Communications in Combinatorics and Optimization
Vol. ??, No. ?? (????), pp. 1-12
DOI: 10.22049/CCO.2023.27817.1360

Research Article

PI index of bicyclic graphs

Manju Sankaramalil Chithrabhanu[1], Kanagasabapathi Somasundaram[2,*]

[1]Department of Mathematics, School of Physical Sciences, Kochi, Amrita Vishwa Vidyapeetham, India
manjuchithrabhanu@gmail.com

[2]Department of Mathematics, Amrita School of Physical Sciences, Coimbatore, Amrita Vishwa Vidyapeetham, India
*s_sundaram@cb.amrita.edu

Received: 17 May 2022; Accepted: 18 January 2023
Published Online: 24 January 2023

Abstract: The PI index of a graph G is given by $PI(G) = \sum_{e \in E(G)}(|V(G)| - N_G(e))$, where $N_G(e)$ is the number of equidistant vertices for the edge e. Various topological indices of bicyclic graphs have already been calculated. In this paper, we obtained the exact value of the PI index of bicyclic graphs. We also explore the extremal graphs among all bicyclic graphs with respect to the PI index. Furthermore, we calculate the PI index of a cactus graph and determine the extremal values of the PI index among cactus graphs.

Keywords: PI index, Unicyclic graphs, Bicyclic graphs, Extremal values

AMS Subject classification: 05C09, 05C38

1. Introduction

A topological index is a real number related to a molecular graph that gives some structural properties of the molecules. There are different types of topological indices, i.e., distance-based, degree-based, and neighborhood-based topological indices. Some examples of topological indices are the Wiener index, Szeged index, Zagreb index, Padmakar - Ivan (PI) index, weighted PI index, etc., which have applications in the field of chemical graph theory. The Wiener index is the oldest and most widely studied topological index [16]. After the success of the Wiener and Szeged indices in 2000,

* *Corresponding Author*

Khadikar proposed another index, the Padmakar-Ivan index, abbreviated as the PI index, in [7]. It is defined as,

$$PI_e(G) = \sum_{e=uv\in E(G)} (m_u(e|G) + m_v(e|G)),$$

where $m_u(e|G)$ is the number of edges in G lying closer to the vertex u than to the vertex v.

After a few years, Khalifeh introduced the vertex version of this index and, using this notion, computed the exact expression for the PI index of the Cartesian product of graphs in [10].

The vertex Padmakar-Ivan (PI) index of a graph G is defined by,

$$PI\,(G) = \sum_{e-(u,v)\in E(G)} \left(n_u(e) + n_v(e)\right), \tag{1}$$

where $n_u(e)$ denote the number of vertices of G whose distance to the vertex u is smaller than the distance to the vertex v.

A vertex $w \in V(G)$ is said to be an equidistant vertex of an edge $e = uv$ if $d(u,w) = d(v,w)$. The set of equidistant vertices of an edge $e = uv$ is denoted by $D(e)$ and is defined as, $D(e) = \{w \in G : d(u,w) = d(v,w)\}$. $N_G(e)$ denotes the number of equidistant vertices of e, $N_G(e) = |D(e)|$.

In equation (1), $n_u(e) + n_v(e) = |V(G)| - N_G(e)$. So vertex PI index of a graph G is also given by,

$$PI(G) = \sum_{e\in E(G)} (|V(G)| - N_G(e)).$$

The former is the edge PI index and the latter is the vertex PI index. Ilić and Milosavljević introduced another topological index, the weighted vertex PI index in [5], and computed the exact expressions for the weighted vertex PI index of the Cartesian product of graphs. The weighted PI index of a graph G is given by,

