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Research Article

Maximizing the indices of a class of signed complete graphs

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Abstract: The index of a signed graph is the largest eigenvalue of its adjacency matrix. Let $\mathfrak{U}_{n,k,4}$ be the set of all signed complete graphs of order n whose negative edges induce a unicyclic graph of order k and girth at least 4. In this paper, we identify the signed graphs achieving the maximum index in the class $\mathfrak{U}_{n,k,4}$.

Keywords: Index, Signed Complete Graph, Unicyclic

AMS Subject classification: 05C22, 05C50

1. Introduction

The edge set and the vertex set of a simple graph G are denoted by $E(G)$ and $V(G)$, respectively. The *order* of G is the cardinality of $V(G)$. The *girth* of G is the order of a shortest cycle in G. Let $z \in V(G)$. The degree and the neighborhood of z are denoted by $d_G(z)$ and $N_G(z)$, respectively. As usual, K_n denotes the *complete graph* of order n and $K_{1,r}$ denotes the *star graph* of order $r+1$. By C_r, we denote a cycle of order r. A connected graph with exactly one cycle is called a *unicyclic* graph.

Let G be a simple graph, and $\sigma : E(G) \longrightarrow \{-,+\}$ be a mapping defined on the edge set of G. Then $\Psi = (G, \sigma)$ is called a *signed graph* with the underlying graph G. If all edges of a signed graph $\Psi = (G, \sigma)$ are positive, then Ψ is denoted by $(G, +)$. A *positive* (resp. *negative*) cycle is a signed cycle containing an even (resp. odd) number of negative edges. A signed graph is said to be *balanced* if all of its cycles, if any, are positive. Otherwise, it is said to be *unbalanced*. Let (K_n, H^-) denote a signed

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complete graph whose negative edges induce a subgraph H. By $\mathfrak{U}_{n,k,g}$, we denote the set of all signed complete graphs (K_n, U^-), where U is a unicyclic graph of order k and girth at least g. If $A(G) = (a_{ij})$ is the adjacency matrix of a simple graph G, then the *adjacency matrix* of $\Psi = (G, \sigma)$ is defined as $A(\Psi) = (a^{\sigma}_{ij})$, where $a^{\sigma}_{ij} = \sigma(w_i w_j) a_{ij}$. The *characteristic polynomial* of a signed graph Ψ is the characteristic polynomial of $A(\Psi)$ and is denoted by $\varphi(\Psi)$. Also, the spectrum of Ψ is the spectrum of $A(\Psi)$. The *index* of Ψ is the largest eigenvalue of Ψ. For some results on the spectrum of signed graphs see [3, 8, 10, 11].

Let $\Psi = (G, \sigma)$ be a signed graph and $S \subset V(\Psi)$. Let Ψ' be a graph obtained from Ψ by changing the signs of all edges between S and $V(\Psi) - S$. Then we call two graphs Ψ and Ψ' are *switching equivalent*, and we write $\Psi \sim \Psi'$. It is easy to see that two matrices $A(\Psi)$ and $A(\Psi')$ are similar and hence they have the same eigenvalues [12].

A classical problem in the spectral graph theory is the identification of extremal graphs with respect to the index in a given class of graphs. In [4], Brunetti and Ciampella detected the signed graphs with minimum index in the class of signed bicyclic graphs of order n. Signed graphs achieving the extremal index in the set of all unbalanced connected signed graphs with a fixed number of vertices have been studied in [6]. Brunetti and Stanić [5] established the first few signed graphs ordered decreasingly by the index in classes of connected signed graphs, connected unbalanced signed graphs, and signed complete graphs with n vertices. In [8], the signed graph whose largest eigenvalue is maximum among all graphs in $\mathfrak{U}_{n,k,3}$ is determined. In this paper, we find the signed graphs with maximum index in the class $\mathfrak{U}_{n,k,4}$. This result leads to a conjecture on $\mathfrak{U}_{n,k,g}$, for $g > 4$.

