psi_1, \Im_1^{\eta_1 - 3} \times \Psi_1, (0) \times \Psi_1, \mathcal{S}_1 \times (0)\}$ induces $K_{3,5}$ as a subgraph of $\mathcal{EG}(\mathcal{S})$, which is a contradiction by Lemma 1. Hence $\eta_1 \leq 3$.
Suppose $\eta_1 = 3$ and $\mathcal{J}$ ba a nonzero proper ideal of $\mathcal{S}_1$ such that $\mathcal{J} \neq \Im_1, \Im_1^2$. One can see that the set $\{\Im_1 \times (0), \Im_1^2 \times (0), \mathcal{S}_1 \times (0)\} \cup \{(0) \times \Psi_1, \Im_1 \times \Psi_1, \Im_1^2 \times \Psi_1, \mathcal{J} \times (0), \mathcal{J} \times \Psi_1\}$ induces $K_{3,5}$ as a subgraph of $\mathcal{EG}(\mathcal{S})$, a contradiction. Hence $\mathcal{S}_1$ has only two nonzero proper ideals given by $\Im_1$ and $\Im_1^2$.
Suppose $\eta_1 = 2$ and $\mathcal{K}$ be a nonzero proper ideal of $\mathcal{S}_1$ with $\mathcal{K} \neq \Im_1$. Then by [13, Proposition 2.7] there exist at least three nonzero proper ideals $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ of $\mathcal{S}_1$ such that $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 \neq \Im_1$. One can see that the set $\{\Im_1 \times (0), (0) \times \Psi_1, \Im_1 \times \Psi_1\} \cup \{\mathcal{K}_1 \times (0), \mathcal{K}_2 \times (0), \mathcal{K}_3 \times (0), \mathcal{K} \times (0), \mathcal{S}_1 \times (0)\}$ induces $K_{3,5}$ as a subgraph of $\mathcal{EG}(\mathcal{S})$, a contradiction. Hence $\mathcal{S}_1$ has a unique nonzero proper ideal given by $\Im_1$. Thus, $\mathcal{EG}(\mathcal{S})$ is planar from [10, Theorem 4.10], a contradiction.
Conversely, if (1) holds, then the projective embedding of $\mathcal{EG}(\mathcal{S})$ is shown in Figure 1. If (2) holds, then the projective embedding of $\mathcal{EG}(\mathcal{S})$ is shown in Figure 2, where

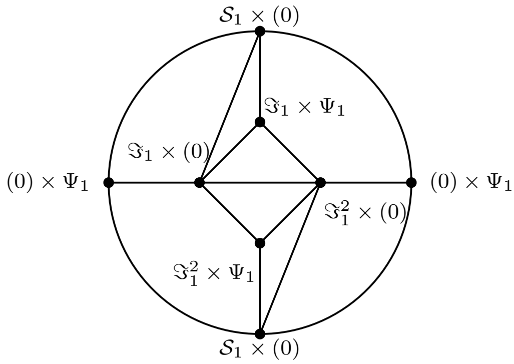

Figure 1. Projective embedding of $\mathcal{EG}(\mathcal{S}_1 \times \Psi_1)$, where $\Im_1$ and $\Im_1^2$ are only nonzero proper ideals of $\mathcal{S}_1$

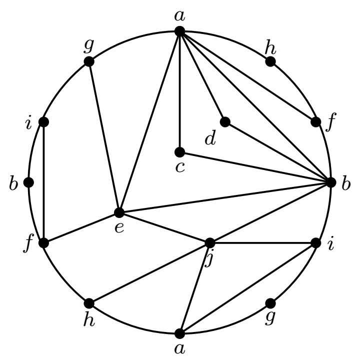

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Figure 2. Projective embedding of $\mathcal{EG}(\mathcal{S}_1 \times \Psi_1 \times \Psi_2)$, where $\Im_1$ is the only nonzero proper ideal of $\mathcal{S}_1$

$a = \Im_1 \times (0) \times (0)$, $b = \mathcal{S}_1 \times (0) \times (0)$, $c = \Im_1 \times \Psi_1 \times \Psi_2$, $d = (0) \times \Psi_1 \times \Psi_2$, $e = \Im_1 \times (0) \times \Psi_2$, $f = (0) \times \Psi_1 \times (0)$, $g = \mathcal{S}_1 \times \Psi_1 \times (0)$, $h = \mathcal{S}_1 \times (0) \times \Psi_2$, $i = (0) \times (0) \times \Psi_2$, $j = \Im_1 \times \Psi_1 \times (0)$.