$$PI_w(G) = \sum_{e\in E(G)} \left((d_G(u) + d_G(v))(|V(G)| - N_G(e))\right).$$

The topological indices of some molecular graphs are studied in [1] and [16]. Khadikar et al. investigated the chemical and biological applications of PI index in [9] and [8]. Indulal et al. constructed a class of non-bipartite graphs possessing PI-invariant edges in [6]. Manju and Somasundaram [3] obtained the PI index for some classes of perfect graphs like co-bipartite graphs, line graphs, and prismatic graphs. They also calculated the exact value of the PI and weighted PI indices of powers of paths, cycles, and their complements in [2]. Gopika et al. obtained the weighted PI index for the direct and strong product of certain classes of graphs in [4]. In [15] and [11], authors calculated the sharp lower and upper bounds on the edge PI index of connected bicyclic graphs with the constant number of vertices. They also characterize the case of equality for both bounds. In [12], Gang Ma et al. obtained the upper and lower bounds on the edge PI index of connected unicyclic and bicyclic graphs with given girth and characterized the corresponding extremal graphs. The computation of the upper bound on the edge PI index of connected bicyclic graphs with an even number of edges has been done by the same authors in [14]. In [13], the upper and lower bounds on the weighted vertex PI index of bicyclic graphs are obtained and the corresponding extremal graphs are also given. Motivated by this, we discussed the exact value of the vertex PI index of bicyclic graphs, and the corresponding extremal graphs in this paper. We also studied the PI index of cactus graphs.

The contraction of an edge $e = uv$ in a graph G results in a new graph in which edge e is replaced by a new vertex, which is adjacent to all vertices that are adjacent to u or v in G. It may contain parallel edges. $G * e$ is the new graph by excluding all parallel edges.

If G is a bipartite graph with n vertices and m edges, $PI(G) = nm$ [6]. In particular, if T_n is a tree with n vertices, then $PI(T_n) = n(n-1)$ and $PI(T_n * e) = (n-1)(n-2)$. The subdivision of any graph is a bipartite graph. If G is a graph with n vertices and m edges, then the subdivision of G has $n + m$ vertices and $2m$ edges. Therefore the PI index of the subdivision of G is $2m(n + m)$.

2. PI index of unicyclic graphs

Throughout this section, we assume that G is a unicyclic graph with n vertices and m edges with the unique cycle C_k. It is easy to see that number of vertices and edges are the same in a unicyclic graph. Also $PI(C_n)= \begin{cases} n(n-1) & \text{if } n \text{ is odd} \\ n^2 & \text{if } n \text{ is even.} \end{cases}$

We observed an important property that the contribution of edges of an odd cycle C_k to $N_G(e)$ gives a partition of n.

Lemma 1. *Let G be a unicycle graph with n vertices with an odd cycle C_{2k+1}. Then*

$$\sum_{e \in E(C_{2k+1})} N_G(e) = n.$$

Proof. If $G \simeq C_{2n+1}$, then $\sum N_G(e) = n$. Otherwise, let $T_i, i = 1, 2, \ldots, k$ be the trees in G rooted on each vertex v_i of C_{2k+1} (some T_i can be empty). For any edge in the cycle, there is only one vertex (say v_i) in the cycle as the equidistant vertex, and hence all the vertices of T_i are also equidistant. Therefore, the sum of the number of equidistant vertices corresponding to edges in C_{2k+1} is n. □

Lemma 2. *Let G be a unicycle graph with n vertices with an odd cycle C_{2k+1}. The edges in C_{2k+1} contribute $2kn$ to the $PI(G)$.*

From Lemma 1, it is easy to see that the edges in C_{2k+1} contribute $2kn$ to the PI index of G.

