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Research Article

Signed total Italian k-domination in digraphs

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Abstract: Let $k \geq 1$ be an integer, and let D be a finite and simple digraph with vertex set $V(D)$. A signed total Italian k-dominating function (STIkDF) on a digraph D is a function $f : V(D) \to \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x \in N^-(v)} f(x) \geq k$ for each vertex $v \in V(D)$, where $N^-(v)$ consists of all vertices of D from which arcs go into v, and (ii) each vertex u with $f(u) = -1$ has an in-neighbor v for which $f(v) = 2$ or two in-neighbors w and z with $f(w) = f(z) = 1$. The weight of an STIkDF f is $\omega(f) = \sum_{v \in V(D)} f(v)$. The signed total Italian k-domination number $\gamma^k_{stI}(D)$ of D is the minimum weight of an STIkDF on D. In this paper we initiate the study of the signed total Italian k-domination number of digraphs, and we present different bounds on $\gamma^k_{stI}(D)$. In addition, we determine the signed total Italian k-domination number of some classes of digraphs.

Keywords: Digraph, Signed total Italian k-dominating function, Signed total Italian k-domination number, Signed total Roman k-dominating function, Signed total Roman k-domination number

AMS Subject classification: 05C20, 05C69

1. Terminology and introduction

In this paper we continue the study of signed total Roman domination and signed total Italian domination in graphs and digraphs. For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [9]. Specifically, let G be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integer $n = n(G) = |V(G)|$ is the *order* of the graph G. The *open neighborhood* of vertex v is $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d_G(v) = d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively.

Cockayne, Dreyer, S.M. Hedetniemi and S.T. Hedetniemi [6] introduced the concept of *Roman domination* in graphs, and since then a lot of related variations and generalizations in graphs and digraphs have been studied (see, [2–5]). In this paper we continue the study of Roman and Italian dominating functions in graphs and digraphs. If $k \geq 1$ is an integer, then Volkmann [13] defined the *signed total Roman k-dominating function* (STRkDF) on a graph G as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $f(N(v)) = \sum_{x \in N(v)} f(x) \geq k$ for every $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$. The weight of an STRkDF f on a graph G is $\omega(f) = \sum_{v \in V(G)} f(v)$. The *signed total Roman k-domination number* $\gamma^k_{stR}(G)$ of G is the minimum weight of an STRkDF on G. The special case $k = 1$ was introduced and investigated by Volkmann [10, 11].
If $k \geq 1$ is an integer, then Volkmann [15] defined the *signed total Italian k-dominating function* (STIkDF) on a graph G as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ having the property $f(N(v)) \geq k$ for every $v \in V(G)$, and each vertex u with $f(u) = -1$ is adjacent to a vertex v with $f(v) = 2$ or to two vertices w and z with $f(w) = f(z) = 1$. Note that in the case $k \geq 2$ or $\delta(G) \geq 2$, the second condition is superfluous. The weight of an STIkDF f is $\omega(f) = \sum_{v \in V(G)} f(v)$. The *signed total Italian k-domination number* $\gamma^k_{stI}(G)$ of G is the minimum weight of an STIkDF on G. The special case $k = 1$ was introduced and investigated by Volkmann [14].
The signed total Italian k-domination number exists when $\delta(G) \geq \frac{k}{2}$. The definitions lead to $\gamma^k_{stI}(G) \leq \gamma^k_{stR}(G)$. Therefore each lower bound of $\gamma^k_{stI}(G)$ is also a lower bound of $\gamma^k_{stR}(G)$.

Let now D be a finite and simple digraph with vertex set $V(D)$ and arc set $A(D)$. The integer $n = n(D) = |V(D)|$ is the *order* of the digraph D. The sets $N^+_D(v) = N^+(v) = \{x|(v, x) \in A(D)\}$ and $N^-_D(v) = N^-(v) = \{x|(x, v) \in A(D)\}$ are called the *out-neighborhood* and *in-neighborhood* of the vertex v. Likewise $N^+_D[v] = N^+[v] = N^+(v) \cup \{v\}$ and $N^-_D[v] = N^-[v] = N^-(v) \cup \{v\}$. We write $d^+_D(v) = d^+(v) = |N^+(v)|$ for the *out-degree* and $d^-_D(v) = d^-(v) = |N^-(v)|$ for the *in-degree* of the vertex v. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. The *minimum* and *maximum in-degree* are $\delta^-(D) = \delta^-$ and $\Delta^-(D) = \Delta^-$ and the *minimum* and *maximum out-degree* are $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$. For an arc $(x, y) \in A(D)$, the vertex y is an out-neighbor of x and x is an in-neighbor of y, and we also say that x dominates y and y is dominated by x.
If $k \geq 1$ is an integer, then Volkmann [12] defined the *signed total Roman k-dominating function* (STRkDF) on a digraph D as a function $f : V(D) \longrightarrow \{-1, 1, 2\}$ such that $f(N^-(v)) = \sum_{x \in N^-(v)} f(x) \geq k$ for every $v \in V(D)$, and every vertex u for which $f(u) = -1$ has an in-neighbor v for which $f(v) = 2$. The weight of an STRkDF f on a digraph D is $\omega(f) = \sum_{v \in V(D)} f(v)$. The *signed total Roman k-domination number* $\gamma^k_{stR}(D)$ of D is the minimum weight of an STRkDF on D. Amjadi and Soroudi [1], Dehgardi and Volkmann [7] and Volkmann [12] studied the signed total Roman k-domination number in digraphs.
If $k \geq 1$ is an integer, then a *signed total Italian k-dominating function* (STIkDF) on