2. Main result

In [8], the authors proved that among all signed complete graphs of order $n > 5$ whose negative edges induce a unicyclic graph of order k, the signed graph (K_n, U_1^-) has the maximum index, where U_1 is shown in Fig. 1. Here, we determine $(K_n, U^-) \in \mathfrak{U}_{n,k,4}$ with maximum index. We begin with the following theorem well known as Interlacing theorem for signed graphs, which is a consequence of [7, Theorem 1.3.11].

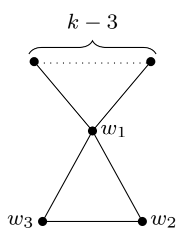

Figure 1. **The unicyclic graph U_1.**

Theorem 1. *Let Ψ be a signed graph of order n, and Ψ' be an induced subgraph of Ψ with m vertices. If the eigenvalues of Ψ and Ψ', respectively, are $\lambda_1 \geq \cdots \geq \lambda_n$ and $\lambda'_1 \geq \cdots \geq \lambda'_m$, then $\lambda_{n-m+i} \leq \lambda'_i \leq \lambda_i$ for $i = 1, \ldots, m$.*

Let $Y = (y_1, \ldots, y_n)^T$ be an eigenvector associated with the eigenvalue λ of a signed graph $\Psi = (G, \sigma)$. Assume that the entry y_w corresponds to the vertex w. The *eigenvalue equation* for w is as follows:

$$\lambda y_w = \sum_{v \in N_G(w)} \sigma(vw) y_v.$$

Lemma 1. [9, Lemma 5.1(i)] *Let Ψ be a signed graph and $w_1, w_2, w_3 \in V(\Psi)$ be distinct vertices, and let $Y = (y_1, \ldots, y_n)^T$ be an eigenvector associated with the index $\lambda_1(\Psi)$. Let Ψ' be the graph obtained by replacing the sign of the positive edge w_1w_2 and the negative edge w_1w_3. If $y_{w_1} \geq 0$, $y_{w_2} \leq y_{w_3}$ or $y_{w_1} \leq 0$, $y_{w_2} \geq y_{w_3}$, then $\lambda_1(\Psi) \leq \lambda_1(\Psi')$. If at least one inequality is strict, then $\lambda_1(\Psi) < \lambda_1(\Psi')$.*

If w_1, w_2 and w_3 are as above, then $\mathfrak{R}(w_1, w_2, w_3)$ denotes the relocation explained in Lemma 1.

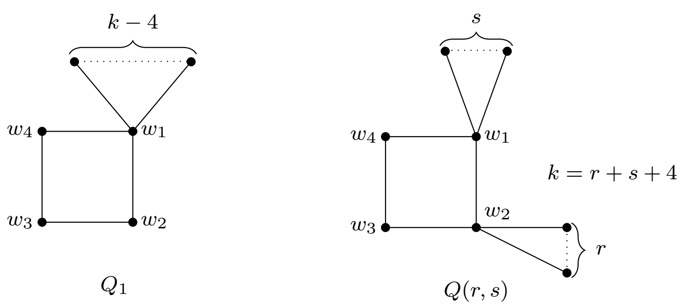

Figure 2. **The unicyclic graphs Q_1, $Q(r,s)$.**

Now, we can prove the main theorem of the article.

Theorem 2. *Let $(K_n, U^-) \in \mathfrak{U}_{n,k,4}$. Then*

$$\lambda_1(K_n, U^-) \leq \lambda_1(K_n, Q_1^-),$$

where Q_1 is shown in Fig. 2. Moreover, the equality holds if and only if $U = Q_1$.