Theorem 1. *Let G be a unicycle graph with n vertices with unique cycle C_k. Then*

$$PI(G) = \begin{cases} n^2 & \text{if } k \text{ is even} \\ n(n-1) & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Let G be the unicyclic graph with n vertices and the unique cycle C_k. If k is even, then G is a bipartite graph and so $PI(G) = n^2$. Assume that k is odd. Let E_1 be the edges of C_k and $E_2 = E(G) - E_1$. Now $PI(G) = \sum_{e \in E_1} (|V(G)| - N_G(e)) + \sum_{e \in E_2} (|V(G)| - N_G(e)) = (k-1)n + \sum_{e \in E_2} (|V(G)| - N_G(e))$ (by Lemma 2).
The set E_2 has $(n-k)$ edges, and each edge in E_2 has no equidistant vertices in G. Thus $PI(G) = (k-1)n + (n-k)n = n(n-1)$. □

By Theorem 1, the PI index of G does not depend on k. So we can easily say that the PI index of a unicyclic graph G is either $n(n-1)$ or n^2, it attains its lower bound if the cycle in G is odd and attains its upper bound if the cycle is even.
If G is a unicyclic graph, then $G{*}e$ is a unicyclic graph. Suppose G is a unicyclic graph with n vertices and the cycle C_3 then $G{*}e$ is a tree, and hence $PI(G{*}e) = (n-1)(n-2)$. The following corollary is an easy consequence of Theorem 1.

Corollary 1. *Let G be the unicyclic graph with n vertices and the unique cycle $C_k, k \geq 4$. Then* $PI(G * e) = \begin{cases} (n-1)^2 & \text{if } k \text{ is odd and } e \in C_k \text{ or if } k \text{ is even and } e \notin C_k \\ (n-1)(n-2) & \text{if } k \text{ is odd and } e \notin C_k \text{ or if } k \text{ is even and } e \in C_k. \end{cases}$

3. PI Index of Bicyclic Graphs

A simple connected graph G is bicyclic if its number of edges is equal to one more than the number of vertices in G. Let $G = C(p, q, k)$ be a bicyclic graph with n vertices, which has two cycles C_p and C_q (throughout this paper we assume that $p \leq q$) and the two cycles share k edges $(k \geq 0)$ $e_1, e_2, \ldots, e_k$. Then G has three cycles, C_p, C_q, and C_{p+q-2k}. Let $C_p : u_1u_2 \ldots u_pu_1$, $C_q = v_1v_2 \ldots v_qv_1$ and C_{p+q-2k} : $u_1u_pu_{p-1}u_{p-2} \ldots u_{k+1}v_{k+2}v_{k+3} \ldots v_qu_1$ with $u_1 = v_1, u_2 = v_2, \ldots, u_{k+1} = v_{k+1}$. Let T_i be the tree rooted at the vertex u_i of C_p and T'_i be the tree rooted at v_i of C_q. Let $|V(T_i)| = t_i, |V(T'_i)| = t'_i$ (including the vertices u_i and v_i). We can call the edges of $C_p \cup C_q$ as the cyclic part and the remaining edges as the non-cyclic part of G. An example of a bicyclic graph is shown in Fig. 1.
We can easily see that a vertex u is an equidistant vertex of an edge e, then the vertices of trees rooted on u are also equidistant of e. we call such trees, equidistant trees of e. The equidistant trees corresponding to edges $e_i \in C_p \cap C_q$ has more importance. Let ρ be the total number of equidistant vertices corresponding to the edges common to C_p and C_q.

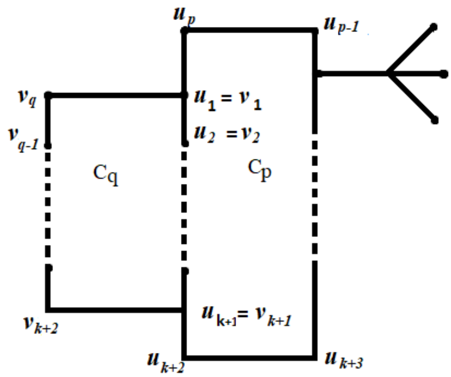

Figure 1. **Example of a Bicyclic Graph**

Lemma 3. *Let $G = C(p, q, k)$ be a bicyclic graph with n vertices. If G has an odd cycle of length r then* $\sum_{e \in C_r} (|V(G)| - N_G(e)) = (r-1)n$.