a digraph D is defined as a function $f : V(D) \longrightarrow \{-1, 1, 2\}$ satisfying the conditions that (i) $f(N^-(v)) \geq k$ for every $v \in V(D)$, and (ii) each vertex u for which $f(u) = -1$ has an in-neighbor v with $f(v) = 2$ or two in-neighbors w and z with $f(w) = f(z) = 1$. Note that in the case $k \geq 2$ or $\delta^-(D) \geq 2$, the second condition is superfluous. The weight of an STIkDF f is $\omega(f) = \sum_{v \in V(D)} f(v)$. The *signed total Italian k-domination number* $\gamma^k_{stI}(D)$ of D is the minimum weight of an STIkDF on D. The special case $k = 1$ was introduced and investigated by Volkmann [16]. A $\gamma^k_{stI}(D)$-function is a signed total Italian k-dominating function on D of weight $\gamma^k_{stI}(D)$. For an STIkDF f on D, let $V_i = V_i(f) = \{v \in V(D) : f(v) = i\}$ for $i = -1, 1, 2$. A signed total Italian k-dominating function $f : V(D) \longrightarrow \{-1, 1, 2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(D)$.

The signed total Italian k-domination number exists when $\delta^-(D) \geq \frac{k}{2}$. The definitions lead to $\gamma^k_{stI}(D) \leq \gamma^k_{stR}(D)$. Therefore each lower bound of $\gamma^k_{stI}(D)$ is also a lower bound of $\gamma^k_{stR}(D)$.

For an integer $q \geq 1$, a subset S of vertices of a digraph D is a *total q-dominating set* if every vertex $x \in V(D)$ has at least q in-neighbors in S. The *total q-domination number* $\gamma_{tq}(D)$ is the minimum cardinality of a total q-dominating set of D.

Our purpose in this work is to initiate the study of the signed total Italian k-domination number in digraphs. We present basic properties and sharp bounds on $\gamma^k_{stI}(D)$. In particular, we show that many lower bounds on $\gamma^k_{stR}(D)$ are also valid for $\gamma^k_{stI}(D)$. Some of our results are extensions of well-known properties of the signed total Roman k-domination number and the signed total Italian domination number $\gamma_{stI}(G) = \gamma^1_{stI}(G)$, given by Volkmann [11, 13, 14].

The *associated digraph* $D(G)$ of a graph G is the digraph obtained from G when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}(v) = N_G(v)$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation holds.

Observation 1. *If $D(G)$ is the associated digraph of a graph G, then $\gamma^k_{stI}(D(G)) = \gamma^k_{stI}(G)$ and $\gamma^k_{stR}(D(G)) = \gamma^k_{stR}(G)$.*

Let K_n and K^*_n be the complete graph and complete digraph of order n, respectively. In [15], the author determines the signed total Italian k-domination number of complete graphs.

Proposition 1. ([15]) If $k \geq 1$ and $n \geq 2$ are integers such that $2n - 2 \geq k$, then it holds:

(i) If $k \geq n$, then $\gamma^k_{stI}(K_n) = k + 2$.

(ii) If $k \leq n - 1$ and $n - k$ is odd, then $\gamma^k_{stI}(K_n) = k + 1$.

(iii) If $k \leq n - 1$ and $n - k$ is even, then $\gamma^k_{stI}(K_n) = k + 2$.

Using Observation 1 and Proposition 1, we obtain the signed total Italian k-domination number of complete digraphs.

Corollary 1. If $k \geq 1$ and $n \geq 2$ are integers such that $2n-2 \geq k$, then it holds:

(i) If $k \geq n$, then $\gamma^k_{stI}(K^*_n) = k+2$.

