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Research Article

Bounds on Sombor Index for Corona Products on R-Graphs

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Abstract: Operations in the theory of graphs has a substantial influence in the analytical and factual dimensions of the domain. In the realm of chemical graph theory, topological descriptor serves as a comprehensive graph invariant linked with a specific molecular structure. The study on the Sombor index is initiated recently by Ivan Gutman. The triangle parallel graph comprises of the edges of subdivision graph along with the edges of the original graph. In this paper, we make use of combinatorial inequalities related with the vertices, edges and the neighborhood concepts as well as the other topological descriptors in the computations for the determination of bounds of Sombor index for certain corona products involving the triangle parallel graph.

Keywords: Sombor index, compounds graph, triangle parallel graphs, graph operations

AMS Subject classification: 05C09, 05C76, 05C92

1. Introduction

Molecular topology is necessarily a non-numerical mathematical entity. Specific numerics often facilitate in the determination of certain quantifiable molecular traits. A chemical compound is ought to alter itself to a molecular network such that the molecular atoms correspond to vertices and the atomic edges are depicted to be the edges. Let G be a molecular graph with vertex set $V(G)$ and edge set $E(G)$, possessing p vertices and q edges, i.e., $|V(G)| = p$ and $|E(G)| = q$. The degree of

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vertex w in G is represented by $d_G(w)$and $e = vw$ is the edge joining the vertex v with vertex w. For the graph, G the minimum and maximum degree are depicted by δ_G and Δ_G respectively.

Initiated by Gutman ,the first and second Zagreb index are the extensively researched topological invariants while stipulating the π-electron energy of the molecules [12–14, 21]. Randić index is one among the primitive and widely explored molecular descriptors [9, 25]. Since then, a great deal of progress and advancements have been carried out in the investigation of molecular descriptors in this realm of chemical graph theory.
The sum-connectivity index, χ was proposed by Bo Zhou and Nenad Trinajstić. Subsequently, numerous characteristics and combinatorial inequalities for corresponding invariant were determined in [28].

$$\chi(G) = \sum_{vw \in E(G)} (d_G(v) + d_G(w))^{-1/2}.$$

Consequently, on [29] generalizing sum-connectivity index with the first Zagreb index embarked the study on another descriptor called the general sum-connectivity index, χ_α $(\alpha \in \mathbb{R})$. Certain theoretical underpinnings for the invariant have also been determined in relation to certain graph operations [1].

$$\chi_\alpha(G) = \sum_{yz \in E(G)} (d_G(y) + d_G(z))^{\alpha}.$$

Another topological descriptor known as the first general Zagreb index, M_1^α $(\alpha \in \mathbb{R})$ was introduced on implementing specific generalizations by Xueliang Li and Jie Zheng [16]. Specific properties along with some graph edge operations have been observed related to the descriptor [18].

$$M_1^\alpha(G) = \sum_{w \in V(G)} (d_G(w))^{\alpha}.$$

Notably, the notions and implementations on Sombor index(SO) were established by Gutman [10] and substantial research on the graph invariant [4, 11, 22, 24] is being explored. SO index for graph G is described:

$$SO(G) = \sum_{yz \in E(G)} \sqrt{d_G(y)^2 + d_G(z)^2}.$$

The study on several graph operations have always provided a wide scope for research in the related areas of the discipline. The subdivision graph $S(G)$ of the graph G is generated by replacing every edge of G with a vertex of degree 2, keeping the original vertices unchanged. Thus, $|V(S(G))| = p + q$ and $|E(S(G))| = 2q$.

Definition 1. The triangle parallel graph $R(G)$ of the graph G is the graph with same vertex set as $S(G)$, including the edges of both G and $S(G)$. Thus, $V(R(G)) = V(S(G))$ and $E(R(G)) = E(G) \cup E(S(G))$, implying that $|V(R(G))| = p + q$ and $|E(R(G))| = 3q$.

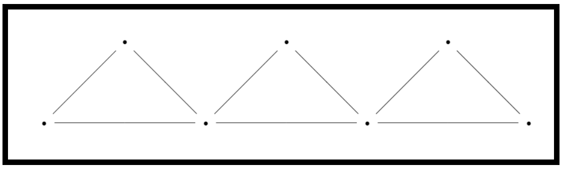


Figure 1. **Triangle parallel graph of P_4, $R(P_4)$**

Four new graph operational series related to the triangle parallel graphs have been initiated by Jie Lan and Bo Zhou [15]. Combinatorial bounds have been obtained for the general sum-connectivity index and the first entire Zagreb index with respect to the corona products [2, 19]. Also inequalities related to the subdivision graph and the corona product variants have been determined related to SK index in [27]. Related works have also been presented for the Sombor indices[6, 17, 23, 30]. Discrete inequalities related to some graph operation formulations have been analysed for certain other molecular descriptors[3, 5, 7, 8, 20, 26].