Proof. Let G be a bicyclic graph with n vertices and let C_p and C_q be the two cycles in G and we assume that $p \leq q < p + q - 2k$. Also assume $k \leq d, d$ is the diameter of smaller cycle C_p. We consider three cases.

Case 1. p and q are odd.
For proving $\sum_{e \in C_p} (|V(G)| - N_G(e)) = (p-1)n$, it needs to show that $\sum_{e \in C_p} N_G(e) = n$.
If u and v are two vertices of C_p, then the distance $d(u, v)$ in G is equal to $d(u, v)$ in C_p. As $k \leq d$ above is the same in the case of C_q. Since p is odd, every vertex on C_p must be an equidistant vertex exactly one edge in C_p. So $\cup_{e \in C_p} D(e)$ includes all the vertices on C_p. Let C be the set of edges common to C_p and C_q and $|C| = k$.
Let $z_1, z_2, \ldots, z_k$ be the vertices on C_q which are equidistant to the edges in C.

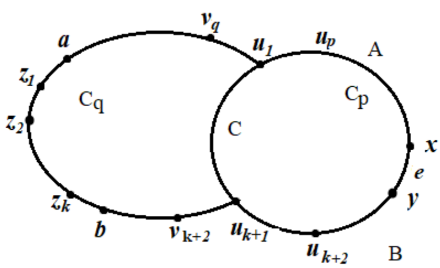

Figure 2. **Cyclic part of G used in the proof of Lemma 3 (Case 1)**

For the vertex u_1 there is an edge $e = xy$ in C_p such that u_1 is equidistant to e. Let a be the vertex on C_q such that (a, z_1) is an edge of C_q as shown in Fig. 2. By calculating the length of all paths connecting x and $a, d(x,a) = d(x,u_1) + d(u_1,a) = \frac{p-1}{2} + (\frac{q-1}{2} - (k-1))$ and $d(y,a) = d(y,u_{k+1}) + d(u_{k+1},a) = \frac{p-1}{2} - k + \frac{q-1}{2} + 1, a \in D(e)$. Let A be the path from x to a including the vertices u_p, u_1, v_q. Since a and u_1 belong to $D(e)$, all vertices of C_q that lies between u_1 and a (along the path A) are also in $D(e)$. Similarly, the vertices $u_{k+1}, v_{k+2}, \ldots, b$ are in $D(e')$, where e' is the edge in C_p, such that vertex u_{k+1} is equidistant to e' and b is the vertex next to z_k as shown in Fig. 2. If a vertex $v \in D(e)$ then the trees rooted on v also in $D(e)$. Hence $\sum\limits_{e\in C_p} N_G(e) = n$. Similarly, we can prove that $\sum\limits_{e\in C_q} N_G(e) = n$.

Case 2. p is even and q is odd.
Assume that $p < q \le p + q - 2k$. Two odd cycles in G are C_q and C_{p+q-2k}. As in the above case, $\cup_{e\in C_q} D(e)$ includes all the vertices on C_q.