(ii) If $k \leq n-1$ and $n-k$ is odd, then $\gamma^k_{stI}(K^*_n) = k+1$.

(iii) If $k \leq n-1$ and $n-k$ is even, then $\gamma^k_{stI}(K^*_n) = k+2$.

Let $K_{p,q}$ and $K^*_{p,q}$ be the complete bipartite graph and complete bipartite digraph with partite sets X and Y, where $|X| = p$ and $|Y| = q$.

Proposition 2. ([15] If $k \geq 1$ and $q \geq p \geq 2$ are integers such that $p \geq k/2$, then $\gamma^k_{stI}(K_{p,q}) = 2k$.

Using Observation 1 and Proposition 2, we obtain the next result.

Corollary 2. If $k \geq 1$ and $q \geq p \geq 2$ are integers such that $p \geq k/2$, then $\gamma^k_{stI}(K^*_{p,q}) = 2k$.

2. Preliminary results

In this section we present basic properties of the signed total Italian k-dominating functions and the signed total Italian k-domination numbers in digraphs.

Proposition 3. Let $k \geq 1$ be an integer, and let D be a digraph of order n with $\delta^-(D) \geq \lceil \frac{k}{2} \rceil$. If $f = (V_{-1}, V_1, V_2)$ is an STIkDF on D, then

(a) $|V_{-1}| + |V_1| + |V_2| = n$.

(b) $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$.

(c) If $\delta^-(D) \geq \lceil \frac{k}{2} \rceil + t$ with an integer $t \geq 0$, then $V_1 \cup V_2$ is a total $\lceil \frac{3k+2t}{6} \rceil$-dominating set of D.

Proof. Since (a) and (b) are immediate, we only prove (c). Suppose on the contrary, that there exists a vertex v with at most $\lceil \frac{3k+2t}{6} \rceil - 1$ in-neighbors in $V_1 \cup V_2$. Then v has at least

$$\begin{aligned}\delta^-(D) - \left(\left\lceil \frac{3k+2t}{6} \right\rceil - 1\right) &\geq \left\lceil \frac{k}{2} \right\rceil + t - \left(\left\lceil \frac{3k+2t}{6} \right\rceil - 1\right)\\ &\geq \frac{k}{2} + t - \left(\frac{3k+2t+5}{6} - 1\right) = \frac{2t}{3} + \frac{1}{6}\end{aligned}$$

and so at least $\lceil \frac{2t}{3} + \frac{1}{6} \rceil$ in-neighbors in V_{-1}. It follows that

$$\begin{aligned} k \leq f(N^-(v)) &\leq 2\left(\left\lceil \frac{3k+2t}{6} \right\rceil - 1\right) - \left\lceil \frac{2t}{3} + \frac{1}{6} \right\rceil \\ &\leq 2\left(\frac{3k+2t+5}{6} - 1\right) - \frac{2t}{3} - \frac{1}{6} = k - \frac{1}{2}, \end{aligned}$$

which is a contradiction. Consequently, $V_1 \cup V_2$ is a total $\lceil \frac{3k+2t}{6} \rceil$-dominating set of D. □

Corollary 3. If D is a digraph of order n and minimum in-degree $\delta^- \geq \lceil \frac{k}{2} \rceil + s$ with an integer $s \geq 0$, then $\gamma_{stI}^k(D) \geq 2\gamma_{t\lceil \frac{3k+2s}{6} \rceil}(D) - n$.

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{stI}^k(D)$-function. Then it follows from Proposition 3 that

$$\begin{aligned} \gamma_{stI}^k(D) &= |V_1| + 2|V_2| - |V_{-1}| = 2|V_1| + 3|V_2| - n \\ &\geq 2|V_1 \cup V_2| - n \geq 2\gamma_{t\lceil \frac{3k+2s}{6} \rceil}(D) - n. \end{aligned}$$

□

The digraphs qK_2^* and qK_4^* as well as qK_3^* show that Corollary 3 is sharp for $k = 1$ as well as for $k = 2$. The proof of the next proposition is identically with the proof of Theorem 1 in [7] and is therefore omitted.

Proposition 4. Let $k \geq 1$ be an integer, and let D be a digraph of order n with minimum in-degree $\delta^- \geq k/2$, maximum out-degree Δ^+ and minimum out-degree δ^+. If $f = (V_{-1}, V_1, V_2)$ is an STIkDF on D, then

(i) $(2\Delta^+ - k)|V_2| + (\Delta^+ - k)|V_1| \geq (\delta^+ + k)|V_{-1}|$.