2. Methodology

In this paper, certain combinatorial bounds linked with corona graph products predicated mainly on the triangle parallel graph, are proposed. We have included the corona operations correlated with vertex, edge, vertex neighborhood and edge neighborhood products for our computations.

2.1. R-Vertex Corona Product

Definition 2. [15] Let G be a simple graph with p_G vertices and q_G edges. Let H be another simple graph with p_H vertices and q_H edges. The R-vertex corona product of G and H, denoted by $R(G) \odot H$, is constructed by taking one copy of $R(G)$ and p_G copies of H, connecting each vertex belonging to $V(G)$ with all vertices of a distinct copy of H. Thus, $|V(R(G) \odot H)| = p_G + q_G + p_G p_H$ and $|E(R(G) \odot H)| = 3q_G + p_G q_H + p_G p_H$.

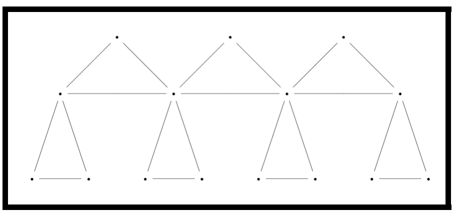


Figure 2. **R-vertex corona product $R(P_4) \odot P_2$**

The degree behaviour of the vertices is:

$$d_{R(G)\odot H}(x)=\begin{cases}2d_G(x)+p_H & ;\text{ if } x\in V(G)\\ 2 & ;\text{ if } x\in I(G)\\ d_H(x)+1 & ;\text{ if } x\in V(H).\end{cases}$$

Theorem 1. *Assume that G and H are arbitrary graphs. Then*

$$\begin{aligned}SO(R(G)\odot H) \quad &\leq \quad 2(SO(G)+M_1(G))+p_G SO(H)+2\sqrt{p_H}\Big[M_1^{1/2}(G)+M_1^{3/2}(G)\\ &+\chi_{1/2}(G)\Big]+\sqrt{2}p_G\Big[M_1^{1/2}(H)+\chi_{1/2}(H)+(\sqrt{2}+1)q_H\Big]\\ &+2q_G\sqrt{p_H^2+4}+p_H\Big(p_G\sqrt{p_H^2+1}+(4+\sqrt{2})q_G\Big).\end{aligned}$$

Equality holds if and only if the components of the graph G and H are isolated vertices.

Proof.

$$\begin{aligned}SO(R(G)\odot H) \quad &= \sum_{xy\in E(R(G)\odot H)}\sqrt{d_{R(G)\odot H}(x)^2+d_{R(G)\odot H}(y)^2}\\ &=\sum_{\substack{xy\in E(R(G))\\ x,y\in V(G)}}\sqrt{(2d_G(x)+p_H)^2+(2d_G(y)+p_H)^2}\\ &+\sum_{\substack{xy\in E(R(G))\\ x\in V(G),y\in I(G)}}\sqrt{(2d_G(x)+p_H)^2+4}\\ &+p_G\sum_{xy\in E(H)}\sqrt{(d_H(x)+1)^2+(d_H(y)+1)^2}\\ &+\sum_{\substack{x\in V(R(G))\\ x\in V(G)}}\sum_{y\in V(H)}\sqrt{(2d_G(x)+p_H)^2+(d_H(y)+1)^2}\\ &=\sum f_1+\sum f_2+\sum g+\sum h\end{aligned}$$

where,

$$\sum f_1=\sum_{\substack{xy\in E(R(G))\\ x,y\in V(G)}}\sqrt{(2d_G(x)+p_H)^2+(2d_G(y)+p_H)^2}$$

$$\sum f_2=\sum_{\substack{xy\in E(R(G))\\ x\in V(G),y\in I(G)}}\sqrt{(2d_G(x)+p_H)^2+4}$$

$$\sum g=p_G\sum_{xy\in E(H)}\sqrt{(d_H(x)+1)^2+(d_H(y)+1)^2}$$

$$\sum h=\sum_{\substack{x\in V(R(G))\\ x\in V(G)}}\sum_{y\in V(H)}\sqrt{(2d_G(x)+p_H)^2+(d_H(y)+1)^2}.$$