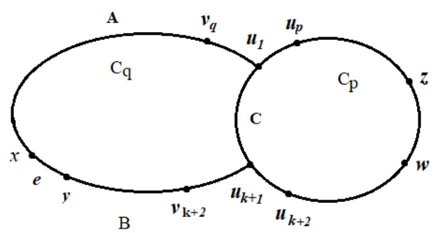

Figure 3. **Cyclic part of G used in the proof of Lemma 3 (Case 2)**

For the vertex u_1 there is an edge $e = xy$ in C_q such that u_1 is equidistant to e. Let z be the vertex on C_p such that $d(u_{k+1}, z) = \frac{p}{2}$ (there exist such a vertex because C_p is an even cycle).
Let A be the path from x to z containing u_1, u_p, and B be the path from y to z containing u_{k+1}, u_{k+2} (not including the edges in path C). By considering all the paths from y to z, $d(y,z) = d(y,u_{k+1}) + d(u_{k+1},z) = q - k + \frac{p}{2}$ and $d(x,z) = d(x,u_1) + d(u_1,z) = q + \frac{p}{2} - k$. From this, we can say that $z \in D(e)$, so all the vertices on C_p that lies between u_1 and z(along A) are equidistant to e. Let $e' = x'y'$ be the edge on C_q such that u_{k+1} is equidistant to e'. Also, let w be the vertex on C_p such that $d(u_1, w) = \frac{p}{2}$. So the vertices between u_{k+1} and w (along B) are in $D(e')$. Therefore we can easily say that the remaining vertices between z and w are equidistant to the edges of C_q lies between e and e'. If a vertex $v \in D(e)$ and hence the trees are rooted on v also in $D(e)$. Hence $\sum_{e\in C_q} N_G(e) = n$.
Two edges $e = xy$ and $e' = uv$ are equidistant if $d(x,u) = d(y,v)$ or $d(x,v) = d(y,u)$, i.e. the distance between those edges are equal. Now consider the cycle C_{p+q-2k}. $E(C_{p+q-2k}) = (E_1 \cup E_2) \setminus E_3$, where $E_1 = E(C_p), E_2 = E(C_q)$ and $E_3 = E(C_p) \cap E(C_q)$. So we only need to consider $E_1 \setminus E_3$. Since E_1 and E_3 are parts of an even

cycle, corresponding to each edge e in E_3 there is an equidistant edge e' in E_1 and $N_G(e) = N_G(e')$, $\sum_{e \in C_{p+q-2k}} N_G(e) = n$.

Case 3. p is odd and q is even.
It is easy to prove.
Using the same procedure we can prove the Lemma for $k > d$. □

Theorem 2. *Let $G = C(p, q, k)$ be a bicyclic graph with n vertices. Then*

$$PI(G) = \begin{cases} n(n-1) + \rho & \text{if } p \text{ and } q \text{ are odd} \\ n(n+1) & \text{if } p \text{ and } q \text{ are even} \\ n^2 - \rho & \text{otherwise.} \end{cases}$$

Proof. We prove this theorem by considering two cases.

Case 1. Assume that $k \le d$.
We distinguish three situations.

Subcase 1.1. Both p and q are odd.
Graph G has exactly two odd cycles C_p, C_q, and an even cycle C_{p+q-2k}. From Lemma 3, C_p and C_q contribute $(p-1)n$ and $(q-1)n$ to the PI index of G, respectively. Consider the edge $e_i \in C_p \cap C_q$, there is one vertex u_j at distance d_p in C_p belongs to $D(e_i)$ and hence the vertices of the tree T_j are also belong to $D(e_i)$. Similarly there exist v_m in C_q and hence $V(T'_m)$ are also belongs to $D(e_i)$. Therefore $N_G(e_i) = t_j + t'_m$. Thus $\cup_{r=0}^{k-1} V(T_{j+r}) \in D(e_i)$ and $\cup_{r=0}^{k-1} V(T'_{m+r}) \in D(\bigcup_{e_i \in C_p \cap C_q} e_i)$. So the edges common to C_p and C_q contribute $kn - \left(\sum_{r=0}^{k-1} t_{j+r} + \sum_{r=0}^{k-1} t'_{m+r} \right) = kn - \rho$ to the PI index of G. If we consider the non-cyclic part, each tree T_i and T'_i contributes $n(t_i - 1)$ and $n(t'_i - 1)$ to $PI(G)$. Hence

$$\begin{aligned} PI(G) &= (p-1)n + (q-1)n - (kn - \rho) + n\sum_{i=1}^{p}(t_i - 1) + n\sum_{i=k+2}^{q}(t'_i - 1) \\ &= n\left(p + q - 2 - k + \sum_{i=1}^{p}(t_i - 1) + \sum_{i=k+2}^{q}(t'_i - 1)\right) + \rho \\ &= n\left(p + q - k - 2 + \left(\sum_{i=1}^{p} t_i - p\right) + \left(\sum_{i=k+2}^{q} t'_i - (q - k - 1)\right)\right) + \rho \\ &= n(n-1) + \rho. \end{aligned}$$

Subcase 1.2. Both p and q are even.
In this case, G is bipartite since all the three cycles in G are even in length and therefore $PI(G) = n(n+1)$.