(ii) $(2\Delta^+ + \delta^+)|V_2| + (\Delta^+ + \delta^+)|V_1| \geq (\delta^+ + k)n$.

(iii) $(\Delta^+ + \delta^+)\omega(f) \geq (\delta^+ - \Delta^+ + 2k)n + (\delta^+ - \Delta^+)|V_2|$.

(iv) $\omega(f) \geq (\delta^+ - 2\Delta^+ + 2k)n/(2\Delta^+ + \delta^+) + |V_2|$.

3. Bounds on the signed total Italian k-domination number

We start with a general upper bound, and we characterize all extremal digraphs.

Theorem 1. Let D be a digraph of order n with $\delta^-(D) \geq \lceil \frac{k}{2} \rceil$. Then $\gamma_{stI}^k(D) \leq 2n$, with equality if and only if k is even, $\delta^-(D) = \frac{k}{2}$, and each vertex of D has an out-neighbor of minimum in-degree.

Proof. Define the function $g : V(D) \longrightarrow \{-1, 1, 2\}$ by $g(x) = 2$ for each vertex $x \in V(D)$. Since $\delta^-(D) \geq \lceil \frac{k}{2} \rceil$, the function g is an STIkDF on D of weight $2n$ and thus $\gamma^k_{stI}(D) \leq 2n$.
Now let k be even, $\delta^-(D) = \frac{k}{2}$, and assume that each vertex of D has an out-neighbor of minimum in-degree. Let f be an STIkDF on D, and let $x \in V(D)$ be an arbitrary vertex. Then x has an out-neighbor v with $d^-(v) = \frac{k}{2}$. Therefore the condition $f(N^-(v)) \geq k$ implies $f(x) = 2$. Thus f is of weight $2n$, and we obtain $\gamma^k_{stI}(D) = 2n$.
Conversely, assume that $\gamma^k_{stI}(D) = 2n$. If $k = 2p + 1$ is odd, then $\delta^-(D) \geq p + 1$. Define the function $h : V(D) \longrightarrow \{-1, 1, 2\}$ by $h(w) = 1$ for an arbitrary vertex w and $h(x) = 2$ for each vertex $x \in V(D) \setminus \{w\}$. Then

$$h(N^-(v)) = \sum_{x \in N^-(v)} f(x) \geq 2(p + 1) - 1 = 2p + 1 = k$$

for each vertex $v \in V(D)$. Thus the function h is an STIkDF on D of weight $2n - 1$, and we obtain the contradiction $\gamma^k_{stI}(D) \leq 2n - 1$.
Let now k even, and assume that there exists a vertex w such that $d^-(x) \geq \frac{k}{2} + 1$ for each $x \in N^+(w)$. Define the function $h_1 : V(D) \longrightarrow \{-1, 1, 2\}$ by $h_1(w) = 1$ and $h_1(x) = 2$ for each vertex $x \in V(D) \setminus \{w\}$. Then $h_1(N^-(w)) \geq k$, $h_1(N^-(x)) \geq 2(\frac{k}{2} + 1) - 1 = k + 1$ for each $x \in N^+(w)$ and $h_1(N^-(y)) \geq k$ for each $y \notin N^+[w]$. Hence the function h_1 is an STIkDF on D of weight $2n - 1$, a contradiction to the assumption $\gamma^k_{stI}(D) = 2n$. This completes the proof. □

The proof of Theorem 1 also leads to the next result.

Theorem 2. Let D be a digraph of order n with $\delta^-(D) \geq \lceil \frac{k}{2} \rceil$. Then $\gamma^k_{stR}(D) \leq 2n$, with equality if and only if k is even, $\delta^-(D) = \frac{k}{2}$, and each vertex of D has an out-neighbor of minimum in-degree.

Observation 2. *If D is a digraph of order n with $\delta^-(D) \geq k$, then $\gamma^k_{stI}(D) \leq \gamma^k_{stR}(D) \leq n$.*

Proof. Define the function $f : V(D) \longrightarrow \{-1, 1, 2\}$ by $f(x) = 1$ for each vertex $x \in V(D)$. Since $\delta^-(D) \geq k$, the function f is an STRkDF on D of weight n and thus $\gamma^k_{stI}(D) \leq \gamma^k_{stR}(D) \leq n$. □

A digraph D is *out-regular* or r-*out-regular* if $\delta^+(D) = \Delta^+(D) = r$. As an application of Proposition 4 (iii) and (iv), we obtain the following lower bound on the signed total Italian k-domination number.