□

3. Coloring of essential annihilating-ideal graph

In this section, we will discuss about the coloring of $\mathcal{EG}(\mathcal{S})$. We prove that $\mathcal{EG}(\mathcal{S})$ is weakly perfect for an Artinian ring $\mathcal{S}$. Moreover, the exact value of $\chi(\mathcal{EG}(\mathcal{S}))$ is given.

Theorem 5. *Let $\mathcal{S}$ be a commutative Artinian ring. Then the following hold:*

1. *If $\mathcal{S}$ is a local ring, then $\omega(\mathcal{EG}(\mathcal{S})) = \chi(\mathcal{EG}(\mathcal{S})) = |\mathbb{I}^*(\mathcal{S})|$.*
2. *If $\mathcal{S}$ is a non-local ring, then $\omega(\mathcal{EG}(\mathcal{S})) = \chi(\mathcal{EG}(\mathcal{S})) = |N^*(\mathcal{S})| + |Max(\mathcal{S})|$, where*

$N(\mathcal{S})$ is the set of nilpotent ideals of $\mathcal{S}$.

Proof. (1) Since $\mathcal{S}$ is local, then by [10, Lemma 3.2], $\mathcal{EG}(\mathcal{S})$ is a complete graph. Hence $\omega(\mathcal{EG}(\mathcal{S})) = \chi(\mathcal{EG}(\mathcal{S})) = |\mathbb{I}^*(\mathcal{S})|$.
(2) Since $\mathcal{S}$ is non-local Artinian ring, $\mathcal{S} \cong \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$, where each $\mathcal{S}_i$ is a local Artinian ring and $n \geq 2$. Define

$$A = \{\mathcal{I}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_n \in A^*(\mathcal{S}) : \mathcal{I}_i \text{ is a nilpotent ideal of } \mathcal{S}_i \text{ for each } 1 \leq i \leq n\}$$

and

$$B = \{\mathcal{I}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_n \in A^*(\mathcal{S}) : \mathcal{I}_i \text{ is not a nilpotent ideal of } \mathcal{S}_i \text{ for some } 1 \leq i \leq n\}.$$

It is easy to see that $A^*(\mathcal{S}) = A \cup B$ and $A \cap B = \emptyset$. Consider the following claims:
Claim(i): $\mathcal{EG}(\mathcal{S})[A]$ is a complete subgraph of $\mathcal{EG}(\mathcal{S})$. Let $\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_n \in A$, then $\mathcal{I}_i$ is a nilpotent ideal of $\mathcal{S}_i$ for each $1 \leq i \leq n$. This implies that $\mathcal{I}$ is also a nilpotent ideal of $\mathcal{S}$ and hence is adjacent with every other vertex of $\mathcal{EG}(\mathcal{S})$ by [10, Lemma 3.1]. Thus, our claim is proved.
Claim(ii): $\mathcal{EG}(\mathcal{S})[B]$ is a multipartite subgraph of $\mathcal{EG}(\mathcal{S})$. Define

$$B_1 = \{\mathcal{S}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_n \in B : \mathcal{I}_i \trianglelefteq \mathcal{S}_i \text{ for each } 2 \leq i \leq n\}$$

and for each $2 \leq i \leq n$

$$B_i = \{\mathcal{I}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_{i-1} \times \mathcal{S}_i \times \mathcal{I}_{i+1} \times \cdots \times \mathcal{I}_n \in B : \mathcal{I}_k \triangleleft \mathcal{S}_k \text{ for each } 1 \leq k \leq i-1 \text{ and } \mathcal{I}_l \trianglelefteq \mathcal{S}_l \text{ for each } i+1 \leq l \leq n\}.$$