Subcase 1.3. p is even and q is odd.
Graph G has one even cycle C_p and two odd cycles C_q and C_{p+q-2k}. Partition the edge set $E(G)$ into three sets, $E_1 = E(C_{p+q-2k}), E_2 = C_p \cap C_q = \{e_1, e_2, \ldots, e_k\}$ and E_3 is the union of edges in the non-cyclic part. From the Lemma 3, E_1 contributes $(p+q-2k-1)n$ to PI index of G.
For the edge e_i in E_2, there is exactly one vertex v_m in C_q at distance d_q belongs to $D(e_i)$ and thus $V(T'_m) \subseteq D(e_i)$. So $N_G(e_i) = t'_m$. Thus the edges in $C_p \cap C_q$ contributes $kn - \sum_{r=0}^{r=k-1} t'_{m+r} = kn - \rho$ to PI index of G. The non-cyclic part E_3 contributes the same as in Subcase 1.1. Thus

$$\begin{aligned}
PI(G) &= (p+q-2k-1)n + (kn-\rho) + n(\sum_{i=1}^{p}(t_i - 1)) + n(\sum_{i=k+2}^{q}(t'_i - 1)) \\
&= (p+q-2k-1+k)n - \rho + n(\sum_{i=1}^{p} t_i - p) + n(\sum_{i=k+2}^{q} t'_i - (q-k-1)) \\
&= (p+q-2k-1+k+\sum_{i=1}^{p} t_i + \sum_{i=k+2}^{q} t'_i - p - q + k + 1))n - \rho \\
&= n^2 - \rho.
\end{aligned}$$

Case 2. Assume $k > d$.

We consider three situations.

Subcase 2.1. Both p and q are odd.
In this case, we partition the edge set into four sets. Let $E_1 = E(C_p), E_2 = E(C_q), E_3 = E(C_p \cap C_q)$, and E_4 be the union of edges in the non-cyclic part of G. E_1 and E_2 contribute $(p-1)n$ and $(q-1)n$ to the PI index of G, respectively. Also, E_4 contributes the same as in the above case. Now, we have to find the number of equidistant vertices corresponding to the common edges. Consider the set with common edges $C = \{e_1, e_2, \ldots, e_{k-\frac{p-1}{2}}, e_{k-\frac{p-1}{2}+1}, \ldots, e_{\frac{p-1}{2}}, e_{\frac{p-1}{2}+1}, \ldots, e_k\}$. Each edge in C has an equidistant vertex in C_p. Since it is a part of C_q, possible equidistant vertex is at d_q. For an edge $e_i = u_i u_{i+1}$ in C, if v is the equidistant vertex in C_q, then length of the shortest $u_i - v$ path is d_q. Such a vertex exist if any other $u_i - v$ path (not along C_q) greater than d_q, that is, $d(u_i, v) = d(u_i, u_1) + d(u_1, u_{k+1}) + d(u_{k+1}, v) = (i-1) + (p-k) + (\frac{q-1}{2} - (k-i)) > \frac{q-1}{2}$ implies $i > k - \frac{p-1}{2}$. Similarly, $d(u_{i+1}, v) = d_q$, it is the distance of the shortest path along C_q. If we consider any other path, the length should be greater than d_q, that is, $d(u_{i+1}, z) = d(u_{i+1}, u_{k+1}) + d(u_{k+1}, u_1) + d(u_1, v) = (k-i) + (p-k) + (\frac{q-1}{2} - (i-1)) > \frac{q-1}{2}$ implies $i \leq \frac{p-1}{2}$. Therefore, there exist equidistant vertices at distance d_q for the edges $e_{k-\frac{p-1}{2}+1}, e_{k-\frac{p-1}{2}+2}, \ldots, e_{\frac{p-1}{2}}$. The total number of vertices equidistant to edges in C is ρ. Thus we have $PI(G) = (p-1)n + (q-1)n + n(\sum_{i=1}^{p} t_i - p) + n(\sum_{i=k+2}^{q} t'_i - (q-k-1)) - (kn - \rho) = n(n-1) + \rho$.