Corollary 4. If D is a digraph of order n, minimum in-degree $\delta^- \geq \frac{k}{2}$, maximum out-degree Δ^+ and minimum out-degree δ^+, then

$$\gamma^k_{stI}(D) \geq \left\lceil \frac{2\delta^+ + 3k - 2\Delta^+}{2\Delta^+ + \delta^+} n \right\rceil .$$

Proof. If D is an r-out-regular digraph, then the lower bound is an immediate consequence of Proposition 4 (iii). Let now $\Delta^+ > \delta^+$. Multiplying both sides of the inequality in Proposition 4 (iv) by $\Delta^+ - \delta^+$ and adding the resulting inequality to the inequality in Proposition 4 (iii), we obtain the desired lower bound. □

Since $\gamma^k_{stR}(D) \ge \gamma^k_{stI}(D)$, the lower bound of Corollary 4 is also valid for $\gamma^k_{stR}(D)$ (see [7]). Corollary 4 and Observation 1 lead to the next known bound.

Corollary 5. ([13, 15]) If G is a graph of order n, minimum degree $\delta \ge \frac{k}{2}$ and maximum degree Δ, then

$$\gamma^k_{stR}(G) \ge \gamma^k_{stI}(G) \ge \left\lceil \frac{2\delta + 3k - 2\Delta}{2\Delta + \delta} n \right\rceil.$$

Examples 12 and 13 in [13] demonstrate that Corollary 5 is sharp and therefore Corollary 4 is sharp too. The special case $k = 1$ of Corollary 4 can be found in [14]. A digraph D is *r-regular* if $\Delta^-(D) = \Delta^+(D) = \delta^-(D) = \delta^+(D) = r$.

Example 1. *If H is a k-regular digraph of order n, then it follows from Corollary 4 that $\gamma^k_{stI}(H) \ge n$ and thus $\gamma^k_{stI}(H) = n$, according to Observation 2.*

Example 1 shows that Observation 2 and Corollary 4 are both sharp.

Example 2. *If C_n is an oriented cycle of length n, then C_n is 1-regular. Hence Example 1 implies $\gamma_{stI}(C_n) = n$. In addition, Theorem 1 leads to $\gamma^2_{stI}(C_n) = 2n$. These are further examples showing the sharpness of Corollary 4.*

Theorem 3. If D is a digraph of order n with $\delta^-(D) \ge \frac{k}{2}$, then

$$\gamma^k_{stI}(D) \ge k + 2 + \delta^-(D) - n.$$

If in addition $\delta^-(D) - k$ is odd, then $\gamma^k_{stI}(D) \ge k + 3 + \delta^-(D) - n$.

Proof. Let f be a $\gamma^k_{stI}(D)$-function. Then there exists a vertex w with $f(w) \ge 1$. It follows from the definitions that

$$\begin{aligned} \gamma^k_{stI}(D) &= \sum_{x \in V(D)} f(x) = f(w) + \sum_{x \in N^-(w)} f(x) + \sum_{x \in V(D) - N^-[w]} f(x) \\ &\ge 1 + k - (n - d^-(w) - 1) = k + 2 + d^-(w) - n \\ &\ge k + 2 + \delta^-(D) - n. \end{aligned}$$

Now assume that $\delta^-(D) - k$ is odd. If there exists a vertex w with $f(w) = 2$ or a vertex v with $f(v) = 1$ and $d^-(v) \ge \delta^-(D) + 1$, then the inequality chain above leads to $\gamma^k_{stI}(D) \ge k + 3 + \delta^-(D) - n$. So assume that $f(x) \in \{-1, 1\}$ for each vertex

$x \in V(D)$, and each vertex y with $f(y) = 1$ has in-degree $d^-(y) = \delta^-(D)$. Let now u be a vertex of minimum degree with $f(u) = 1$. Since $\delta^-(D) - k$ is odd, we observe that

$$\sum_{x \in N^-(u)} f(x) \geq k + 1$$

and thus

$$\begin{aligned}
\gamma^k_{stI}(D) &= \sum_{x \in V(D)} f(x) = f(u) + \sum_{x \in N^-(u)} f(x) + \sum_{x \in V(D) - N^-[u]} f(x) \\
&\geq 1 + k + 1 - (n - d^-(u) - 1) = k + 3 + d^-(u) - n \\
&= k + 3 + \delta^-(D) - n.
\end{aligned}$$

This completes the proof. □

Corollary 1 shows that Theorem 3 is sharp for $k \leq n - 1$. If $\Delta^-(D) \geq \delta^-(D) + 3$, then the next lower bound is an improvement of Theorem 3.