Then one can see that $B = \bigcup_{i=1}^{n} B_i$ and $\bigcap_{i=1}^{n} B_i = \emptyset$. Let $\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_n$ and $\mathcal{J} = \mathcal{J}_1 \times \mathcal{J}_2 \times \cdots \times \mathcal{J}_n \in B_i$ for some $1 \leq i \leq n$. Since $Ann(\mathcal{I}_i\mathcal{J}_i) = Ann(\mathcal{S}_i) = 0$, $Ann(\mathcal{I}\mathcal{J}) \not\leq_e \mathcal{S}$. This implies that $\mathcal{I} \nsim \mathcal{J}$ in $\mathcal{EG}(\mathcal{S})$. Hence, no two elements of B_i are adjacent for each $1 \leq i \leq n$. This prove our claim.
Also, we can see in Claim(i) that each element of $\mathcal{EG}(\mathcal{S})[A]$ is adjacent with every element of $\mathcal{EG}(\mathcal{S})[B]$. Hence $\mathcal{EG}(\mathcal{S}) = \mathcal{EG}(\mathcal{S})[A] \vee \mathcal{EG}(\mathcal{S})[B]$, which implies that

$$\omega(\mathcal{EG}(\mathcal{S})) = \chi(\mathcal{EG}(\mathcal{S})) = |A| + n = |N^*(\mathcal{S})| + |Max(\mathcal{S})|, \text{ where } n = |Max(\mathcal{S})|.$$

□

Corollary 1. *Let $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$ be a commutative ring, where $\mathcal{S}_i$ is a local Artinian ring for each $1 \leq i \leq n$ and $n \geq 2$. If $\mathcal{S}_i$ has finitely many ideals for each $1 \leq i \leq n$, then*

$$\omega(\mathcal{EG}(\mathcal{S})) = \chi(\mathcal{EG}(\mathcal{S})) = n - 1 + \prod_{i=1}^{n} m_i,$$

where m_i is the number of proper ideals of $\mathcal{S}_i$ for each $1 \leq i \leq n$.

Proof. Since $\mathcal{S}_i$ is a local Artinian ring for each $1 \leq i \leq n$, $|Max(\mathcal{S})| = n$. Also, $\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_n$ is a nilpotent ideal of $\mathcal{S}$ if $\mathcal{I}_i \lhd \mathcal{S}_i$ for each $1 \leq i \leq n$. Thus, $|N^*(\mathcal{S})| = \prod_{i=1}^{n} m_i - 1$. Hence, by Theorem 5,

$$\omega(\mathcal{EG}(\mathcal{S})) = \chi(\mathcal{EG}(\mathcal{S})) = n - 1 + \prod_{i=1}^{n} m_i.$$

□

Corollary 2. *Let $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ be the prime decomposition of n, where $m \geq 2$. Then*

$$\omega(\mathcal{EG}(\mathbb{Z}_n)) = \chi(\mathcal{EG}(\mathbb{Z}_n)) = m + k_1 k_2 \cdots k_m - 1.$$

Corollary 3. *Let $\mathcal{S} = \Psi_1 \times \Psi_2 \times \cdots \times \Psi_n$ be a commutative ring, where Ψ_i is a field for each $1 \leq i \leq n$. Then*

$$\omega(\mathcal{EG}(\mathcal{S})) = \chi(\mathcal{EG}(\mathcal{S})) = n.$$

We look at $\mathcal{EG}(\mathcal{S})$ with finite chromatic number in the next two results.

Theorem 6. *Let $\mathcal{S}$ be a commutative non-reduced ring. If $\omega(\mathcal{EG}(\mathcal{S})) < \infty$, then these statements are equivalent:*

1. *$Z(\mathcal{S}) = Nil(\mathcal{S})$.*
2. *$\mathcal{S}$ is an Artinian local ring.*