For computation of $\sum f_1$ and $\sum f_2$,

$$\begin{aligned}\sum f_1 &= \sum_{\substack{xy\in E(R(G))\\ x,y\in V(G)}} \sqrt{(2d_G(x)+p_H)^2+(2d_G(y)+p_H)^2}\\ &= \sum_{\substack{xy\in E(R(G))\\ x,y\in V(G)}} \sqrt{4(d_G(x)^2+d_G(y)^2)+4p_H(d_G(x)+d_G(y))+2p_H^2}\\ \implies \sum f_1 &\leq 2SO(G)+\sqrt{2p_H}\Big(\sqrt{2}\chi_{1/2}(G)+q_G\sqrt{p_H}\Big).\end{aligned}$$

Similarly for $\sum f_2$,we have

$$\sum f_2 \leq 2\Big[M_1(G)+\sqrt{p_H}M_1^{3/2}(G)+q_G\sqrt{p_H^2+4}\Big].$$

In order to determine $\sum g$,

$$\sum g = p_G\sum_{xy\in E(H)}\sqrt{(d_H(x)+1)^2+(d_H(y)+1)^2} \leq p_G\Big[SO(H)+\sqrt{2}\Big(\chi_{1/2}(H)+q_H\Big)\Big].$$

For the determination of $\sum h$,

$$\sum h = \sum_{\substack{x\in V(R(G))\\ x\in V(G)}}\sum_{y\in V(H)}\sqrt{(2d_G(x)+p_H)^2+(d_H(y)+1)^2}$$

$$\text{and so } \sum h \leq \sqrt{2}\Big(\sqrt{2p_H}M_1^{1/2}(G)+p_GM_1^{1/2}(H)\Big)+p_G\Big(p_H\sqrt{p_H^2+1}+2q_H\Big)+4q_Gp_H.$$

From all the computations,

$$\begin{aligned}SO(R(G)\odot H) &\leq 2(SO(G)+M_1(G))+p_GSO(H)+2\sqrt{p_H}\Big[M_1^{1/2}(G)+M_1^{3/2}(G)\\ &\quad+\chi_{1/2}(G)\Big]+\sqrt{2}p_G\Big[M_1^{1/2}(H)+\chi_{1/2}(H)+(\sqrt{2}+1)q_H\Big]\\ &\quad+2q_G\sqrt{p_H^2+4}+p_H\Big(p_G\sqrt{p_H^2+1}+(4+\sqrt{2})q_G\Big).\end{aligned}$$

The equality holds if and only if the components of the graph G and H are isolated vertices. This concludes the proof. □

2.2. R-Edge Corona Product

Definition 3. [15]The R-Edge Corona Product for $R(G)$ and H as indicated by $R(G)\ominus H$ is obtained by taking a copy of distinct vertex graph $R(G)$ and q_G copies of H and linking a vertex of $I(G)$ on i_{th} location in $R(G)$ to each vertex in the i_{th} copy of H.$|V(R(G)\ominus H)| = p_G+q_G+q_Gp_H$ and $|E(R(G)\ominus H)| = 3q_G+q_Gq_H+q_Gp_H$.

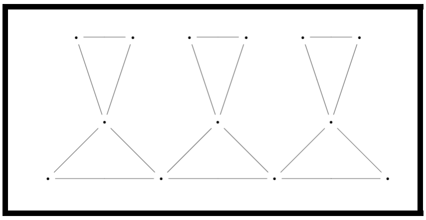


Figure 3. **R-edge corona product** $R(P_4) \ominus P_2$

The degree behaviour of the vertices is:

$$d_{R(G)\ominus H}(x) = \begin{cases} 2d_G(x) & ; \text{ if } x \in V(G) \\ 2 + p_H & ; \text{ if } x \in I(G) \\ d_H(x) + 1 & ; \text{ if } x \in V(H). \end{cases}$$

Theorem 2. *Assume that G and H are arbitrary graphs. Then*

$$SO(R(G) \ominus H) \leq 2SO(G) + q_G SO(H) + \sqrt{2}\left[\sqrt{2}M_1(G) + q_G\left(M_1^{1/2}(H) + \chi_{1/2}(H)\right)\right]$$
$$+ q_G\left[\sqrt{2}\left(\sqrt{2}(p_H+2) + q_H(\sqrt{2}+1)\right) + p_H\left(\sqrt{p_H^2 + 4p_H + 5}\right)\right].$$

Equality holds if and only if the components of the graph G and H are isolated vertices.