Proof. Suppose that $\Psi = (K_n, U^-) \in \mathfrak{U}_{n,k,4}$ attains the maximum index. Let $\lambda_1 = \lambda_1(\Psi)$. Clearly, $(K_n, Q_1^-) - \{w_1, w_3\} = (K_{n-2}, +)$, see Fig. 2. Thus by Theorem 1, we have $n - 3 \leq \lambda_1(K_n, Q_1^-) \leq \lambda_1$. Assume that U contains a cycle C of order $g > 3$ and some trees attached at vertices of C. Let $V(U) = \{w_1, \ldots, w_k\}$ and $V(C) = \{w_1, \ldots, w_g\}$. We assume that the vertices of C have been labelled consecutively, i.e.

$w_i w_{i+1} \in E(C)$ for all $i \in \{1, \ldots, g-1\}$. Let $Y = (y_1, \ldots, y_n)^T$ be a unit eigenvector associated with λ_1.

First, we show that $y_i \neq 0$ for some i, $1 \leq i \leq g$. Suppose to the contrary that $y_i = 0$, for $i = 1, \ldots, g$. Let $w_p w_q \in E(C)$ and $w_p w_j \in E(U)$, where $1 \leq p, q \leq g$ and $g < j \leq k$. If $y_j \neq 0$, then by Lemma 1, the relocation $\mathfrak{R}(w_j, w_q, w_p)$ would contradict the maximality of λ_1. By repeating the same procedure, we get $y_i = 0$ for $i = 1, \ldots, k$. Hence $k < n$. Let y_t be a component of Y such that $|y_i| \leq |y_t|$, for $i = 1, \ldots, n$. Assume that $y_t > 0$ (otherwise, consider $-Y$ instead of Y). By the eigenvalue equation for w_t, we find that $\sum_{i=k+1, i\neq t}^{n} y_i = \lambda_1 y_t$. Consequently, $\lambda_1 \leq n - k - 1$, a contradiction.

Assume to the contrary that $g > 4$. We may suppose that $y_1 > 0$. If $y_3 \leq y_2$, then the possibility of $\mathfrak{R}(w_1, w_3, w_2)$ contradicts the maximality of λ_1. So $y_2 < y_3$. Now, assume that $y_2 \geq 0$. If $y_g < y_1$, then the relocation $\mathfrak{R}(w_2, w_g, w_1)$ gives a contradiction and hence $0 < y_1 \leq y_g$. If $y_2 \leq y_1$ (resp. $y_1 \leq y_2$), then the relocation $\mathfrak{R}(w_g, w_2, w_1)$ (resp. $\mathfrak{R}(w_3, w_1, w_2)$) contradicts the maximality of λ_1. Therefore, $y_2 < 0$. If $y_g \geq 0$, then by $\mathfrak{R}(w_g, w_2, w_1)$, we find a contradiction. Hence $y_g < 0$. So if $y_{g-1} \geq y_3$ (resp. $y_3 \geq y_{g-1}$), then $\mathfrak{R}(w_2, w_{g-1}, w_3)$ (resp. $\mathfrak{R}(w_g, w_3, w_{g-1})$), gives the final contradiction. It follows that $g = 4$.

Let $V(C) = \{w_1, \ldots, w_4\}$. If $k = 4$, we are done. Assume that $k > 4$ and $w_1 w_5 \in E(U)$. If $y_5 = 0$, then the relocation $\mathfrak{R}(w_5, w_i, w_1)$ implies that $y_i = y_1$, for $i = 2, 3, 4$. We may assume that $y_1 > 0$ (otherwise, consider $-Y$ instead of Y). Now, the relocation $\mathfrak{R}(w_3, w_5, w_4)$ contradicts the maximality of λ_1. Hence $y_5 \neq 0$.

Assume that $y_5 > 0$ (otherwise, consider $-Y$ instead of Y). Thus the relocation $\mathfrak{R}(w_5, w_i, w_1)$ implies that $y_1 < y_i$, for $i = 2, 3, 4$. Note that if $5 < q \leq k$ and $w_1 w_q \in E(U)$, then $\mathfrak{R}(w_q, w_4, w_1)$ yields that $y_q > 0$. We consider two cases.