Subcase 2.2. p and q are even.
In this case G is bipartite graph and hence $PI(G) = n(n+1)$.

Subcase 2.3. p is even, and q is odd.
The edge set of G can be partitioned as $E_1 = E(C_{p+q-2k}), E_2 = E(C_p) \cap E(C_q), E_3$ is the union of edges in the non-cyclic part. E_1 contributes $(p+q-2k-1)n$ to the PI index of G. Now, we have to find the equidistant vertices corresponding to the common edges. Let the common edges be $C = \{e_1, e_2, \dots, e_{k-\frac{p}{2}}, e_{k-\frac{p}{2}+1}, \dots, e_{\frac{p}{2}}, e_{\frac{p}{2}} + 1, \dots, e_k\}$. Since the common edge is a part of C_q, a possible equidistant vertex is at d_q. Similarly, as we have done in Case 1, there exist equidistant vertices at distance d_q for the edges $e_{k-\frac{p}{2}+1}, e_{k-\frac{p}{2}+2}, \dots, e_{\frac{p}{2}}$. The total number of vertices equidistant to edges in C is ρ. Therefore,

$$\begin{aligned} PI(G) &= (p+q-2k-1)n + n(\sum_{i=1}^{p}(t_i - 1)) + n(\sum_{i=k+2}^{q}(t'_i - 1)) + (nk - \rho) \\ &= (p+q-2k-1)n + n(\sum_{i=1}^{p}(t_i - p)) + n(\sum_{i=k+2}^{q}(t'_i - (q-k-1)) + (nk - \rho) \\ &= n(p+q-2k-1+n-p-(q-k-1)+k) - \rho = n^2 - \rho. \end{aligned}$$

□

From the above theorem, we conclude that for any bicyclic graph, the PI index depends on the number of vertices and the number of equidistant vertices ρ corresponding to the common edges of two cycles C_p and C_q.
Next, we consider extremal graphs among bicyclic graphs. Let G be a bicyclic graph with n vertices and two odd cycles, C_p, and C_q. From the Theorem 2 $PI(G) = n(n-1) + \rho$. Here, the minimum PI index is $n(n-1)$, which is attained by graphs with $\rho = 0$. G_2 in Fig. 4 is such a graph. The maximum is $n(n-1) + n - 1$ obtained when ρ is maximum.
If G has one even and one odd cycle, $PI(G) = n^2 - \rho$. Here, the minimum PI index is $n^2 - \rho = n^2 - (n-2)$, which is attained by such graphs which have maximum ρ. Maximum PI index n^2 obtained when $\rho = 0$. G_1 and G_2 in Fig. 4 are examples of extremal graphs, and such graphs are not unique.

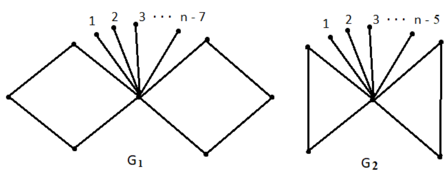

Figure 4. **Extremal Graphs**

Next, we consider those graphs G such that, the resulting graph $G*e$ has the following property. The number of edges common to the cycles in $G*e$ should be less than the diameter of the shortest cycle in $G*e$, and $p,q \geq 4$. We can partition the edge set of G as $E_1 = C_p$ (not in C_q), $E_2 = C_q$ (not in C_p), $E_3 = E(C_p) \cap E(C_q)$ and $E_4 = \cup_{i=0}^{k-1} E(T_{j+i}), E_5 = \cup_{i=0}^{k-1} E(T'_{m+i})$ and E_6 represent the remaining edges. t and t' are the number of equidistant vertices (for the common edges) on C_p and C_q respectively, $*$ denote the same in $G*e$. The following corollary is an easy consequence of Theorem 2.