Proof.

$$\begin{aligned} SO(R(G) \ominus H) &= \sum_{xy \in E(R(G)\ominus H)} \sqrt{d_{R(G)\ominus H}(x)^2 + d_{R(G)\ominus H}(y)^2} \\ &= \sum_{\substack{xy \in E(R(G)) \\ x,y \in V(G)}} \sqrt{(2d_G(x))^2 + (2d_G(y))^2} + \sum_{\substack{xy \in E(R(G)) \\ x \in V(G), y \in I(G)}} \sqrt{(2d_G(x))^2 + (p_H+2)^2} \\ &+ q_G \sum_{xy \in E(H)} \sqrt{(d_H(x)+1)^2 + (d_H(y)+1)^2} \\ &+ \sum_{\substack{x \in V(R(G)) \\ x \in I(G)}} \sum_{y \in V(H)} \sqrt{(p_H+2)^2 + (d_H(y)+1)^2}. \end{aligned}$$

So,

$$SO(R(G) \ominus H) = \sum f_1 + \sum f_2 + \sum g + \sum h$$

where,

$$\sum f_1 = \sum_{\substack{xy \in E(R(G)) \\ x,y \in V(G)}} \sqrt{(2d_G(x))^2 + (2d_G(y))^2}$$

$$\sum f_2 = \sum_{\substack{xy\in E(R(G))\\ x\in V(G), y\in I(G)}} \sqrt{(2d_G(x))^2 + (p_H+2)^2}$$

$$\sum g = q_G \sum_{xy\in E(H)} \sqrt{(d_H(x)+1)^2 + (d_H(y)+1)^2}$$

$$\sum h = \sum_{\substack{x\in V(R(G))\\ x\in I(G)}} \sum_{y\in V(H)} \sqrt{(p_H+2)^2 + (d_H(y)+1)^2}.$$

Now, for the computation of $\sum f_1$ and $\sum f_2$,

$$\begin{aligned}\sum f_1 &= \sum_{\substack{xy\in E(R(G))\\ x,y\in V(G)}} \sqrt{(2d_G(x))^2 + (2d_G(y))^2}\\ &= 2 \sum_{\substack{xy\in E(R(G))\\ x,y\in V(G)}} \sqrt{d_G(x)^2 + d_G(y)^2}\\ &= 2SO(G)\end{aligned}$$

and,

$$\sum f_2 \quad \leq \quad 2\Big[M_1(G) + q_G(p_H+2)\Big].$$

To determine $\sum g$,

$$\begin{aligned}\sum g &= q_G \sum_{xy\in E(H)} \sqrt{(d_H(x)+1)^2 + (d_H(y)+1)^2}\\ &= q_G \sum_{xy\in E(H)} \sqrt{\Big(d_H(x)^2 + d_H(y)^2\Big) + 2\Big(d_H(x) + d_H(y)\Big) + 2}\end{aligned}$$

$$\sum g \quad \leq \quad q_G\Big[SO(H) + \sqrt{2}\Big(\chi_{1/2}(H) + q_H\Big)\Big].$$

For the determination of $\sum h$,

$$\sum h = \sum_{\substack{x\in V(R(G))\\ x\in I(G)}} \sum_{y\in V(H)} \sqrt{(p_H+2)^2 + (d_H(y)+1)^2}$$

$$\sum h \quad \leq \quad q_G\Big[\sqrt{2}\Big(M_1^{1/2}(H) + \sqrt{2}q_H\Big) + p_H\big(\sqrt{p_H^2 + 4p_H + 5}\big)\Big].$$

Hence from all the computations,

$$\begin{aligned}SO(R(G)\ominus H) \leq 2SO(G) + q_G SO(H) + \sqrt{2}\Big[\sqrt{2}M_1(G) + m_G\Big(M_1^{1/2}(H) + \chi_{1/2}(H)\Big)\Big]\\ + q_G\Big[\sqrt{2}\Big(\sqrt{2}(p_H+2) + q_H(\sqrt{2}+1)\Big) + p_H\Big(\sqrt{p_H^2 + 4p_H + 5}\Big)\Big].\end{aligned}$$

The equality holds if and only if the components of the graph G and H are isolated vertices. This concludes the proof. $\square$

2.3. R - Vertex Neighborhood Corona Product

Definition 4. [15]The R-Vertex Neighborhood Corona Product for $R(G)$ and H as indicated by $R(G) \boxdot H$ is obtained by taking one copy of distinct vertex graph $R(G)$ and p_G copies of H and linking the adjacent or neighboring vertex of G in $R(G)$ on the i_{th} location in $R(G)$ to each vertex in the i_{th} copy of H. $|V(R(G) \boxdot H)| = p_G + q_G + p_G p_H$ and $|E(R(G) \boxdot H)| = 3q_G + p_G q_H + 4q_G q_H$.