Case 1. $y_1 \geq 0$. Since $y_1 < y_i$, we have $y_i > 0$ for $i = 2, 3, 4$. If $y_5 \leq y_4$, then $\mathfrak{R}(w_3, w_5, w_4)$ leads to a contradiction. Hence $y_4 < y_5$ and thus $y_1 < y_5$.

Let T be the tree attached at vertex w_1 in U. First, we prove that T is a star. For proving this, let $w_5 w_p \in E(U)$, where $p > 5$. If $y_p \geq 0$, then $\mathfrak{R}(w_p, w_1, w_5)$ yields a contradiction. So $y_p < 0$. Therefore, $\mathfrak{R}(w_2, w_p, w_3)$ contradicts the maximality of λ_1. This completes the assertion.

Next, we claim that $d_U(w_2) = d_U(w_3) = d_U(w_4) = 2$. By contrary assume that $w_i w_p \in E(U)$, for some $i \in \{2, 3, 4\}$ and $p > 5$. Again, if $y_p \geq 0$, then $\mathfrak{R}(w_p, w_1, w_i)$ gives a contradiction and hence $y_p < 0$. Finally, the relocation $\mathfrak{R}(w_5, w_p, w_1)$ concludes a contradiction. The claim is proved. Thus $U = Q_1$, and $\Psi = (K_n, Q_1^-)$.

Case 2. $y_1 < 0$. Note that $y_1 < y_i$, for $i = 2, 3, 4$.

I. We first show that the tree attached at vertex w_1 in U is a star. Suppose to the contrary that $w_5 w_p \in E(U)$, where $p > 5$. Similar to the Case 1, we deduce that $y_p < 0$, since otherwise the relocation $\mathfrak{R}(w_p, w_1, w_5)$ gives a contradiction. Hence $\mathfrak{R}(w_p, w_4, w_5)$ implies that $y_5 > y_4$ and so by $\mathfrak{R}(w_3, w_5, w_4)$, we find that

$y_3 > 0$. On the other hand, if $y_4 \geq 0$, then $\mathfrak{R}(w_4, w_p, w_3)$ would contradict the maximality of λ_1. Therefore, $y_4 < 0$. Let Ψ' be obtained by applying two relocations $\mathfrak{R}(w_3, w_1, w_2)$ and $\mathfrak{R}(w_4, w_5, w_1)$ on Ψ. If A and A' are adjacency matrices of Ψ and Ψ', then we have

$$\lambda_1(\Psi') - \lambda_1(\Psi) = \max_{||X||=1} X^T A' X - Y^T A Y$$

$$\geq Y^T(A' - A)Y = 4y_3(y_2 - y_1) + 4y_4(y_1 - y_5) > 0,$$

a contradiction.

II. Next, we prove that $d_U(w_3) = 2$. By contrary assume that two vertices w_3 and w_p are adjacent in U and $p > 5$. By $\mathfrak{R}(w_p, w_1, w_3)$, we have $y_p < 0$, so $\mathfrak{R}(w_p, w_i, w_3)$ yields that $y_i < y_3$, for $i = 2, 4, 5$. Thus $y_3 > 0$. Suppose that $y_4 \leq 0$. Similar to the Part I, by applying two relocations $\mathfrak{R}(w_3, w_1, w_2)$ and $\mathfrak{R}(w_4, w_5, w_1)$, we find a contradiction. Now, assume that $y_4 \geq 0$. Here, two relocations $\mathfrak{R}(w_1, w_3, w_2)$ and $\mathfrak{R}(w_4, w_p, w_3)$ give the final contradiction.