Proof. (1) $\Longrightarrow$ (2) Let $\mathcal{I}$ be a nonzero nilpotent ideal of $\mathcal{S}$. Then we claim that $\mathcal{I}$ is finitely generated. Suppose on contrary that $\mathcal{I}$ is generated by $\{x_i : i \in \wedge\}$ with $|\wedge| = \infty$. Since $x_i\mathcal{S} \subseteq Nil(\mathcal{S})$ for each $i \in \wedge$, $\{x_i\mathcal{S} : i \in \wedge\}$ is an infinite clique by [10, Lemma 3.1], a contradiction. Hence our claim is proved. Define $X = \{\mathcal{I} \in \mathcal{S} : \mathcal{I}$ is nilpotent ideal of $\mathcal{S}\}$. Then again by [10, Lemma 3.1], the graph induced by X is a complete subgraph of $\mathcal{EG}(\mathcal{S})$ and thus $|X| < \infty$. This together with $Z(\mathcal{S}) = Nil(\mathcal{S})$ implies that $\mathcal{S}$ is an Artinian ring. Also, $Z(\mathcal{S}) = Nil(\mathcal{S})$ shows that $\mathcal{S}$ is local.
(2) $\Longrightarrow$ (1) is clear. □

Theorem 7. *Let $\mathcal{S}$ be a commutative Artinian ring. Then $\omega(\mathcal{EG}(\mathcal{S})) = \chi(\mathcal{EG}(\mathcal{S})) < \infty$ if and only if one of the following holds:*

1. *$\mathcal{S}$ is a reduced ring.*
2. *$\mathcal{S}$ has finite number of ideals.*

Proof. One way is clear. Suppose that $\omega(\mathcal{EG}(\mathcal{S})) = \chi(\mathcal{EG}(\mathcal{S})) < \infty$ and $\mathcal{S}$ is not a reduced ring. Since $\mathcal{S}$ is an Artinian ring, $\mathcal{S} \cong \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$, where $(\mathcal{S}_i, \Im_i)$ is a local Artinian ring with $\Im_i \neq 0$ for each i. Define $X = \{\mathcal{I} \times (0) \times \cdots \times (0) \in \mathbb{I}(\mathcal{S}) : \mathcal{I}$ is a nilpotent ideal of $\mathcal{S}_1\}$. Then by [10, Lemma 3.1], the graph induced by the set X is a complete subgraph of $\mathcal{EG}(\mathcal{S})$, which implies that $|X| < \infty$. Since $\mathcal{S}_1$ is local, $Z(\mathcal{S}_1) = Nil(\mathcal{S}_1)$ and so $\mathcal{S}_1$ contains finitely many ideals. Let $x \in Nil(\mathcal{S}_1)$ with $x^2 = 0$. Then the set $\{x\mathcal{S}_1 \times \mathcal{I}_2 \times \mathcal{I}_3 \times \cdots \times \mathcal{I}_n \in \mathbb{I}(\mathcal{S}) : \mathcal{I}_2 \in A^*(\mathcal{S}_2)$ and $\mathcal{I}_i = (0)$ for each $3 \leq i \leq n\}$ is a clique of $\mathcal{EG}(\mathcal{S})$ and so $\mathcal{S}_2$ has finitely many ideals. Applying the same argument one can shows that $\mathcal{S}$ also have finitely many ideals. □

4. Domination number of essential annihilating-ideal graph

In this section, we will discuss about the domination number $\mathcal{EG}(\mathcal{S})$. The following result shows that the domination number of $\mathcal{EG}(\mathcal{S})$ may be any arbitrary number.

Proposition 1. *Let $\mathcal{S} = \Psi_1 \times \Psi_2 \times \cdots \times \Psi_n$ be a commutative ring, where each Ψ_i is a field and $n \geq 3$ is a fixed integer. Then $\lambda(\mathcal{EG}(\mathcal{S})) = n$.*