Proposition 5. If D is a digraph of order n with $\delta^-(D) \geq \frac{k}{2}$, then

$$\gamma^k_{stI}(D) \geq k + \Delta^-(D) - n.$$

Proof. Let $w \in V(G)$ be a vertex of maximum in-degree, and let f be a $\gamma^k_{stI}(D)$-function. Then the definitions imply

$$\begin{aligned}
\gamma^k_{stI}(D) &= \sum_{x \in V(D)} f(x) = \sum_{x \in N^-(w)} f(x) + \sum_{x \in V(D) - N^-(w)} f(x) \\
&\geq k + \sum_{x \in V(D) - N^-(w)} f(x) \geq k - (n - \Delta^-(D)) \\
&= k + \Delta^-(D) - n,
\end{aligned}$$

and the proof of the desired lower bound is complete. □

Theorem 4. Let $k \geq 2$ be an integer. If D is a digraph of order n with $\delta^-(D) \geq \lceil \frac{k}{2} \rceil$, then

$$\gamma^k_{stI}(D) \geq k + 3 + \left\lceil \frac{k}{2} \right\rceil - n.$$

Proof. If $\delta^-(D) \geq \lceil \frac{k}{2} \rceil + 1$, then Theorem 3 implies the desired bound. Let now $\delta^-(D) = \lceil \frac{k}{2} \rceil$, and let f be a $\gamma^k_{stI}(D)$-function. We show that there exists a vertex w with $f(w) = 2$. Suppose on the contrary that $f(x) \in \{-1, 1\}$ for each vertex $x \in V(D)$. Since $\delta^-(D) = \lceil \frac{k}{2} \rceil$ and $k \geq 2$, we obtain the contradiction

$$\sum_{x \in N^-(u)} f(x) \leq \left\lceil \frac{k}{2} \right\rceil < k$$

for every vertex u of minimum in-degree. Hence there exists a vertex w with $f(w) = 2$, and as in the proof of Theorem 3, it follows that

$$\begin{aligned}\gamma^k_{stI}(D) &= \sum_{x\in V(D)} f(x) = f(w) + \sum_{x\in N^-(w)} f(x) + \sum_{x\in V(D)-N^-[w]} f(x)\\ &\geq 2 + k - (n - d^-(w) - 1) \geq k + 3 + \delta^-(D) - n\\ &= k + 3 + \left\lceil \frac{k}{2} \right\rceil - n.\end{aligned}$$

□

Example 3. *Let $k \geq 2$ and $n \geq \lceil \frac{k}{2} \rceil + 1$ be integers. Let the digraph H consists of $H_1 = K^*_{\lceil \frac{k}{2} \rceil+1}$ and further vertices $v_1, v_2, \dots, v_{n-1-\lceil \frac{k}{2} \rceil}$ such that every vertex of H_1 dominates v_i for $1 \leq i \leq n - 1 - \lceil \frac{k}{2} \rceil$.*
If k is even, then define the function $g : V(H) \longrightarrow \{-1, 1, 2\}$ by $g(x) = 2$ for $x \in V(H_1)$ and $g(v_i) = -1$ for $1 \leq i \leq n - 1 - \lceil \frac{k}{2} \rceil$. We observe that $g(N^-(x)) = k$ for $x \in V(H_1)$ and $g(N^-(v_i)) = k + 2$ for $1 \leq i \leq n - 1 - \lceil \frac{k}{2} \rceil$. Therefore g is an STRkDF on H of weight $\omega(g) = k + 3 + \lceil \frac{k}{2} \rceil - n$.
If k is odd, then define the function $h : V(H) \longrightarrow \{-1, 1, 2\}$ by $h(w) = 1$ for one vertex $w \in V(H_1)$, $h(x) = 2$ for $x \in V(H_1) \setminus \{w\}$ and $g(v_i) = -1$ for $1 \leq i \leq n - 1 - \lceil \frac{k}{2} \rceil$. We observe that $h(N^-(x)) \geq k$ for $x \in V(H_1)$ and $h(N^-(v_i)) = k + 2$ for $1 \leq i \leq n - 1 - \lceil \frac{k}{2} \rceil$. Therefore h is an STRkDF on H of weight $\omega(h) = k + 3 + \lceil \frac{k}{2} \rceil - n$.
Hence Theorem 4 implies $\gamma^k_{stI}(H) = k + 3 + \lceil \frac{k}{2} \rceil - n$ in both cases and thus Theorem 4 is sharp.

A set $S \subseteq V(D)$ is a *2-packing* of the digraph D if $N^-[u] \cap N^-[v] = \emptyset$ for any two distinct vertices $u, v \in S$. The maximun cardinality of a 2-packing in D is the *2-packing number* of D, denoted by $\rho(D)$. Analogously to Theorem 4 in [7], one can prove the next lower bound on the signed total Italian k-domination number.