III. We now prove that $d_U(w_2) = 2$ or $d_U(w_4) = 2$. By contrary, assume that $w_2w_p, w_4w_q \in E(U)$ and $p, q > 5$. By $\mathfrak{R}(w_p, w_1, w_2)$ and $\mathfrak{R}(w_q, w_1, w_4)$, respectively, we deduce that $y_p, y_q < 0$. Therefore, if $y_2 \geq y_4$, then $\mathfrak{R}(w_q, w_2, w_4)$ implies a contradiction. Otherwise, $\mathfrak{R}(w_p, w_4, w_2)$ contradicts the maximality of λ_1.

IV. Finally, we show that if T is the tree attached at w_2 in U, then T is a star. Suppose to the contrary that $w_2w_p, w_pw_q \in E(U)$, where $p, q > 5$. Since $y_1 < y_2$, so $\mathfrak{R}(w_p, w_1, w_2)$ yields that $y_p < 0$. Thus $y_5 > 0 > y_p$, and $\mathfrak{R}(w_q, w_5, w_p)$ concludes that $y_q > 0$. Consequently, if $y_1 \geq y_p$, then $\mathfrak{R}(w_5, w_p, w_1)$ would contradict the maximality of λ_1. Otherwise, $\mathfrak{R}(w_q, w_1, w_p)$ gives a contradiction. Similarly, we can show that the tree attached at w_4 in U, if any, is a star.

Therefore, $U = Q_1$ or $U = Q(r, s)$ and the candidates as maximizers are: (K_n, Q_1^-) and $(K_n, Q(r, s)^-)$, see Fig. 2. By [8, Corollary 1], we have $\lambda_1(K_n, Q(r, s)^-) < \lambda_1(K_n, Q_1^-)$. The proof is complete. □

Lemma 2. [2, Lemma 3] *Let $(K_n, K_{1,k}^-)$ be a signed complete graph. Then*

$$\varphi(K_n, K_{1,k}^-) = (\lambda + 1)^{n-3}\Big(\lambda^3 + (3 - n)\lambda^2 + (3 - 2n)\lambda + 4(n - k - 1)k + 1 - n\Big).$$

Corollary 1. *Let $\Psi = (K_n, Q_1^-)$ be a graph with $4 \leq k$ negative edges, where Q_1 is shown in Fig. 2. Then $n - 3 \leq \lambda_1(\Psi) \leq n - 1$. Moreover, $\lambda_1(\Psi) = n - 3$ if and only if $\Psi = (K_6, C_4^-)$. Also, $\lambda_1(\Psi) = n - 1$ if and only if $\Psi = (K_4, C_4^-) \sim (K_4, +)$.*

Proof. By [1, Theorem 2.5], $\lambda_1(\Psi) \leq n-1$. If the equality holds, then by [11, Lemma 2.1], Ψ is balanced. Thus $n = k = 4$ and $\Psi = (K_4, C_4^-) \sim (K_4, +)$. On the other hand, $\Psi - w_1 = (K_{n-1}, K_{1,2}^-)$ and $\Psi - \{w_1, w_3\} = (K_{n-2}, +)$, see Fig. 2. Therefore,

$$\lambda_1(K_{n-2}, +) = n-3 \leq \lambda_1(K_{n-1}, K_{1,2}^-) \leq \lambda_1(\Psi).$$

If $\lambda_1(\Psi) = n-3$, then $\lambda_1(K_{n-1}, K_{1,2}^-) = n-3$. By Lemma 2, one can deduce that $n = 6$. By a computer search, if $k = 6$, then $\lambda_1(K_6, Q_1^-) = 4.1$, and if $k = 5$, then $\lambda_1(K_6, Q_1^-) = 3.5$. But $\lambda_1(K_6, C_4^-) = 3$ which completes the proof. □

We close the article with the following conjecture. According to Theorem 2 and [8, Theorem 4], the conjecture is true for $g = 3, 4$.

Conjecture 1. *Among all graphs of order $n > 5$ in $\mathfrak{U}_{n,k,g}$, the graph with the maximum index is the signed graph whose negative edges induce a cycle C_g with $k-g$ pendant vertices attached at the same vertex of the cycle.*

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