Proof. We claim that $\mathcal{A} = \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n\}$ is a dominating set of $\mathcal{EG}(\mathcal{S})$, where $\mathcal{E}_i = (0) \times (0) \times \cdots \times (0) \times \Psi_i \times (0) \times \cdots \times (0)$ for each $1 \leq i \leq n$. Let $\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_n$ be any vertex of $\mathcal{EG}(\mathcal{S})$, then there exists at least one $1 \leq j \leq n$ such that $\mathcal{I}_j = (0)$. Since $\mathcal{I} \cdot \mathcal{E}_j = 0$, $\mathcal{I}$ is adjacent with $\mathcal{E}_j$ in $\mathcal{EG}(\mathcal{S})$. This prove our claim. Hence $\lambda(\mathcal{EG}(\mathcal{S})) \leq n$.
To show that $\lambda(\mathcal{EG}(\mathcal{S})) \geq n$, let $\overline{\mathcal{A}}$ be any $\lambda(\mathcal{EG}(\mathcal{S}))$-set and $\overline{\mathcal{E}}_i = \Psi_1 \times \Psi_2 \times \cdots \times \Psi_{i-1} \times (0) \times \Psi_{i+1} \times \cdots \times \Psi_n$ for each $1 \leq i \leq n$. Observe that $\overline{\mathcal{E}}_i$ is only adjacent with $\mathcal{E}_i$ for each $1 \leq i \leq n$. Thus, $|\overline{\mathcal{A}} \cap \{\mathcal{E}_i, \overline{\mathcal{E}}_i\}| \geq 1$ for each $1 \leq i \leq n$, which implies that $|\overline{\mathcal{A}}| \geq n$. Hence $\lambda(\mathcal{EG}(\mathcal{S})) = n$. □

Theorem 8. *Let $\mathcal{S}$ be a commutative ring. Then $\lambda(\mathcal{EG}(\mathcal{S})) = 1$ if and only if one of the following holds:*

1. *There exists a nonzero ideal $\mathcal{I}$ of $\mathcal{S}$ such that $Ann(\mathcal{I}) \leq_e \mathcal{S}$.*
2. *$\mathcal{S} \cong \Psi \times \mathcal{D}$, where Ψ is a field and $\mathcal{D}$ is an integral domain.*

Proof. Suppose there exists $\mathcal{I}$ such that $Ann(\mathcal{I}) \leq_e \mathcal{S}$. Since $Ann(\mathcal{I}) \subseteq Ann(\mathcal{I}\mathcal{J})$ for each $\mathcal{J} \in A^*(\mathcal{S}) \setminus \{\mathcal{I}\}$, $Ann(\mathcal{I}\mathcal{J}) \leq_e \mathcal{S}$. This implies that $\mathcal{J}$ is adjacent with $\mathcal{I}$ for every $\mathcal{J} \in A^*(\mathcal{S}) \setminus \{\mathcal{I}\}$ and hence $\lambda(\mathcal{EG}(\mathcal{S})) = 1$. If $\mathcal{S} \cong \Psi \times \mathcal{D}$, where Ψ is a field and $\mathcal{D}$ is an integral domain. Then $\{\Psi \times (0)\}$ is a dominating set of $\mathcal{EG}(\mathcal{S})$. Hence $\lambda(\mathcal{EG}(\mathcal{S})) = 1$.
Conversely, suppose that $\lambda(\mathcal{EG}(\mathcal{S})) = 1$. Consider the following cases:

Case(i) If $\mathcal{S}$ is a non-reduced ring with $a \in Nil^*(\mathcal{S})$, then by [10, Lemma 3.1], $Ann(a\mathcal{S}) \leq_e \mathcal{S}$, where $a\mathcal{S}$ denotes the ideal generated by a in $\mathcal{S}$.

Case(ii) If $\mathcal{S}$ is a reduced ring, then $\mathcal{EG}(\mathcal{S}) = AG(\mathcal{S})$ by [10, Theorem 2.5]. Hence by [11, Corollary 2.2], $\mathcal{S} \cong \Psi \times \mathcal{D}$, where Ψ is a field and $\mathcal{D}$ is an integral domain. $\square$

We end this section with the classification of domination number of $\mathcal{EG}(\mathcal{S})$ for an Artinian ring $\mathcal{S}$.

Corollary 4. *Let $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$ be a commutative ring, where each $\mathcal{S}_i$ is a local Artinian ring and $n \geq 2$. Then one of the following holds:*

1. *If $\mathcal{S}_i$ is a field for each i and $n \geq 3$, then $\lambda(\mathcal{EG}(\mathcal{S})) = n$.*
2. *If $n = 2$, $\mathcal{S}_1$ and $\mathcal{S}_2$ both are fields, then $\lambda(\mathcal{EG}(\mathcal{S})) = 1$.*
3. *If $\mathcal{S}_i$ is not a field for some i, then $\lambda(\mathcal{EG}(\mathcal{S})) = 1$.*

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