Theorem 5. If D is a digraph of order n with $\delta^-(D) \geq \frac{k}{2}$, then

$$\gamma^k_{stI}(D) \geq \rho(D)(k + \delta^-(D)) - n.$$

Observation 1, Theorem 5 and the fact that $\gamma^k_{stI}(G) \leq \gamma^k_{stR}(G)$ lead to the following known result.

Corollary 6. ([13]) If G is a graph of order n with $\delta(G) \geq \frac{k}{2}$, then

$$\gamma^k_{stR}(G) \geq \rho(G)(k + \delta(G)) - n.$$

In [13], the author presents an infinite family of graphs achieving equality in Corollary 6. Thus Corollary 6 and Theorem 5 are sharp.

The *complement* $\overline{D}$ of a digraph D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u and v the arc (u,v) belongs to $\overline{D}$ if and only if (u,v) does not belong to D. Using Corollary 4, one can prove the following Nordhaus-Gaddum type inequality analogously to Theorem 6 in [7].

Theorem 6. If D is an r-regular digraph of order n such that $r \geq \frac{k}{2}$ and $n-r-1 \geq \frac{k}{2}$, then

$$\gamma^k_{stI}(D) + \gamma^k_{stI}(\overline{D}) \geq \frac{4kn}{n-1}.$$

If n is even, then $\gamma^k_{stI}(D) + \gamma^k_{stI}(\overline{D}) \geq 4k(n-1)/(n-2)$.

Let $k \geq 1$ be an integer, and let H and $\overline{H}$ be k-regular digraphs of order $n = 2k+1$. By Example 1, we have $\gamma^k_{stI}(H) = \gamma^k_{stI}(\overline{H}) = n$. Consequently,

$$\gamma^k_{stI}(H) + \gamma^k_{stI}(\overline{H}) = 2n = \frac{4kn}{n-1}.$$

Thus the Nordhaus-Gaddum bound of Theorem 6 is sharp.

4. Contrafunctional digraphs

The *underlying graph* of a digraph is that graph obtained by replacing each arc (u,v) or symmetric pairs (u,v), (v,u) of arcs by the edge uv. A digraph is *connected* if its underlying graph is connected. A rooted tree is a connected digraph with a vertex r of in-degree 0, called the root, such that every vertex different from the root has in-degree 1. A digraph D is *contrafunctional* if each vertex of D has in-degree one. In [8], Harary, Norman and Cartwright have shown that every connected contrafunctional digraph has a unique oriented cycle and the removal of any arc of the oriented cycle results in a rooted tree.
We start with a general upper bound.

Theorem 7. Let D be a digraph of order n such that $\delta^-(D) \geq \lceil \frac{k}{2} \rceil$, and let t be the number of vertices $x \in V(D)$ with $d^+(x) = 0$. Then $\gamma^k_{stI}(D) \leq \gamma^k_{stR}(D) \leq 2n - 3t$, and if $k \geq 3$ is odd, then $\gamma^k_{stI}(D) \leq \gamma^k_{stR}(D) \leq 2n - 3t - 1$.

Proof. Let $X \subset V(D)$ be the set of vertices with the property that $d^+(x) = 0$ for $x \in X$. Define the function $g : V(D) \longrightarrow \{-1, 1, 2\}$ by $g(x) = -1$ for $x \in X$ and $g(x) = 2$ for $x \in V(D) \setminus X$. Since $N^-(v) \cap X = \emptyset$ for each $v \in V(D)$, it follows that $g(N^-(v)) = \sum_{x \in N^-(v)} g(x) \geq 2\lceil \frac{k}{2} \rceil \geq k$ for each vertex $v \in V(D)$. As each vertex x with $g(x) = -1$ has in-neighbor v with $f(v) = 2$, we deduce that g is an STRkDF on D of weight $\omega(g) = 2(n-t) - t = 2n - 3t$ and thus $\gamma^k_{stI}(D) \leq \gamma^k_{stR}(D) \leq 2n - 3t$.
Let now $k = 2p+1 \geq 3$ be odd. Then $\lceil \frac{k}{2} \rceil = p+1$. Define the function $h : V(D) \longrightarrow \{-1, 1, 2\}$ by $h(x) = -1$ for $x \in X$, $h(w) = 1$ for an arbitrary vertex $w \in V(D) \setminus X$

and $h(x) = 2$ for $x \in V(D) \setminus (X \cup \{w\})$. Then $h(N^-(w)) = \sum_{x\in N^-(w)} h(x) \geq 2\lceil \frac{k}{2} \rceil = 2(p+1) = k+1$ and $h(N^-(v)) = \sum_{x\in N^-(v)} h(x) \geq 2\lceil \frac{k}{2} \rceil - 1 = k$ for $v \in V(D) \setminus \{w\}$. Since $k \geq 3$, each vertex x with $g(x) = -1$ has an in-neighbor v with $f(v) = 2$. Therefore h is an STRkDF on D of weight $\omega(h) = 2(n-t) - 1 - t = 2n - 3t - 1$ and thus $\gamma^k_{stI}(D) \leq \gamma^k_{stR}(D) \leq 2n - 3t - 1$ in this case. □