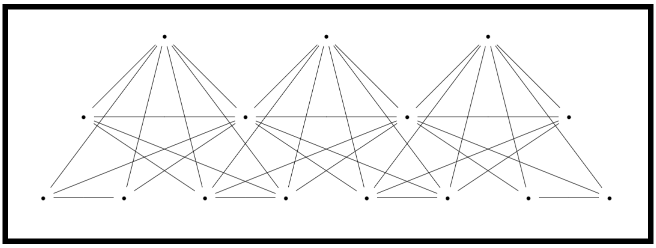


Figure 4. **R-vertex neighborhood corona product** $R(P_4) \boxdot P_2$

The degree behaviour of the vertices is:

$$d_{R(G)\boxdot H}(x) = \begin{cases} (2+p_H)d_G(x) & ; \text{ if } x \in V(G) \\ 2(p_H+1) & ; \text{ if } x \in I(G) \\ d_H(x) + 2d_G(y) & ; \text{ if } x \in V(H), y \in V(G). \end{cases}$$

Theorem 3. *Assume that G and H are arbitrary graphs. Then*

$$\begin{aligned} SO(R(G) \boxdot H) \quad \leq \quad & (p_H+2)SO(G) + p_G SO(H) + M_1(G)[p_H(p_H+7)+2] \\ & + 2\left[M_1^{1/2}(G)\chi_{1/2}(H) + 2M_1^{3/2}(G)M_1^{1/2}(H)\right] \\ & + 4q_G\left[p_H(p_H+2) + \sqrt{2}(\sqrt{2}+1)q_H + 1\right]. \end{aligned}$$

Equality holds if and only if the components of the graph G and H are isolated vertices.

Proof.

$$\begin{aligned} SO(R(G) \boxdot H) \ = \ & \sum_{xy \in E(R(G))} \sqrt{d_{R(G)}(x)^2 + d_{R(G)}(y)^2} \\ & + p_G \sum_{xy \in E(H)} \sqrt{d_H(x)^2 + d_H(y)^2} + \sum_{x \in V(R(G))} \sum_{y \in V(H)} \sqrt{d_{R(G)}(x)^2 + d_H(y)^2} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{xy\in E(R(G))\\ x,y\in V(G)}} \sqrt{(2+p_H)^2 d_G(x)^2 + (2+p_H)^2 d_G(y)^2} \\
&\quad + \sum_{\substack{xy\in E(R(G))\\ x\in V(G)\\ y\in I(G)}} \sqrt{(2+p_H)^2 d_G(x)^2 + 4(p_H+1)^2} \\
&\quad + p_G \sum_{xy\in E(H)} \sqrt{(d_H(x)+2d_G(w_i))^2 + (d_H(y)+2d_G(w_i))^2} \\
&\quad + \sum_{\substack{x\in V(G)\\ w_i\in N_G(x)\\ w_i\in V(G)}} \sum_{y\in V(H)} \sqrt{((p_H+2)^2 d_G(x)^2 + (2d_G(w_i)+d_H(y))^2)} \\
&\quad + \sum_{\substack{x\in V(G)\\ w_i\in N_G(x)\\ w_i\in I(G)}} \sum_{y\in V(H)} \sqrt{4(p_H+1)^2 + (2d_G(x)+d_H(y))^2} \\
&= \sum f_1 + \sum f_2 + \sum g + \sum h_1 + \sum h_2
\end{aligned}
$$

where,

$$
\sum f_1 = \sum_{\substack{xy\in E(R(G))\\ x,y\in V(G)}} \sqrt{(2+p_H)^2 d_G(x)^2 + (2+p_H)^2 d_G(y)^2}
$$

$$
\sum f_2 = \sum_{\substack{xy\in E(R(X))\\ x\in V(X)\\ y\in I(X)}} \sqrt{(2+p_H)^2 d_G(x)^2 + 4(p_H+1)^2}
$$

$$
\sum g = p_G \sum_{xy\in E(H)} \sqrt{(d_H(x)+2d_G(w_i))^2 + (d_H(y)+2d_G(w_i))^2}
$$

$$
\sum h_1 = \sum_{\substack{x\in V(G)\\ w_i\in N_G(x)\\ w_i\in V(G)}} \sum_{y\in V(H)} \sqrt{((p_H+2)^2 d_G(x)^2 + (2d_G(w_i)+d_H(y))^2)}
$$

$$
\sum h_2 = \sum_{\substack{x\in V(G)\\ w_i\in N_G(x)\\ w_i\in I(G)}} \sum_{y\in V(H)} \sqrt{4(p_H+1)^2 + (2d_G(x)+d_H(y))^2}.
$$