Corollary 2. *Let G be a bicyclic graph then the PI index of $G*e$ is as follows.*

1. $PI(G*e) = (n-1)^2 - k$,

$$k = \begin{cases} t (or\ t') & \text{if } p \text{ and } q \text{ are odd and } e \in E_1 (or\ E_2) \text{ or if } p \text{ is odd and } q \text{ is even and} \\ & e \in E_5 \bigcup E_6 \\ t*' \ (or\ t*) & \text{if } p \text{ and } q \text{ are even and } e \in E_2 (or\ E_1) \text{ or if } p \text{ is odd and } q \text{ is even and} \\ & e \in E_3 \\ t-1 & \text{if } p \text{ is odd and } q \text{ is even and } e \in E_4 \end{cases}$$

2. $PI(G*e) = (n-1)(n-2) + k$

$$k = \begin{cases} t+t'-1 & \text{if } p \text{ and } q \text{ are odd and } e \in E_4 \bigcup E_5 \\ t+t' (or\ t*+t*') & \text{if } p \text{ and } q \text{ are odd and } e \in E_6 \ (\text{or if } p \text{ and } q \text{ are even and } e \in E_3) \\ t+t*' & \text{or if } p \text{ is odd and } q \text{ is even and } e \in E_2 \end{cases}$$

3. $PI(G*e) = n(n-1)$, *otherwise.*

4. Cactus graph

A cactus graph is a simple connected graph in which every block is an edge or a cycle. That is, every cycle has at most one vertex in common.

Theorem 3. *Let G be a cactus graph with n vertices, m edges, and p odd cycles. Then $PI(G) = n(m-p)$.*

Proof. Let $C_{k_1}, C_{k_2}, \ldots, C_{k_p}$ be the odd cycles of length $k_1, k_2, \ldots, k_p$ in G. We claim that each C_{k_i} contributes $(k_i - 1)n$ to the PI index of G. Let H be the graph obtained by deleting all edges of odd cycle C_{k_i}, it has k_i components. Each vertex v on C_{k_i} is an equidistant vertex corresponding to some edges (exactly one) e in C_{k_i}. All vertices of the component containing v also belong to $D(e)$. Thus $\sum_{e \in C_{k_i}} N_G(e) = n$. The remaining edges (which are not parts of an odd cycle) contribute $n(m - (k_1 + k_2 + \cdots + k_p))$ to the PI of G. Thus we have

$$\begin{aligned} PI(G) &= ((k_1 - 1)n + (k_2 - 1)n + \cdots + (k_p - 1)n) + (n(m - (k_1 + k_2 + \cdots + k_p))) \\ &= (k_1 + k_2 + \cdots + k_p - p)n + n(m - (k_1 + k_2 + \cdots + k_p)) \\ &= n(m-p). \end{aligned}$$

□

From the above theorem, we can easily say that the PI index of a cactus graph G is maximum and is equal to nm when $p = 0$ or G has no odd cycles. Also, the PI index of G is minimum when p is maximum. Since the maximum number of edge-disjoint triangles among m edges is $\lfloor \frac{m}{3} \rfloor$, the maximum feasible value of p is $\lfloor \frac{m}{3} \rfloor$. So the minimum PI index is $n(m - \lfloor \frac{m}{3} \rfloor)$.

Acknowledgement
The authors wish to thank the anonymous reviewers for their valuable suggestions.

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