Example 3 demonstrates that Theorem 7 is sharp for $k \geq 2$.

Theorem 8. Let D be a connected contrafunctional digraph of order n. If t is the number of vertices $x \in V(D)$ with $d^+(x) = 0$, then $\gamma^2_{stR}(D) = \gamma^2_{stI}(D) = 2n - 3t$.

Proof. Theorem 7 implies $\gamma^2_{stI}(D) \leq \gamma^2_{stR}(D) \leq 2n - 3t$.
Now let $X \subset V(D)$ be the set of vertices with the property that $d^+(x) = 0$ for $x \in X$, and let f be a $\gamma^2_{stI}(D)$-function. Since every vertex of D has in-degree one, we observe that $f(x) = 2$ for each vertex x with $d^+(x) \geq 1$. Clearly, $f(x) \geq -1$ for $x \in X$ and so $\omega(f) \geq 2(n-t) - t = 2n - 3t$. Consequently, $\gamma^2_{stR}(D) = \gamma^2_{stI}(D) = 2n - 3t$. □

In view of Theorem 8, we see that the connected contrafunctional digraphs is a further family of digraphs which show that Theorem 7 is sharp for $k = 2$.

Theorem 9. Let D be a connected contrafunctional digraph of order n. If $X \subset V(D)$ is the set of vertices with $d^+(x) = 0$ for $x \in X$, and Y the set of in-neighbors of X, then

$$\gamma_{stR}(D) = \gamma_{stI}(D) = n + |Y| - 2|X|.$$

Proof. First we note that the sets X and Y are disjoint. Define the function $g : V(D) \longrightarrow \{-1, 1, 2\}$ by $g(x) = -1$ for $x \in X$ and $g(x) = 2$ for $y \in Y$ and $g(u) = 1$ for $u \in V(D) \setminus (X \cup Y)$. Then g is an STRDF on D of weight $\omega(g) = n - |X| - |Y| + 2|Y| - |X| = n + |Y| - 2|X|$ and thus $\gamma_{stI}(D) \leq \gamma_{stR}(D) \leq n + |Y| - 2|X|$.
Conversely, let f be a $\gamma_{stI}(D)$-function. Since every vertex of D has in-degree one, we see that $f(x) \geq 1$ for each vertex $x \in V(D) \setminus (X \cup Y)$. In addition, if $w \in Y$ is the unique in-neighbor of a vertex $x_1 \in X$ with $d^+(x_1) = 0$ and $x_1, x_2, \ldots, x_p$ are all out-neighbors of w such that $d^+(x_i) = 0$ for $1 \leq i \leq p$, then we observe that

$$f(w) + \sum_{i=1}^{p} f(x_i) \geq 2 - p.$$

Using this observation, we deduce that $f(X \cup Y) \geq 2|Y| - |X|$. Combining this with $f(x) \geq 1$ for each vertex $x \in V(D) \setminus (X \cup Y)$, we obtain $\omega(f) \geq n - |X| - |Y| + 2|Y| - |X| = n - 2|X| + |Y|$, and so $\gamma_{stR}(D) \geq \gamma_{stI}(D) \geq n + |Y| - 2|X|$. Consequently, $\gamma_{stR}(D) = \gamma_{stI}(D) = n + |Y| - 2|X|$. □

Let $C_n = v_1 v_2 \dots v_p v_1$ be an oriented cycle of order $p \geq 2$, and let $u_1, u_2, \dots, v_p$ be p further vertices such that v_i dominates u_i for $1 \leq i \leq p$. The resulting digraph H is a contrafunctional digraph of order $n(H) = 2p$ with $|X| = |Y| = p$. Hence it follows from Theorem 9 that

$$\gamma_{stR}(H) = \gamma_{stI}(H) = n(H) + |Y| - 2|X| = 2p + p - 2p = p = 2n(H) - 3|X|.$$

This class of digraphs shows that Theorem 7 is sharp for $k = 1$ too.
Note that Example 2 is a special case of Theorems 8 and 9.

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