Now for the computation of $\sum f_1$ and $\sum f_2$,

$$
\sum f_1 \leq \quad (p_H+2)SO(G)
$$

$$
\sum f_2 \leq \quad (p_H+2)M_1(G) + 4q_G(p_H+1).
$$

Also for the computation of $\sum g$,

$$\sqrt{(d_H(x)+2d_G(w_i))^2+(d_H(y)+2d_G(w_i))^2}$$
$$= d_H(x)^2+d_H(y)^2+8d_G^2(w_i)+4d_G(w_i)\Big[d_H(x)+d_H(y)\Big]$$

$$\Longrightarrow \sum g \quad \leq \quad p_G SO(H)+2\left[M_1^{1/2}(G)\chi_{1/2}(H)+2\sqrt{2}q_G q_H\right].$$

To determine $\sum h_1$,

$$\sum h_1 = \sum_{\substack{x\in V(G)\\ w_i\in N_G(x)\\ w_i\in V(G)}} \sum_{y\in V(H)} \sqrt{((p_H+2)^2 d_G(x)^2+(2d_G(w_i)+d_H(y))^2)}$$
$$= \sum_{\substack{x\in V(G)\\ w_i\in N_G(x)\\ w_i\in V(G)}} \sum_{y\in V(H)} \sqrt{(p_H+2)^2 d_G(x)^2+d_H(y)^2+4d_G(w_i)^2+4d_G(w_i)d_H(y)}$$
$$= \sum_{x\in V(G)} \sum_{w_i\in N_G(x)} \sum_{y\in V(H)} \sqrt{(p_H+2)^2 d_G(x)^2+d_H(y)^2+4d_G(w_i)^2+4d_G(w_i)d_H(y)}.$$

Hence,

$$\sum h_1 \quad \leq \quad p_H\big(p_H+2\big)M_1(G)+2\left[p_H M_1(G)+M_1^{1/2}(H)M_1^{3/2}(G)\right]+4q_G q_H.$$

To determine $\sum h_2$,

$$\sum h_2 = \sum_{\substack{x\in V(G)\\ w_i\in N_G(x)\\ w_i\in I(G)}} \sum_{y\in V(H)} \sqrt{4(p_H+1)^2+(2d_G(x)+d_H(y))^2}$$
$$= \sum_{x\in V(G)} \sum_{y\in V(H)} \sqrt{4(p_H+1)^2+d_H(y)^2+4d_G(x)d_H(y)+4d_G(x)^2}.$$

Hence,

$$\sum h_2 \quad \leq \quad 2\left[M_1^{3/2}(G)M_1^{1/2}(H)+p_H M_1(G)+2q_G\Big(p_H(p_H+1)+q_H\Big)\right].$$

From all the computations,

$$\begin{aligned} SO(R(G)\boxdot H) \quad \leq \quad & (p_H+2)SO(G)+p_G SO(H)+M_1(G)[p_H(p_H+7)+2] \\ & +2\left[M_1^{1/2}(G)\chi_{1/2}(H)+2M_1^{3/2}(G)M_1^{1/2}(H)\right] \\ & +4q_G\left[p_H(p_H+2)+\sqrt{2}(\sqrt{2}+1)q_H+1\right]. \end{aligned}$$

The equality holds if and only if the components of the graph G and H are isolated vertices. This concludes the proof. □

2.4. R-Edge Neighborhood Corona Product

Definition 5. [15] The R-Edge Neighborhood Corona Product for $R(G)$ and H as indicated by $R(G) \boxminus H$is obtained by taking copy of distinct vertex graph $R(G)$ and q_G copies of H and linking the adjacencies or neighboring vertices of $I(G)$ in $R(G)$ on the i_{th} location in $R(G)$ to each vertex in i_{th} copy of H. $|V(R(G) \boxminus H)| = p_G + q_G + q_G q_H$ and $|E(R(G) \boxminus H)| = 3q_G + q_G q_H + 2q_G p_H$.

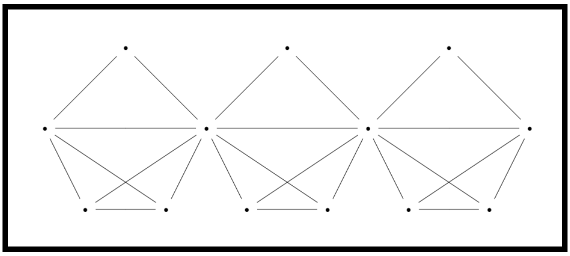


Figure 5. **R-edge neighborhood corona product** $R(P_4) \boxminus P_2$

The degree behaviour of the vertices is:

$$d_{R(G)\boxminus H}(x) = \begin{cases} (2+p_H)d_G(x) & ; \text{if } x \in V(G) \\ 2 & ; \text{if } x \in I(G) \\ d_H(x)+2 & ; \text{if } x \in V(H). \end{cases}$$

Theorem 4. *Assume that G and H are arbitrary graphs. Then*

$$\begin{aligned} SO(R(G) \boxminus H) \quad &\leq \quad (p_H+2)SO(G) + q_G SO(H) + M_1(G)\Big[p_H(p_H+3)+2\Big] \\ &+ 2q_G\Big[\chi_{1/2}(H) + 2M_1^{1/2}(H) + \sqrt{2}(\sqrt{2}+1)q_H + 2(p_H+1)\Big]. \end{aligned}$$

Equality holds if and only if the components of the graph G and H are isolated vertices.

Proof.

$$\begin{aligned} SO(R(G) \boxminus H) = &\sum_{xy \in E(R(G))} \sqrt{d_{R(G)}(x)^2 + d_{R(G)}(y)^2} \\ + \sum_{i=1}^{q_G} \sum_{xy \in E(H)} &\sqrt{d_H(x)^2 + d_H(y)^2} + \sum_{x \in V(R(G))} \sum_{v \in V(H)} \sqrt{d_{R(G)}(x)^2 + d_H(y)^2} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{xy\in E(R(G))\\ x,y\in V(G)}} \sqrt{(p_H+2)^2 d_G(x)^2 + (p_H+2)^2 d_G(y)^2}\\
&+ \sum_{\substack{xy\in E(R(G))\\ x\in V(G)\\ y\in I(G)}} \sqrt{(p_H+2)^2 d_G(x)^2 + 4} + \sum_{i=1}^{q_G} \sum_{xy\in E(H)} \sqrt{(d_H(x)+2)^2 + (d_H(y)+2)^2}\\
&+ \sum_{\substack{x\in I(G)\\ w_i\in N_G(x)\\ w_i\in V(G)}} \sum_{y\in V(H)} \sqrt{(p_H+2)^2 d_G(w_i)^2 + (d_H(y)+2)^2}\\
&= \sum f_1 + \sum f_2 + \sum g + \sum h.
\end{aligned}
$$

where,

$$
\begin{aligned}
\sum f_1 &= \sum_{\substack{xy\in E(R(G))\\ x,y\in V(G)}} \sqrt{(p_H+2)^2 d_G(x)^2 + (p_H+2)^2 d_G(y)^2}\\
\sum f_2 &= \sum_{\substack{xy\in E(R(G))\\ x\in V(G)\\ y\in I(G)}} \sqrt{(p_H+2)^2 d_G(x)^2 + 4}\\
\sum g &= \sum_{i=1}^{q_G} \sum_{xy\in E(H)} \sqrt{(d_H(x)+2)^2 + (d_H(y)+2)^2}\\
\sum h &= \sum_{\substack{x\in I(G)\\ w_i\in N_G(x)\\ w_i\in V(G)}} \sum_{y\in V(H)} \sqrt{(p_H+2)^2 d_G(w_i)^2 + (d_H(y)+2)^2}.
\end{aligned}
$$

For the computation of $\sum f_1$,

$$
\begin{aligned}
\sum f_1 &= \sum_{\substack{xy\in E(R(G))\\ x,y\in V(G)}} \sqrt{(p_H+2)^2 d_G(x)^2 + (p_H+2)^2 d_G(y)^2}\\
&= (p_H+2) \sum_{\substack{xy\in E(R(G))\\ x,y\in V(G)}} \sqrt{d_G(x)^2 + d_G(y)^2}\\
&= (p_H+2) SO(G).
\end{aligned}
$$

Similarly for $\sum f_2$,

$$
\begin{aligned}
\sum f_2 &= \sum_{\substack{xy\in E(R(G))\\ x\in V(G)\\ y\in I(G)}} \sqrt{(p_H+2)^2 d_G(x)^2 + 4}\\
&= \sum_{x\in V(G)} \left(\sqrt{(p_H+2)^2 d_G(x)^2 + 4}\right) d_G(x)
\end{aligned}
$$

Hence $\sum f_2 \quad \leq \quad (p_H+2)M_1(G)+4q_G$. To determine $\sum g$,

$$\sum g = \sum_{i=1}^{q_G} \sum_{xy \in E(H)} \sqrt{(d_H(x)+2)^2+(d_H(y)+2)^2}$$
$$= q_G \sum_{xy \in E(H)} \sqrt{(d_H(x)^2+d_H(y)^2)+4(d_H(x)+d_H(y))+8)}$$
$$\Longrightarrow \sum g \quad \leq \quad q_G\left[SO(H)+2(\chi_{1/2}(H)+\sqrt{2}q_H)\right].$$

Also for $\sum h$, we have,

$$\sum h \quad \leq \quad p_H(p_H+2)M_1(G)+4q_G\left[M_1^{1/2}(H)+p_H+q_H\right].$$

Hence from all the computations,

$$\begin{aligned} SO(R(G) \boxminus H) \quad \leq \quad & (p_H+2)SO(G)+q_G SO(H)+M_1(G)\left[p_H(p_H+3)+2\right] \\ & +2q_G\left[\chi_{1/2}(H)+2M_1^{1/2}(H)+\sqrt{2}(\sqrt{2}+1)q_H+2(p_H+1)\right]. \end{aligned}$$

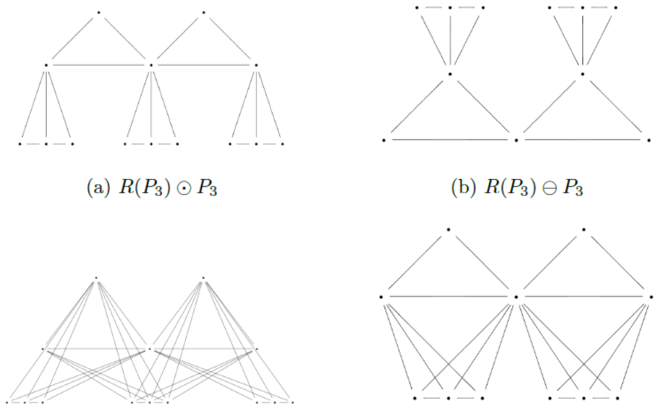
The equality holds if and only if the components of the graph G and H are isolated vertices. This concludes the result. □

3. Discussion

In this section, the formulae of the upper bounds of the Sombor index of the corona graph product variants on the triangle parallel graph for path graphs have been presented.

Example: For m,n $\geq$ 2, the upper bounds of Sombor index of the path graphs for the corona products defined on R-graph are given as:

$$\begin{aligned} SO(R(P_n) \odot P_m) \quad \leq \quad & 2(n-1)\sqrt{m^2+4}+mn\sqrt{m^2+1}+2\sqrt{m}(6.242641n \\ & -7.02118)+16.485281mn-5.414214m+1.471621n-20.026291 \\ SO(R(P_n) \ominus P_m) \quad \leq \quad & m(n-1)\sqrt{m^2+4m+5}+13.071068mn-13.071068m \\ & +5.471621n-11.841057 \\ SO(R(P_n) \boxdot P_m) \quad \leq \quad & 8m^2n-10m^2+76.970563mn-91.67m-16.558332n \\ & +5.949926 \\ SO(R(P_n) \boxminus P_m) \quad \leq \quad & 4m^2n-6m^2+38.142136mn-45.326854m-1.570224n \\ & -4.799213 \end{aligned}$$



(a) $R(P_3) \odot P_3$ (b) $R(P_3) \ominus P_3$

(c) $R(P_3) \boxdot P_3$ (d) $R(P_3) \boxminus P_3$

Figure 6. **Corona Product Variants for P_3 graphs**

For instance if we take $n = 3$, $m = 3$ then

$$\begin{aligned}
SO(R(P_3) \odot P_3) &= 119.547063 \quad (< 199.9495110)\\
SO(R(P_3) \ominus P_3) &= 80.145618119218 \quad (< 113.5943280815567)\\
SO(R(P_3) \boxdot P_3) &= 319.4687775336 \quad (< 499.999996)\\
SO(R(P_3) \boxminus P_3) &= 172.959028762 \quad (< 251.788775)
\end{aligned}$$

Thus, we take the explicit example of standard path graphs for a more profound understanding of the theorems stated.

4. Conclusion

In chemical graph theory, each molecular graph trait is critical for obtaining improvements and the procedure may be aided by the adequate research of topological descriptors.

Accordingly, this paper monitors and analyses the proposed graph operational products notably the R-Vertex, R-Edge, R-Vertex Neighborhood and R-Edge Neighborhood Corona Product of two graphs by exploring the bounds of Sombor index involving the triangle parallel graph or R-graph. The results achieved contributes towards further research for the exploration in the realm of degree, distance dependent descriptors and series of graph operational variants.

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